The couples of bilinear reflexive forms, one of which is non degenerate

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Abstract. Let E be a finite dimensional vectorspace on a field $\mathbb{K}$ with $\mathbb{K}$ algebraically closed or $\mathbb{K} = \mathbb{R}$ and let a and b be two bilinear reflexive forms over E, with a non degenerate. We give a set of basis of E in which a and b take simultaneously a "preferred" form. This generalizes the fact that, if $K = \mathbb{R}$ and if a and b are symmetric, with a positive definite, some basis make a and b simultaneously diagonal. Besides, we decompose the subgroup of $\mathrm{GL}(E)$ preserving a and b as a semi-direct product $R \ltimes N$ with N nilpotent and, if $\mathbb{K} = \mathbb{R}$, R reductive.

Résumé. Soit E un espace vectoriel sur un corps $\mathbb{K}$ avec $\mathbb{K}$ algébriquement clos ou $\mathbb{K} = \mathbb{R}$ et a et b deux formes bilinéaires réflexives sur E, avec a non-dégénérée. Nous donnons un ensemble de bases de E où a et b prennent simultanément une forme "préférée". Ceci généralise le fait que, si $\mathbb{K} = \mathbb{R}$ et si a et b sont symétriques avec a définie positive, a et b admettent des bases où elles sont simultanément diagonales. Par ailleurs, nous décomposons le sous-groupe de $\mathrm{GL}(E)$ préservant a et b en un produit semi-direct $R \ltimes N$ avec N nilpotent et, si $\mathbb{K} = \mathbb{R}$, R réductif.

1 Introduction

Let a and b be two bilinear symmetric forms on a finite dimensional real vector space E. If a is positive definite, then a and b admit a standard simultaneous reduction, given by a basis β in which $\mathrm{Mat}_\beta(a) = \mathrm{Id}$ and $\mathrm{Mat}_\beta(b)$ is diagonal. By the way, this shows that the group $\mathrm{O}(a) \cap \mathrm{O}(b)$ is isomorphic to $\prod_{\lambda \in \Lambda} \mathrm{O}_{d_\lambda}(\mathbb{R})$, where Λ is the set of the eigenvalues of $\mathrm{Mat}_\beta(b)$ and d_λ the dimension of the eigenspace associated to λ.

What happens if we drop the assumption "a positive definite" or even assume simply that a and b are bilinear reflexive i.e. symmetric or skew-symmetric? It seems that no both exhaustive and easy to use answer to that question exists in the literature. Many authors dealt with it in the second half of the nineteenth century. The first, and quite complete, work on it is due to Weierstraß [8] but it is difficult to understand and to use, it deals with the general case (a, b not necessarily reflexive) and no version of it in a modern language seems to be available. A simultaneous reduction of a and b is given (§3, formula 38), but to what it corresponds matricially is a bit hidden. Besides, though he studies, at the end of his paper (§7.F), the case where a and b are symmetric and the base field is $\mathbb{R}$ and makes coefficients $\varepsilon_\lambda = \pm 1$ appear, he does not give their sense and does not bring to the fore that a system of signatures appears as an invariant of the couple (a, b) (see here Proposition 3.12). Among others, Darboux [2] also worked on the question; he re-obtains and completes the results of Weierstraß, by another method. A very detailed survey on it can be found in [7][2], pp. 386–518.

More recently, the problem was partially treated by Klingenberg in [6] and, by the way as a needed tool by Hua in [3] (two hermitian forms, §7-10 pp. 546–553), [4] (two bilinear symmetric forms on an algebraically closed field, stated without proof pp. 452–453) and [5] (a bilinear symmetric form and a hermitian form over $\mathbb{C}$, formula 31 p. 516).

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[2]This reference is a French translation, with additions, of the German *Encyclopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, Teubner Verlag, 1898. This survey is an addition, so cannot be found in the German original.

We treat it here, if the base field is algebraically closed and of characteristic different from two or is $\mathbb{R}$, and in the (easy) case where a or b is non degenerate. More precisely:

(i) We give the form of the subgroup of $\mathrm{GL}(E)$ preserving a and b, by letting this subgroup act on geometrical objects, and we give the invariants of the couple (a, b) under conjugation by $\mathrm{GL}(E)$. This geometrical treatment of the question is new in the literature.

(ii) We give a set of basis of E in which a and b take simultaneously a "preferred" form.

In case a and b are both symmetric or both skew-symmetric, [6] gives **(ii)** —in a different way than ours— but nothing in the direction of **(i)**. In case the field is $\mathbb{C}$ and a and b are both hermitian, §8 of [3] gives also **(ii)** and the invariants of such a couple (a, b) under conjugation by $\mathrm{GL}(E)$: the roots of $\lambda \mapsto \det(\lambda a + b)$ and some integers called "the system of signatures of the pair of forms with respect to [the real roots of $\lambda \mapsto \det(\lambda a + b)$]".

In this paper, **(i)** is given, in the different involved cases, by Theorems 3.11 p. 7, 3.13 p. 8, 4.3 p. 13 and 4.7 p. 17; the invariants of the couple (a, b) under conjugation by $\mathrm{GL}(E)$ are given by Propositions 3.12 p. 8, 3.15 p. 11, 4.6 p. 16 and 4.10 p. 21. When the base field is $\mathbb{R}$, **(ii)** i.e. the simultaneaous "Weierstraß" matricial reductions of a and b, based on Proposition 3.6 and its Corollary 3.7, are summed up, in the different cases, in the appendix of §6 pp. 22–25. *This appendix, useful by itself, may be consulted independently of the article.*

A natural complement to this work is a similar classification of the pairs of complex reflexive forms i.e. bilinear symmetric or skew-symmetric, or hermitian. Sesquilinear forms behave similarly, with some differences however. We hope to provide this complement soon.

1.1 Notation Throughout, a and b stand for bilinear reflexive forms on a finite dimensional space E over a field $\mathbb{K}$ of characteristic different from two; $\mathbb{K}$ is algebraically closed or is $\mathbb{R}$, a is supposed to be non degenerate. Therefore, b can be written as $b = a(\,\cdot\,, B\,\cdot\,)$ with B a (skew-)adjoint endomorphism of E. As $\mathbb{K}$ is in all case perfect, B admits a unique decomposition $B = S + T$ with S semi-simple, T nilpotent and $ST = TS$. The *commutant* $\mathrm{C}(M)$ of an endomorphism $M \in \mathrm{End}(E)$ means here the subgroup $\{\gamma \in \mathrm{GL}(E);\, \gamma M = \mathrm{M}\gamma\}$ of $\mathrm{GL}(E)$. One denotes by $\mathrm{Stab}(a)$ the subgroup of $\mathrm{GL}(E)$ preserving a; $\mathrm{Stab}(a) = \mathrm{O}(a)$ if a is symmetric, $\mathrm{Stab}(a) = \mathrm{Sp}(a)$ if a is skew-symmetric.

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2 Recall: the commutant of a nilpotent endomorphism

We recall here classical facts the following is based on.

2.1 Notation If T is a nilpotent endomorphism of a finite dimensional vector space E, we introduce, for $p, q \in \mathbb{N}$, $E_{p,q} = \mathrm{Im}\, T^{n-p} \cap \mathrm{Ker}\, T^q$ where n is the nilpotence index of T.

The $E_{p,q}$ satisfy:

- $q \geq p \Rightarrow E_{p,q} = E_{p,p}$,
- $\forall k \in \mathbb{N}$, $E_{n,k} = \mathrm{Ker}\, T^k$ and $E_{k,k} = \mathrm{Im}\, T^{n-k}$,
- $(p \leq p'$ and $q \leq q') \Rightarrow E_{p,q} \subset Ep', q'$,
- $\forall p.q \in \mathbb{N}$, $T(E_{p,q}) = E_{p-1,q-1}$.

So, they are ordered by $\subset$ in the following way:

$$\begin{array}{ccccccccc}
E_{1,1} & = \operatorname{Im} T^{n-1} & & & & & & & \\
\cap & & & & & & & & \\
E_{2,1} & \subset & E_{2,2} & = \operatorname{Im} T^{n-2} & & & & & \\
\cap & & \cap & & & & & & \\
E_{3,1} & \subset & E_{3,2} & \subset & E_{3,3} & = \operatorname{Im} T^{n-3} & & & \\
\cap & & \cap & & & & & & \\
\vdots & & \vdots & & & \ddots & & & \\
\cap & & \cap & & & & & & \\
E_{n,1} & \subset & E_{n,2} & \subset & \cdots & \subset & E_{n,n} & = \operatorname{Im} T^0 = E & \\
\| & & \| & & & & \| & & \\
\operatorname{Ker} T & & \operatorname{Ker} T^2 & & & & \operatorname{Ker} T^n & &
\end{array}$$

2.2 Notation for $k \leq n$, we denote by $\check{E}_k$ the quotient space $E_{n,k}/(E_{n-1,k} + E_{n,k-1})$.

The $E_{p,q}$ are natural to introduce: they are the only subspaces of E canonically associated to T, in the following sense.

2.3 Proposition *Let F be a subspace of E, then F is stable by $\mathrm{C}(T)$ if and only if it is of the form $F = +_{p,q\in\mathcal{E}} E_{p,q}$ with $\mathcal{E} \subset \{(p,q) \in [\![1,n]\!]^2; p \geq q\}$.*

Proof. The sense $\boxed{\Leftarrow}$ is immediate; to prove $\boxed{\Rightarrow}$, let us take an F which is not of the form $F = +_{p,q\in\mathcal{E}} E_{p,q}$ and build a $u \in \mathrm{C}(T)$ such that $u(F) \neq F$. Suppose that for all (p,q), $E_{p,q} \subset F$ or $F \cap E_{p,q} \subset (E_{p-1,q} + E_{p,q-1})$. Then let $\mathcal{E}_0$ be the set of the (p,q) minimal for $\subset$ such that $E_{p,q} \not\subset F$, then $(p,q) \in \mathcal{E}_0 \Rightarrow F \cap E_{p,q} \subset (E_{p-1,q} + E_{p,q-1})$ i.e. $F = +_{(p,q)\in\mathcal{E}_0}(E_{p-1,q} + E_{p,q-1})$. So, as F is not of the form $+_{p,q\in\mathcal{E}} E_{p,q}$, we find a (p,q) such that $E_{p,q} \not\subset F$ and $F \cap E_{p,q} \not\subset (E_{p-1,q} + E_{p,q-1})$. In other words, we find a (p,q) and $x, y \in E_{p,q} \setminus (E_{p-1,q} + E_{p,q-1})$ such that $x \in F$ and $y \notin F$.

Then, let us take $x' \in (T^{n-p})^{-1}(x) \subset E_{n,q+n-p} \setminus (E_{n-1,q+n-p} + E_{n,q+n-p-1})$ and $y' \in (T^{n-p})^{-1}(y) \subset E_{n,q+n-p} \setminus (E_{n-1,q+n-p} + E_{n,q+n-p-1})$; the family $\varphi = ((T^k(x'))_{k=0}^{q+n-p-1}, (T^k(y'))_{k=0}^{q+n-p-1})$, stable by T by construction, can be completed to form a Jordan basis of T. Let us set $E_1 = \operatorname{span}(\varphi)$ and denote by E_2 a vector space spanned by some additional Jordan basis vectors. Both E_i are T-stable. Now we set u linear and commuting with T, such that $u(x') = y'$, $u(y') = x'$ and $u_{|E_2} = \mathrm{Id}_{E_2}$; u is as wanted: $u(F) \not\subset F$ as $u(x) = y$ and u is bijective. $\square$

Besides, introducing the following notion, we obtain in 2.6 a decomposition of $\mathrm{C}(T)$.

2.4 Definition *A decomposition D of E of the form $E = \oplus_{p=1}^{n} \oplus_{q=1}^{p} D_{p,q}$ is called here adapted to T if*

(i) *each $D_{p,q}$ is a supplement of $E_{p-1,q} + E_{p,q-1}$ in $E_{p,q}$,* (2.1)

(ii) $\forall p \leq n$, $\forall q \leq p$, $D_{p,q} = T^{n-p}(D_{n,q+n-p})$. (2.2)

We denote by $\mathcal{D}_T$ the set of these decompositions.

So, if we mimic the diagram of the $E_{p,q}$, we have:

$$\begin{array}{ccccccccc}
D_{1,1} & & & & & & & & \\
\oplus & & & & & & & & \\
D_{2,1} & \oplus & D_{2,2} & & & & & & \\
\oplus & & \oplus & & & & & & \\
D_{3,1} & \oplus & D_{3,2} & \oplus & D_{3,3} & & & & \\
\oplus & & \oplus & & & & & & \\
\vdots & & \vdots & & & \ddots & & & \\
\oplus & & \oplus & & & & & & \\
D_{n,1} & \oplus & D_{n,2} & \oplus & \cdots & \oplus & D_{n,n} & = E, &
\end{array}$$

the action of T shifting the $D_{p,q}$ of one row upwards and of one column to the left.

2.5 Theorem *(following immediately from the existence of Jordan basis for T) Such decompositions exist.*

2.6 Proposition $\mathrm{C}(T) = R \ltimes N$ *where* $N = \{\gamma \in \mathrm{C}(T);\ \gamma$ *acts trivially on all* $\check{E}_k, k \leq n\}$; N *acts simply transitively on* $\mathcal{D}_T$ *and is nilpotent,* $R = \mathrm{C}(T)/N \simeq \prod_{k=1}^{n} \mathrm{GL}(\check{E}_k)$.

Proof. It is an easy verification. If $\gamma \in \mathrm{C}(T)$, γ acts on each $\check{E}_k$; N is the kernel of this morphism $\mathrm{C}(T) \mapsto \prod_{k=1}^{n} \mathrm{GL}(\check{E}_k)$ so is normal. For each $k \leq n$, let β_k be a basis of $\check{E}_k$; Let $D = (D_{p,q})_{q \leq p \leq n}$ and $\Delta = (\Delta_{p,q})_{q \leq p \leq n}$ be any two decompositions in $\mathcal{D}_T$. For each k, the projection $E_{n,k} \mapsto \check{E}_k$ gives two isomorphisms $D_{n,k} \simeq \check{E}_k$ and $\Delta_{n,k} \simeq \check{E}_k$; by those isomorpisms, β_k is mapped on a basis β_k^D of $D_{n,k}$, respectively a basis β_k^Δ of $\Delta_{n,k}$. By the different powers of T, those β_k^D and β_k^Δ are mapped on basis of each $D_{p,q}$, respectively $\Delta_{p,q}$, providing a basis β^D, respectively β^Δ, of E. The linear application $u : \beta^D \mapsto \beta^\Delta$ commutes with T by construction, as D and Δ are T-adapted decompositions, acts trivially on all $\check{E}_k$ and maps D on Δ, so N acts transitively on $\mathcal{D}_T$. Finally if γ and γ' in N map D on Δ, they map necessarily β^D on β^Δ, so $\gamma = \gamma'$: N acts simply transitively on $\mathcal{D}_T$.

The group N is nilpotent: let β be any basis of E formed by a basis of $E_{1,1}$, completed to form a basis of $E_{2,1}$ ... etc, on a way which preserves the inclusions of the $E_{p,q}$, and let γ be in $\mathrm{C}(T)$. Then, $\mathrm{Mat}_\beta(\gamma)$ is upper block-triangular, each block corresponding to a couple $((p,q),(p',q'))$ of indices of the $E_{p,q}$ (the block $((p,q),(p',q'))$ is null if and only if $E_{p,q} \not\subset E_{p',q'}$, in particular the matrix is block-upper triangular) . Now γ is in N if and only if its action on all quotients $E_{p,q}/(E_{p-1,q} + E_{p,q-1})$ is trivial i.e. if and only if the diagonal blocks, corresponding to the couples $((p,q),(p,q))$, are identity matrices. So, written in β, N is a subgroup of the upper triangular, unipotent matrices, thus is nilpotent.

To show the announced semi-direct product, let us exhibit a section of the third arrow in the exact sequence $\{0\} \to N \to \mathrm{C}(T) \to R \to \{0\}$. Fix any D in $\mathcal{D}_T$ and, to each class ρ in $R = \mathrm{C}(T)/N$, associate the unique $\gamma \in \rho \subset \mathrm{C}(T)$ such that $\gamma(D) = D$. This map is a section as wanted.

Finally, take as above some basis β_k of the $\check{E}_k$ and $D \in \mathcal{D}_T$, we take also $(h_k)_{k=1}^n \in \prod_{k=1}^{n} \mathrm{GL}(\check{E}_k)$ and denote $h(\beta_k)$ by β_k^h. As above, the datum of the β_k, β_k^h and of D gives two basis β^D and $\beta^{h,D}$ of E. Consider γ linear mapping β^D on $\beta^{h,D}$; $\gamma \in \mathrm{C}(T)$ and the quotient action of γ on the $\check{E}_k$ is, by construction, that of $(h_k)_{k=1}^n$. Besides, if $\gamma' \in \mathrm{C}(T)$ acts on the $\check{E}_k$ like $(h_k)_{k=1}^n$, then $\gamma^{-1} \circ \gamma'$ acts trivially on the $\check{E}_k$ i.e. is in N. So $R = \mathrm{C}(T)/N \simeq \prod_{k=1}^{n} \mathrm{GL}(\check{E}_k)$. □

3 When a and b are both symmetric or both skew-symmetric

Both these cases are treated simultaneously, as the endomorphism B is then a-selfadjoint. Besides, in theses cases, the condition "a or b non degenerate" can be replaced without changing the problem by "for some $(\lambda, \mu) \in \mathbb{K}^2$, $\lambda a + \mu b$ is non degenerate". We always suppose in the following that a is non degenerate.

3.a Preliminaries

3.1 Notation P denotes the minimal polynomial of B and $P = \prod_{i=1}^{d} P_i^{n_i}$ is its decomposition in a product of powers of mutually prime irreducible polynomials. With $E_i = \mathrm{Ker}(P_i^{n_i}(B))$ the characteristic subspace of B associated to P_i, $E = \oplus_{i=1}^{d} E_i$.

Before all, we shall do the following standard remark.

3.2 Lemma *The sum $E = \oplus_{i=1}^{d} E_i$ is a- and b-orthogonal. The group $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is equal to $\prod_{i=1}^{d} \mathrm{Stab}(E_i, a_{|E_i}) \cap \mathrm{Stab}(E_i, b_{|E_i})$.*

Proof. The second part of the statement follows from the fact that $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ preserves the E_i. For the first part, take $i \neq j$, $U, V \in \mathbb{K}[X]$ such that $UP_i^{n_i} + VP_j^{n_j} = 1$, $x \in E_i$ and $y \in E_j$. Then:

$$\begin{aligned} a(x,y) &= a((UP_i^{n_i} + VP_j^{n_j})(B)x, y) \\ &= a(UP_i^{n_i}(B)x, y) + a(x, VP_j^{n_j}(B)y) \quad \text{as } B \text{ is } a\text{-selfadjoint} \\ &= a(0,y) + a(x,0) \\ &= 0 \end{aligned}$$

and:

$$\begin{aligned} b(x,y) &= a((UP_i^{n_i} + VP_j^{n_j})(B)x, By) \\ &= a(UP_i^{n_i}(B)x, y) + a(x, VP_j^{n_j}(B)By) \\ &= a(0,y) + a(x,0) \\ &= 0. \end{aligned}$$

□

3.3 Convention Consequently, in the following, we focus on one single of the E_i, i.e., equivalently, we suppose that the minimal polynomial of B is of the form P^n with P irreducible.

To go further, we need the following definition. We recall that S and T are the semi-simple and the nilpotent part of B.

3.4 Definition *A decomposition D of E of the form $E = \oplus_{p=1}^{n} \oplus_{q=1}^{p} D_{p,q}$ is called here adapted to (a, T) if it is adapted to T (see Definition 2.4) and if, with respect to the form a, E is equal to the following sum:*

$$E = \left(\overset{\perp}{\bigoplus_{p+q \leq n}} \left(\underbrace{D_{p,q}}_{\text{tot. isotropic}} \oplus \underbrace{D_{n+1-q,n+1-p}}_{\text{tot. isotropic}} \right) \right) \overset{\perp}{\oplus} \left(\overset{\perp}{\bigoplus_{\frac{n+1}{2} \leq p \leq n}} \underbrace{D_{p,n+1-p}}_{\text{non degenerate}} \right). \tag{3.1}$$

We denote by $\mathcal{D}_{(a,T)}$ the set of these decompositions.

So, if D is adapted to (a,T), in the diagram:

$$
\begin{array}{ccccccccccccc}
D_{1,1} \\
\oplus \\
D_{2,1} & \oplus & D_{2,2} \\
\oplus & & \oplus \\
D_{3,1} & \oplus & D_{3,2} & \oplus & D_{3,3} \\
\oplus & & \oplus & & \oplus \\
\vdots & & \vdots & & \vdots & \ddots \\
\oplus & & \oplus & & \oplus \\
D_{n-2,1} & \oplus & D_{n-2,2} & \oplus & \boxed{D_{n-2,3}} & \oplus & \cdots & \oplus & D_{n-2,n-2} \\
\oplus & & \oplus & & \oplus \\
D_{n-1,1} & \oplus & \boxed{D_{n-1,2}} & \oplus & D_{n-1,3} & \oplus & \cdots & \oplus & D_{n-1,n-2} & \oplus & D_{n-1,n-1} \\
\oplus & & \oplus & & \oplus & & & & \oplus & & \oplus \\
\boxed{D_{n,1}} & \oplus & D_{n,2} & \oplus & D_{n,3} & \oplus & \cdots & \oplus & D_{n,n-2} & \oplus & D_{n,n-1} & \oplus & D_{n,n} = E,
\end{array}
$$

each boxed $D_{p,q}$ is non degenerate, orthogonal to any other term; the other ones are totally isotropic and the sum of two $D_{p,q}$ symmetrically placed with respect to the boxed diagonal is non degenerate, orthogonal to any other $D_{p,q}$.

Such decompositions exist. To show it, we prove a little more, showing by the way how, by induction, these decompositions are built (see the proof of Proposition 3.6). This Proposition 3.6 is in fact the result the whole paper is based on.

3.5 Notation The kernel of the reflexive form $a(\cdot\,,T^k\cdot)$ is $\operatorname{Ker}T^k = E_{n,k}$, so this form induces a non degenerate reflexive form, denoted by b^k, on $E/\operatorname{Ker}T^k$. Besides, T acts on $E/\operatorname{Ker}T^k$, we still denote by T this endomorphism of $E/\operatorname{Ker}T^k$.

3.6 Proposition *If $D^k = (D^k_{p,q})_{k+1\leq q\leq p\leq n}$ is a decomposition of $E/\operatorname{Ker}T^k$ adapted to $(a(\cdot\,,T^k\cdot),T)$, then there exists a decomposition $D^{k-1} = (D^{k-1}_{p,q})_{k-1\leq q\leq p\leq n}$ of $E/\operatorname{Ker}T^{k-1}$ adapted to $(a(\cdot\,,T^{k-1}\cdot),T)$, which projects itself, modulo $\operatorname{Ker}T^k$, onto D^k: if we set $\pi : E/\operatorname{Ker}T^k \to E/\operatorname{Ker}T^{k-1}$, then $k+1\leq q\leq p\leq n \Rightarrow \pi(D^{k-1}_{p,q}) = D^k_{p,q}$.*

3.7 Corollary $\mathcal{D}_{(a,T)} \neq \emptyset$.

Proof of the corollary. For $k=n$, $\operatorname{Ker}T^k = E$ so $E/\operatorname{Ker}T^k = \{0\}$ and $b^n = 0$, so the trivial decomposition of $\{0\}$ is adapted to (b^n,T), thus $\mathcal{D}_{(b^n,T)} \neq \emptyset$. Applying Proposition 3.6 by downward induction on k, we get $\mathcal{D}_{(b^0,T)} \neq \emptyset$. Now $b^0 = a$, so the result. $\square$

Proof of the proposition. Let us take D^k and π as in the proposition; we set $H = \pi^{-1}(\oplus_{q=k+1}^n D_{n,q}) \subset E/\operatorname{Ker}T^{k-1}$, H is b^{k-1}-non degenerate. Indeed (all orthogonalities are with respect to b^{k-1}):

- $H \supset (E_{n,k}/\operatorname{Ker}T^{k-1}) = (E_{n-1,n-1}/\operatorname{Ker}T^{k-1})^\perp$ so $H^\perp \subset (E_{n-1,n-1}/\operatorname{Ker}T^{k-1})$,
- $H + (E_{n-1,n-1}/\operatorname{Ker}T^{k-1}) = E/\operatorname{Ker}T^{k-1}$ so $H^\perp \cap (E_{n-1,n-1}/\operatorname{Ker}T^{k-1})^\perp = H^\perp \cap (E_{n,k}/\operatorname{Ker}T^{k-1}) = \{0\}$,

Thus, $H^\perp \cap H \subset H^\perp \cap \underbrace{(E_{n-1,n-1}/\operatorname{Ker}T^{k-1}) \cap H}_{=E_{n-1,k}/\operatorname{Ker}T^{k-1}} \subset H^\perp \cap (E_{n,k}/\operatorname{Ker}T^{k-1}) = \{0\}$, this is the announced nondegeneracy.

Now in H, we *choose* a totally isotropic complement F of $E_{n,k}/\operatorname{Ker} T^{k-1}$ in H. This is possible as, in H, $E_{n,k}/\operatorname{Ker} T^{k-1}$ contains its orthogonal: $(E_{n,k}/\operatorname{Ker} T^{k-1})^{\perp} \cap H = (E_{n-1,n-1}/\operatorname{Ker} T^{k-1}) \cap H = E_{n-1,k}/\operatorname{Ker} T^{k-1} \subset E_{n,k}/\operatorname{Ker} T^{k-1}$. Then π provides an isomorphism $\pi : F \xrightarrow{\simeq} \oplus_{q=k+1}^{n} D_{n,q}^{k}$; we set, for $q \in [\![k+1, n]\!]$, $D_{n,q}^{k-1} = \pi_{|F}^{-1}(D_{n,q}^{k})$ and, for $k+1 \leq q \leq p \leq n$, $D_{n,q}^{k-1} = T^{n-p}(D_{n,q+n-p}^{k-1})$. This is a decomposition adapted to (b^{k-1}, T), lacking of the term $D_{n,k}^{k-1}$. Let us already prove this, before adding the lacking term. Property (2.2) is satisfied by construction, property (2.1) is satisfied as, for all $q \geq k+1$, $D_{n,q}^{k}$ is a supplement of $(E_{n-1,q} + E_{n,q-1})/\operatorname{Ker} T^{k}$ in $E_{n,q}/\operatorname{Ker} T^{k}$ and $D_{n,q}^{k-1} = (\pi_{|F})^{-1}(D^{k}, n, q)$. Finally, (3.1) is satisfied. Indeed, as D^{k} is adapted to (b^{k}, T):

$$E/\operatorname{Ker} T^{k} = \left(\overset{\perp}{\bigoplus_{\substack{p+q\leq n \\ p>k \\ q>k}}} (\underbrace{D_{p,q}}_{\text{tot. isotropic}} \oplus \underbrace{D_{n+k+1-q,n+k+1-p})}_{\text{tot. isotropic}} \right) \overset{\perp}{\oplus} \left(\overset{\perp}{\bigoplus_{\frac{n+k+1}{2}\leq p\leq n}} \underbrace{D_{p,n+k+1-p}}_{\text{non degenerate}} \right). \tag{3.2}$$

Now take $v, w \in F$ and consider $b^{k-1}(T^{l}(v), T^{m}(w))$.

- If $m \geq 1$, (or $l \geq 1$, the roles of m and l are symmetric) $b^{k-1}(T^{l}(v), T^{m}(w)) = b^{k}(T^{l}(\pi(v)), T^{m-1}(\pi(w))$.
- If $l = m = 0$, $b^{k-1}(v, w) = 0$ as F is totally isotropic.

One checks that this gives relation (3.2) with b^{k-1} replacing b^{k}, "$n+k$" replacing "$n+k+1$", and the term $D_{n,k}^{k-1}$ lacking, which is the wanted result.

To obtain a full decomposition adapted to (b^{k-1}, T), we must add $D_{n,k}^{k-1}$. This $D_{n,k}^{k-1}$ cannot be something else as the b^{k-1}-orthogonal complement of the sum of all the $D_{p,q}^{k-1}$ already built; besides this orthogonal complement is as wanted. □

3.8 Remarks • The proof of Proposition 3.6 shows that the choice of a decomposition adapted to (a, T) amounts, by induction on k, to the choice of b^{k}-totally isotropic complements of some subspaces in some quotient spaces.

• As T is a-(skew)adjoint, the $E_{p,q}$ are stable by $\perp$, in the following sense:

$$\forall p, q,\ E_{p,q}^{\perp} = E_{n-q,n-q} + E_{n,n-p}.$$

3. b The form of Stab(a) ∩ Stab(b), when $\mathbb{K}$ is algebraically closed

3.9 Recall The endomorphism B is defined in 1.1, the quotients $\check{E}_{k}$ in 2.2 and the set $\mathcal{D}_{(a,T)}$ in 3.4. Besides we recall that we made the convention 3.3.

Here, $\mathbb{K}$ is algebraically closed, of characteristic different from two, so the minimal polynomial of B is of the form P^{n}, $P = X - \lambda$ i.e. $B = \lambda \operatorname{Id} + T$, T nilpotent of index n.

3.10 Notation The form $b^{k} = a(\,\cdot\,, T^{k}\cdot\,)$ is defined on $E/\operatorname{Ker} T^{k}$, so also on $E_{n,k+1}/\operatorname{Ker} T^{k}$; its kernel on $E_{n,k+1}/\operatorname{Ker} T^{k}$ is $E_{n-1,k+1}/(\operatorname{Ker} T^{k} \cap E_{n-1,k+1})$, so it defines a non degenerate form, denoted by $\check{b}_{k+1}$, on:

$$(E_{n,k+1}/\operatorname{Ker} T^{k})/(E_{n-1,k+1}/(\operatorname{Ker} T^{k} \cap E_{n-1,k+1})) \simeq E_{n,k+1}/(E_{n,k} + E_{n-1,k+1}) = \check{E}_{k+1}.$$

3.11 Theorem Stab$(a) \cap$ Stab$(b) = R \ltimes N$, *where* $N = \{\gamma \in \operatorname{Stab}(a) \cap \operatorname{Stab}(b)\,;\, \gamma$ *acts trivially on the* $\check{E}_{k}$, $k \leq n\}$; N *acts simply transitively on* $\mathcal{D}_{(a,T)}$ *and is nilpotent,* $R = (\operatorname{Stab}(a) \cap \operatorname{Stab}(b))/N \simeq \prod_{k=1}^{n} \operatorname{Stab}(\check{E}_{k}, \check{b}_{k})$.

Proof. Repeat the proof of Proposition 2.6. The only details to change are the following.

- $u : \beta^D \mapsto \beta^\Delta$ commutes with T *and preserves* a as D and Δ *are* (a,T)*-adapted.*

- At the end, take $(h_k)_{k=1}^n \in \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k)$; the application γ mapping β^D on $\beta^{h,D}$ commutes with T and preseves a as the h_k preserve the $\check{b}_k$ and as D is (a,T)-adapted. $\square$

As (a,T)-adapted decompositions exist (Corollary 3.7) follows from what precedes that:

3.12 Proposition *Let (a,b) be a couple of $\mathbb{K}$-bilinear forms, $\mathbb{K}$ algebraically closed and* $\mathrm{Char}\,\mathbb{K} \neq 2$, *on a finite dimentional vectorspace E, both symmetric or both skew-symmetric; we take B the endomorphism such that $b = a(\,\cdot\,, B\,\cdot\,)$ and $P = \prod_{i=1}^N P_i^{n_i}$, with P_i mutually prime irreducible polynomials, the minimal polynomial of B. Then, the couple (a,b) is characterized, up to conjugation by $\mathrm{GL}(E)$, by the invariants of B modulo conjugation by $\mathrm{GL}(E)$, that is to say its eigenvalues i.e. the roots of P and, for each root, the Jordan invariants of the nilpotent endomorphism $P_i(B)_{|\mathrm{Ker}\,P_i^{n_i}(B)}$ i.e. the dimensions of the $\check{E}_k$.*

3. c The form of Stab$(a) \cap$ Stab(b), when $\mathbb{K} = \mathbb{R}$

For the notation, see the Recall 3.9 above. The minimal polynomial of B is supposed to be P^n with P irreducible, thus of degree one or two.

3.13 Theorem *If a and b are symmetric, then:*

- *either $\deg P = 1$, then $\mathrm{O}(a) \cap \mathrm{O}(b) = R \ltimes N$, where $N = \{\gamma \in \mathrm{O}(a) \cap \mathrm{O}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{O}(a) \cap \mathrm{O}(b))/N \simeq \prod_{k=1}^n \mathrm{O}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1}^n \mathrm{O}(r_k, s_k)$ with $(r_k, s_k)_{k=1}^n$ the signatures of the $(\check{b}_k)_{k=1}^n$; $r_k + s_k = d_k := \dim \check{E}_k$.*

- *or $\deg P = 2$, we denote by $E^{\mathbb{C}}$ the complexification of E, by $S^{\mathbb{C}}$ that of S and set $P = (X - \lambda)(X - \overline{\lambda})$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let us set $E' = \mathrm{Ker}(S^{\mathbb{C}} - \lambda\,\mathrm{Id}_{E^{\mathbb{C}}})$, $\mathrm{O}(a) \cap \mathrm{O}(b)$ acts on E'; $\mathrm{O}(a) \cap \mathrm{O}(b) = R \ltimes N$, where $N = \{\gamma \in \mathrm{O}(a) \cap \mathrm{O}(b)\,;\, \gamma$ acts trivially on the $\check{E}'_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a_{|E'}, T^{\mathbb{C}}_{|E'})}$ and is nilpotent, $R = (\mathrm{O}(a) \cap \mathrm{O}(b))/N \simeq \prod_{k=1}^n \mathrm{O}(\check{E}'_k, \check{b}'_k) \simeq \prod_{k=1}^n \mathrm{O}(d_k, \mathbb{C})$ with $d_k := \dim_{\mathbb{C}} \check{E}'_k = \frac{1}{2} \dim_{\mathbb{R}} \check{E}_k$.*

If a and b are skew-symmetric, then:

- *either $\deg P = 1$, then $\mathrm{Sp}(a) \cap \mathrm{Sp}(b) = R \ltimes N$, where $N = \{\gamma \in \mathrm{Sp}(a) \cap \mathrm{Sp}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{Sp}(a) \cap \mathrm{Sp}(b))/N \simeq \prod_{k=1}^n \mathrm{Sp}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1}^n \mathrm{Sp}(2d_k, \mathbb{R})$ with $d_k := \frac{1}{2} \dim \check{E}_k$.*

- *or $\deg P = 2$, we denote by $E^{\mathbb{C}}$ the complexification of E, by $S^{\mathbb{C}}$ that of S and set $P = (X - \lambda)(X - \overline{\lambda})$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let us set $E' = \mathrm{Ker}(S^{\mathbb{C}} - \lambda\,\mathrm{Id}_{E^{\mathbb{C}}})$, $\mathrm{Sp}(a) \cap \mathrm{Sp}(b)$ acts on E'; $\mathrm{Sp}(a) \cap \mathrm{Sp}(b) = R \ltimes N$, where $N = \{\gamma \in \mathrm{Sp}(a) \cap \mathrm{Sp}(b)\,;\, \gamma$ acts trivially on the $\check{E}'_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a_{|E'}, T^{\mathbb{C}}_{|E'})}$ and is nilpotent, $R = (\mathrm{Sp}(a) \cap \mathrm{Sp}(b))/N \simeq \prod_{k=1}^n \mathrm{Sp}(\check{E}'_k, \check{b}'_k) \simeq \prod_{k=1}^n \mathrm{Sp}(2d_k, \mathbb{C})$ with $2d_k := \dim_{\mathbb{C}} \check{E}'_k = \frac{1}{2} \dim_{\mathbb{R}} \check{E}_k$.*

3.14 Remark In the case where $\deg P = 2$ (a and b both symmetric or both skew-symmetric), one can also build the real quotient spaces $\check{E}_k$, on which acts $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$. If $D' = (D'_{p,q})_{q \leq p \leq n}$ is an $(a^{\mathbb{C}}, T^{\mathbb{C}}_{|E'})$-adapted decomposition of E', then $D^{\mathbb{C}} = (D^{\mathbb{C}}_{p,q})_{q \leq p \leq n} = (D'_{p,q} \oplus \overline{D'_{p,q}})_{q \leq p \leq n}$ is a decomposition of $E^{\mathbb{C}}$ adapted simultaneously to $(a^{\mathbb{C}}, T^{\mathbb{C}})$ and to $(a^{\mathbb{C}}(\,\cdot\,, S^{\mathbb{C}}\cdot\,), T^{\mathbb{C}})$. Taking the real part of $D^{\mathbb{C}}$, one obtains a decomposition D of the real space E, adapted simultaneously to (a,T) and to $(a(\,\cdot\,,S\cdot),T)$. So $\mathcal{D}_{(a,T)}\cap\mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}\neq\emptyset$; N acts simply transitively on it. The real dimension of the quotients $\check{E}_k$ is $2\dim_{\mathbb{C}}\check{E}'_k$ i.e., with the notation of the theorem, $2d_k$ if a and b are symmetric and $4d_k$ if a and b are skew-symmetric. Besides, the action of S on each $\check{E}_k$ is well defined and, denoting it still by S, R acts on $\oplus_{k=1}^n\check{E}_k$ as $\prod_{k=1}^n(\mathrm{Stab}(\check{b}_k)\cap\mathrm{C}(S))$.

If a and b are symmetric, $\check{b}_k$ defined on $\check{E}_k$ is also symmetric, so has a signature, which is (d_k,d_k).

The "preferred" basis. Adapted decompositions and the forms $\check{b}_k$ defined on the quotients $\check{E}_k$ provide a set of "preferred" basis of E, on which $\mathrm{Stab}(a)\cap\mathrm{Stab}(b)$ acts simply transitively, and in which the matrices of a and b take simultaneously a normal form, the "Weierstraß" simultaneaous reduction of a and b, as given, with the vectors of the basis ordered differently, in [8] §3 formula 38. Preferred basis are not unique; they are in particular Jordan basis of B, but *not any* of these. Let us exhibit these basis. Like in the statement of Theorem 3.13, we suppose that the minimal polynomial of B is of the form P^n, P irreducible. In the general case, preferred basis are the concatenation of preferred basis on each $\mathrm{Ker}(P^n(B))$, P running among the prime factors of the minimal polynomial of B, see Lemma 3.2.

Principle of construction. To build a preferred basis β, take $D=(D_{p,q})_{q\le p\le n}$ a decomposition in $\mathcal{D}_{(a,T)}$ (non empty by Corollary 3.7) if $\deg P=1$ or in $\mathcal{D}_{(a,T)}\cap\mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ (non empty by Corollary 3.7 and Remark 3.14) if $\deg P=2$, and a family $\check{\beta}_k$ of basis of the $\check{E}_k$, adapted to $\check{b}_k$: a (pseuso-)orthonormal basis of $(\check{E}_k,\check{b}_k)$ if $\check{b}_k$ is symmetric, a symplectic basis of $(\check{E}_k,\check{b}_k)$ if $\check{b}_k$ is skew-symmetric ... The projection of $E_{n,k}$ on $\check{E}_k=E_{n,k}/(E_{n,k-1}+E_{n-1,k})$ maps $D_{n,k}$ bijectively on $\check{E}_k$, so the pull back of $\check{\beta}_k$ by this projection provides a basis $\beta_{n,k}$ of $D_{n,k}$. One sets $\beta_k=(T^{n-k}(\beta_{n,k}),\dots,T(\beta_{n,k}),\beta_{n,k})$. Then:

- β_k is a basis of $\oplus_{i=0}^{k-1}T^i(D_{n,k})$, denoted by D_k,
- $E=\oplus_{k=1}^nD_k=\oplus_{k=1}^n(\oplus_{i=0}^{k-1}T^i(D_{n,k}))$ and each D_k is stable by T.

Now it is sufficient to give the matricial forms for a and B, in restriction to the D_k, in a basis β_k. It gives the following, when $\mathbb{K}=\mathbb{R}$, in each possible case.

Case a and b symmetric, $\deg P=1$. We remind that $d_k=\dim\check{E}_k$ and that (r_k,s_k) is the signature of $\check{b}_k$, defined on $\check{E}_k$; $r_k+s_k=d_k$, $\dim D_k=kd_k$. We set $P=X-\lambda$. We obtain then the Weierstraß reduction of a and b:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k})=\underbrace{\begin{pmatrix} & & I_{r_k,s_k}\\ & \iddots & \\ I_{r_k,s_k} & & \end{pmatrix}}_{k\text{ blocks}},\quad \mathrm{Mat}_{\beta_{k+1}}(B_{|D_{k+1}})=\underbrace{\begin{pmatrix}\lambda I_{d_k} & I_{d_k} & & \\ & \lambda I_{d_k} & \ddots & \\ & & \ddots & I_{d_k}\\ & & & \lambda I_{d_k}\end{pmatrix}}_{k\text{ blocks}}.$$

The full matrix of a or of B is block-diagonal, with on the diagonal the blocks $\mathrm{Mat}_{\beta_1}(a_{|D_1})$, ..., $\mathrm{Mat}_{\beta_n}(a_{|D_n})$, respectively $\mathrm{Mat}_{\beta_1}(B_{|D_1})$, ..., $\mathrm{Mat}_{\beta_n}(B_{|D_n})$.

Case a and b symmetric, $\deg P=2$. We set $P=(X-\lambda)(X-\overline{\lambda})$ with $\lambda\in\mathbb{C}\setminus\mathbb{R}$. We choose a basis $\check{\beta}_k$ of each $\check{E}_k$ on the following way. Let us recall that $E'=\mathrm{Ker}(S^{\mathbb{C}}-\lambda\,\mathrm{Id})$; $E^{\mathbb{C}}=E'\oplus\overline{E'}$, $d_k=\dim_{\mathbb{C}}\check{E}'_k=\frac{1}{2}\dim_{\mathbb{R}}\check{E}_k$. We take $(z_j)_{j=1}^{d_k}$ a $a^{\mathbb{C}}$-orthonormal basis of $\check{E}'_k$

and set:

$$\check\beta_k = \left(\left(\frac{1-\mathrm{i}}{2} z_j + \frac{1+\mathrm{i}}{2} \overline{z_j} \right)_{j=1}^{d_k}, \left(\frac{1+\mathrm{i}}{2} z_j + \frac{1-\mathrm{i}}{2} \overline{z_j} \right)_{j=1}^{d_k} \right).$$

In such a basis, setting $\lambda = \mu + \mathrm{i}\nu$ with $(\mu, \nu) \in \mathbb{R} \times \mathbb{R}^*$ and still denoting by S the quotient action of S on $\check E_k$:

$$\mathrm{Mat}_{\check\beta_k}(\check b_k) = L_{d_k} := \begin{pmatrix} 0 & I_{d_k} \\ I_{d_k} & 0 \end{pmatrix} \text{ and } \mathrm{Mat}_{\check\beta_k}(S) = \Lambda_{d_k} := \begin{pmatrix} \mu I_{d_k} & -\nu I_{d_k} \\ \nu I_{d_k} & \mu I_{d_k} \end{pmatrix}.$$

Therefore, in the basis β_k of the subspace D_k obtained from such a basis $\check\beta_k$ of $\check E_k$:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & L_{d_k} \\ & \iddots & \\ L_{d_k} & & \end{pmatrix}}_{k \text{ blocks}}, \ \mathrm{Mat}_{\beta_{k+1}}(B_{|D_{k+1}}) = \underbrace{\begin{pmatrix} \Lambda_{d_k} & I_{2d_k} & & \\ & \Lambda_{d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \Lambda_{d_k} \end{pmatrix}}_{k \text{ blocks}}.$$

As above, the full matrix of a or of B is block-diagonal, with on the diagonal the blocks $\mathrm{Mat}_{\beta_1}(a_{|D_1}), \dots, \mathrm{Mat}_{\beta_n}(a_{|D_n})$, respectively $\mathrm{Mat}_{\beta_1}(B_{|D_1}), \dots, \mathrm{Mat}_{\beta_n}(B_{|D_n})$.

Case a and b skew-symmetric, deg $P = 1$. We set $P = X - \lambda$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We choose a basis $\check\beta_k$ of each $\check E_k$ which is symplectic for $\check b_k$, i.e.:

$$\mathrm{Mat}_{\check\beta_k}(\check b_k) = J_{d_k} := \begin{pmatrix} 0 & -I_{d_k} \\ I_{d_k} & 0 \end{pmatrix}, \text{ besides } \mathrm{Mat}_{\check\beta_k}(S) = \lambda I_{2d_k},$$

note that necessarily d_k is even, as $\check\beta_k$ is a non degenerate skew-symmetric form on $\check E_k$. In the basis β_k of the subspace D_k obtained from such a basis $\check\beta_k$ of $\check E_k$:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & J_{d_k} \\ & \iddots & \\ J_{d_k} & & \end{pmatrix}}_{k \text{ blocks}}, \ \mathrm{Mat}_{\beta_{k+1}}(B_{|D_{k+1}}) = \underbrace{\begin{pmatrix} \lambda I_{2d_k} & I_{2d_k} & & \\ & \lambda I_{2d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \lambda I_{2d_k} \end{pmatrix}}_{k \text{ blocks}}.$$

The full matrix of a or of B is block-diagonal with these blocks, as explained above.

Case a and b skew-symmetric, deg $P = 2$. We set $P = (X - \lambda)(X - \overline{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. On each $\check E_k$ is defined the skew-symmetric form $\check b_k$ and the quotient action of S, still denoted by S. We remind that $E' = \mathrm{Ker}(S^{\mathbb{C}} - \lambda\, \mathrm{Id})$, $E^{\mathbb{C}} = E' \oplus \overline{E'}$. Then $\dim_{\mathbb{C}} \check E'_k$ is necessarily even, as the non degenerate bilinear skew-symmetric form, denoted by $\check b'_k$, obtained by the quotient action of $a^{\mathbb{C}}(\,\cdot\,, T^{k-1}\cdot\,)$, is defined on it. So $\dim_{\mathbb{R}} \check E_k$ is multiple of 4. We set $d_k = \frac{1}{2} \dim_{\mathbb{C}} \check E'_k = \frac{1}{4} \dim_{\mathbb{R}} \check E_k$. We choose a basis $\check\beta_k$ of $\check E_k$ obtained as follows. We choose $((z_j)_{j=1}^{d_k}, (z'_j)_{j=1}^{d_k})$ a $\check b'_k$-symplectic basis of $\check E'_k$ i.e. a basis such that

$$\mathrm{Mat}_{\left((z_j)_{j=1}^{d_k}, (z'_j)_{j=1}^{d_k}\right)}(\check b'_k) = J_{d_k} := \begin{pmatrix} 0 & -I_{d_k} \\ I_{d_k} & 0 \end{pmatrix}$$

and we set

$$\check\beta_k = \left((x_j)_{j=1}^{d_k}, (y_j)_{j=1}^{d_k}, (x'_j)_{j=1}^{d_k}, (y'_j)_{j=1}^{d_k} \right) \text{ with}$$

$$\forall j,\ x_j = \frac{1}{\sqrt{2}}(z_j + \overline{z_j}),\ y_j = \frac{1}{\mathrm{i}\sqrt{2}}(-z_j + \overline{z_j}),\ x'_j = \frac{1}{\sqrt{2}}(z'_j + \overline{z'_j}) \text{ and } y'_j = \frac{1}{\mathrm{i}\sqrt{2}}(z'_j - \overline{z'_j}).$$

In such a basis,

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = J_{2d_k} := \begin{pmatrix} 0 & -I_{2d_k} \\ I_{2d_k} & 0 \end{pmatrix} \text{ and } \mathrm{Mat}_{\check{\beta}_k}(S) = \begin{pmatrix} \Lambda_{d_k} & 0 \\ 0 & \overline{\Lambda}_{d_k} \end{pmatrix}$$

where, setting $\lambda = \mu + \mathrm{i}\nu$ with $(\mu,\nu) \in \mathbb{R} \times \mathbb{R}^*$,

$$\Lambda_p := \begin{pmatrix} \mu I_p & -\nu I_p \\ \nu I_p & \mu I_p \end{pmatrix} \text{ and } \overline{\Lambda}_p = \begin{pmatrix} \mu I_p & \nu I_p \\ -\nu I_p & \mu I_p \end{pmatrix}.$$

In the basis β_k of the subspace D_k obtained from such a basis $\check{\beta}_k$ of $\check{E}_k$:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & J_{2d_k} \\ & ⋰ & \\ J_{2d_k} & & \end{pmatrix}}_{k \text{ blocks}},\ \mathrm{Mat}_{\beta_{k+1}}(B_{|D_{k+1}}) = \underbrace{\begin{pmatrix} \widetilde{\Lambda}_{d_k} & I_{4d_k} & & \\ & \widetilde{\Lambda}_{d_k} & \ddots & \\ & & \ddots & I_{4d_k} \\ & & & \widetilde{\Lambda}_{d_k} \end{pmatrix}}_{k \text{ blocks}}$$

with $\widetilde{\Lambda}_{d_k} := \begin{pmatrix} \Lambda_{d_k} & 0 \\ 0 & \overline{\Lambda}_{d_k} \end{pmatrix}$.

Eventually follows from what precedes the following characterization of the conjugaison class of a couple (a,b) under the action of $\mathrm{GL}(E)$.

3.15 Proposition *Let (a,b) be a couple of bilinear forms on a real vectorspace E, both symmetric or both skew-symmetric; we denote by B the endomorphism such that $b = a(\,\cdot\,, B\,\cdot\,)$ and by $\prod_{i=1}^N P_i^{n_i}$, with P_i mutually prime irreducible polynomials, the minimal polynomial of B. Then, the couple (a,b) is characterized, up to conjugation by $\mathrm{GL}(E)$, by:*

- *the $(P_i, n_i)_{i=1}^N$,*
- *the dimensions of the quotients $(\check{E}_k)_{k=1}^{n_i}$ (see Notation 2.2), which are, in the notation previously introduced, denoted by:*

  * *$(d_k)_{k=1}^{n_i}$ if a and b are symmetric and for each P_i of degree one,*

  * *$(2d_k)_{k=1}^{n_i}$ if a and b are symmetric and for each P_i of degree two —they are necessarily even—,*

  * *$(2d_k)_{k=1}^{n_i}$ if a and b are skew-symmetric and for each P_i of degree one —they are necessarily even—,*

  * *$(4d_k)_{k=1}^{n_i}$ if a and b are skew-symmetric and for each P_i of degree two —they are necessarily multiple of four— (the $(2d_k)_{k=1}^{n_i}$ are the complex dimensions of the quotients $(\check{E}'_k)_{k=1}^{n_i}$),*

- *additionally, if a and b are symmetric and for each P_i of degree one, the signatures $(r_k, s_k)_{k=1}^{n_i}$ of the forms $(\check{b}_k)_{k=1}^{n_i}$ defined on the $(\check{E}_k)_{k=1}^{n_i}$; $r_k + s_k = d_k$. Those are, according to the terminilogy of [3] §8, "the system of signatures of the pair of forms with respect to [the root of P_i]".*

4 When one of a, b is symmetric, the other skew-symmetric

We suppose here that one of a, b is symmetric and the other skew-symmetric, so in this case B is a-skew-adjoint. Like in the previous case, we need a first standard preliminary.

4.a A preliminary lemma

4.1 Lemma *Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and $\widetilde{E} = E \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. Then, in $\overline{\mathbb{K}}[X]$, the minimal polynomial of B is of the form $X^{n_0} \prod_{i=1}^{N} P_i^{n_i}$ with $P_i = (X - \lambda_i)(X + \lambda_i)$, $i \neq j \Rightarrow \lambda_i \neq \pm\lambda_j$ and $i \geq 1 \Rightarrow \lambda_i \neq 0$. Besides:*

- *The characteristic subspaces $\mathrm{Ker}(B_i - \lambda_i \,\mathrm{Id})^{n_i}$ of B, associated with nonzero eigenvalues $(i \geq 1)$, are a- and b-totally isotropic.*

- *Denoting $\mathrm{Ker}\, P_i^{n_i}(B) \subset \widetilde{E}$ by $\widetilde{E}_i$ and $\mathrm{Ker}(B^{n_0}) \subset \widetilde{E}$ by E_0, then, with respect to a and to b:*

$$\widetilde{E} = \overset{\perp}{\underset{0 \leq i \leq N}{\oplus}} \widetilde{E}_i.$$

- *The subgroup $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ of $\mathrm{GL}(\widetilde{E})$ equals $\prod_{i=0}^{N} \mathrm{Stab}(\widetilde{E}_i, a_{|\widetilde{E}_i}) \cap \mathrm{Stab}(\widetilde{E}_i, b_{|\widetilde{E}_i})$.*

Proof. As B is a-skew-adjoint, B and $-B$ are conjugated, so, if P is the minimal polynomial of B, $P(X) = P(-X)$; the form of P in $\overline{\mathbb{K}}[X]$ follows. Besides, notice that the semi-simple and nilpotent parts S and T of B are also a-skew-adjoint. Indeed, let S^* and T^* be the adjoints of S and T, S^* is semi-simple and T^* nilpotent, $S^*T^* = (TS)^* = (ST)^* = T^*S^*$ and $S^* + T^* = B^*$, thus S^* and T^* are the semi-simple and nilpotent parts of $B^* = -B$. But these semi-simple and nilpotent parts are also $-S$ and $-T$, so $S^* = -S$ and $T^* = -T$.

Take now $i \geq 1$ and x and y in $\mathrm{Ker}(B - \lambda_i \,\mathrm{Id})^{n_i} = \mathrm{Ker}(S - \lambda_i \,\mathrm{Id})$, then:

$$a(x,y) = \frac{1}{\lambda_i} a(x, \lambda_i y) = \frac{1}{\lambda_i} a(x, Sy) = \frac{1}{\lambda_i} a(-Sx, y) = -\frac{1}{\lambda_i} a(\lambda_i x, y) = -a(x,y)$$

so $\mathrm{Ker}(B - \lambda_i \,\mathrm{Id})^{n_i}$ is a-totally isotropic. It is also b-totally isotropic as $b = a(\,\cdot\,, B\cdot\,)$ and $BS = SB$. This is the first point.

Let us set $P_0 = X \in \overline{\mathbb{K}}[X]$; take $i, j \in [\![0, d]\!]$ with $i \neq j$, $x \in \mathrm{Ker}\, P_i^{n_i}(B)$ and $y \in \mathrm{Ker}\, P_j^{n_j}(B)$. Then:

$$\begin{aligned} a(x,y) &= a((UP_i^{n_i} + VP_j^{n_j})(B).x, y) \\ &= a(UP_i^{n_i}(B).x, y) + a(x, VP_j^{n_j}(B).y) \\ &= a(0,y) + a(x,0) = 0 \quad \text{and:} \end{aligned}$$

$$\begin{aligned} b(x,y) &= b((UP_i^{n_i} + VP_j^{n_j})(B).Bx, y) \\ &= b(B(UP_i^{n_i}(B).x), y) + a(x, B(VP_j^{n_j}(B).y)) \\ &= a(0,y) + a(x,0) \\ &= 0, \end{aligned}$$

this is the second point. The third one follows immediately. □

4.2 Convention In $\mathbb{K}[X]$, the minimal polynomial of B is then of the form $Q_0^{n_0} \prod_i Q_i^{n_i}$ with $Q_0 = X$, the Q_i mutually prime and, for each i, $Q_i(X) = Q_i(-X)$, Q_i irreducible or Q_i of the form $R_i(X)R_i(-X)$ with R_i irreducible and $R_i(X) \wedge R_i(-X) = 1$. **As a consequence**

of the lemma, as in §3, we focus now on one single of the $\mathrm{Ker}\, Q_i^{n_i}(B)$ i.e., equivalently, we suppose that the minimal polynomial of B is of the form Q^n with Q irreducible and $Q(X) = Q(-X)$ or $Q = R(X)R(-X)$ with R irreducible and $R(X)$ and $R(-X)$ mutually prime.

4. b The form of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$, when $\mathbb{K}$ is algebraically closed

For the notation used here, we send back to Recall 3.9. The field $\mathbb{K}$ is supposed to be algebraically closed, so the minimal polynomial of B is of the form P^n with $P = X$ or $P = (X - \lambda)(X + \lambda)$. We always suppose that a is non degenerate. Notice that the nature, symmetric or skew-symmetric, of the forms $\check{b}_k$ depends now on the parity of k. As $\check{b}_k$ comes from a quotient action of $a(\,\cdot\,, T^{k-1}\cdot\,)$, if k is even, T^{k-1} is a-skew-symmetric so $\check{b}_k$ is as b, and if k is odd, T^{k-1} is a-symmetric so $\check{b}_k$ is as a.

4.3 Theorem • *If a is symmetric and b skew-symmetric and degenerate (i.e. $P = X$), $\mathrm{O}(a) \cap \mathrm{Sp}(b) = R \ltimes N$ with $N = \{\gamma \in \mathrm{O}(a) \cap \mathrm{Sp}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{O}(a) \cap \mathrm{Sp}(b))/N \simeq \prod_{k=1}^{n} \mathrm{Stab}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1, k\ odd}^{n} \mathrm{O}(d_k, \mathbb{K}) \times \prod_{k=1, k\ even}^{n} \mathrm{Sp}(2d_k, \mathbb{K})$, with $d_k = \dim \check{E}_k$ if k is odd and $d_k = \frac{1}{2} \dim \check{E}_k$ if k is even.*

• *If a is skew-symmetric and b symmetric and degenerate (i.e. $P = X$), $\mathrm{Sp}(a) \cap \mathrm{O}(b) = R \ltimes N$ with $N = \{\gamma \in \mathrm{Sp}(a) \cap \mathrm{O}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{Sp}(a) \cap \mathrm{O}(b))/N \simeq \prod_{k=1}^{n} \mathrm{Stab}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1, k\ odd}^{n} \mathrm{Sp}(2d_k, \mathbb{K}) \times \prod_{k=1, k\ even}^{n} \mathrm{O}(d_k, \mathbb{K})$, with $d_k = \frac{1}{2} \dim \check{E}_k$ if k is odd and $d_k = \dim \check{E}_k$ if k is even.*

• *If both forms are non degenerate (i.e. $P = (X - \lambda)(X + \lambda)$, $\lambda \neq 0$), then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ acts on $E^+ = \mathrm{Ker}(S - \lambda\,\mathrm{Id})$. This representation of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ in $\mathrm{GL}(E^+)$ is faithful and its image is the commutant $\mathrm{C}(T_{|E^+}) \subset \mathrm{GL}(E^+)$ of $T_{|E^+}$.*

4.4 Remark In the third case, it follows then that, after Proposition 2.6, $\mathrm{Stab}(a) \cap \mathrm{Stab}(b) = R \ltimes N$ where $N = \{\gamma \in \mathrm{C}(T);\ \gamma$ acts trivially on all $\check{E}_k, k \leq n\}$; N acts simply transitively on the set $\mathcal{D}_T^+$ of the decompositions of E^+ adapted to $T_{|E^+}$ and is nilpotent, $R = (\mathrm{Stab}(a) \cap \mathrm{Stab}(b))/N \simeq \prod_{k=1}^{n} \mathrm{GL}(\check{E}_k^+)$.

Proof of the theorem. Both first cases are proven on the same way as Theorem 3.11. For the third case, we must show that $\rho : \mathrm{Stab}(a) \cap \mathrm{Stab}(b) \to \mathrm{GL}(E^+)$ defined by $\rho(\gamma) = \gamma_{|E^+}$ is an isomorphism from $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ onto $\mathrm{C}(T_{|E^+})$. $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is mapped by ρ in $\mathrm{C}(T_{|E^+})$, indeed, if $\gamma \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)$, γ commutes with T so $\gamma_{|E^+}$ commutes with $T_{|E^+}$. Besides the form a induces musical isomorphisms :

$$E \underset{\sharp}{\overset{\flat}{\rightleftarrows}} E^*. \text{ As } E^+ \text{ and } E^- \text{ are } a\text{-totally isotropic by Lemma 4.1: } E^- \underset{\sharp}{\overset{\flat}{\rightleftarrows}} E^{+*}.$$

So, to each $\gamma^+ \in \mathrm{GL}(E^+)$, we associate canonically a $\gamma^- \in \mathrm{GL}(E^-)$ by: $\gamma^- = \sharp \circ {}^t(\gamma^+)^{-1} \circ \flat$. We set $\widetilde{\gamma} = (\gamma^+, \gamma^-) \in \mathrm{GL}(E^+) \times \mathrm{GL}(E^-)$. Then, if $\gamma^+ \in \mathrm{GL}(E^+)$, $\widetilde{\gamma} \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ and $\gamma^+ = \rho(\widetilde{\gamma})$, so ρ is onto. Let us check it.

• $\widetilde{\gamma} \in \mathrm{Stab}(a)$. As E^+ and E^- are $\widetilde{\gamma}$-stable and a-totally isotropic, it is sufficient to check it with $(x,y) \in E^+ \times E^-$.

$$\begin{aligned}
a(\widetilde{\gamma}(y), \widetilde{\gamma}(x)) &= a(({}^t(\gamma^+)^{-1}(y^\flat))^\sharp, \gamma^+(x)) \\
&= \left({}^t(\gamma^+)^{-1}(y^\flat)\right)(\gamma^+(x)) \\
&= a(y, (\gamma^+)^{-1}(\gamma^+(x))) \\
&= a(y,x).
\end{aligned}$$

• $\widetilde{\gamma}$ commutes with B. As $S_{|E^\pm} = \pm\lambda\,\mathrm{Id}_{E^\pm}$ and as E^+ and E^- are $\widetilde{\gamma}$-stable, we have only to check that $\widetilde{\gamma}$ commutes with T, i.e. that γ^- commutes with $T_{|E^-}$. T is a-skew-adjoint, so $\forall z \in E$, ${}^tT(z^\flat) = -T(z)^\flat$. Take $y \in E^-$:

$$\begin{aligned}
[\gamma^-(T(y))]^\flat = {}^t(\gamma^+)^{-1}(T(y)^\flat) &= -{}^t(\gamma^+)^{-1}({}^tT(y^\flat)) = -{}^t(T.(\gamma^+)^{-1})(y^\flat) \\
&= -{}^t((\gamma^+)^{-1}.T)(y^\flat) = -{}^tT.{}^t(\gamma^+)^{-1}(y^\flat) = -{}^tT((\gamma^-(y))^\flat) = (T(\gamma^-(y)))^\flat,
\end{aligned}$$

so $\gamma^- T = T\gamma^-$, the wanted result.

Finally ρ is injective, this is immediate: $\rho(\gamma) = \mathrm{Id}_{E^+} \Rightarrow \gamma_{|E^+} = \mathrm{Id}_{E^+} \Rightarrow \gamma = \mathrm{Id}_E$. $\square$

We may also detail a bit the situation in the third case of Theorem 4.3, in the following corollary.

4.5 Corollary *We suppose that both a and b are non degenerate. The set $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ of the decompositions of E simultaneously adapted to (a,T) and to $(a(\,\cdot\,,S\cdot),T)$ is non empty; it is canonically in bijection with the set $\mathcal{D}_T^+$ of the decompositions of E^+, adapted to T, in the following sense: if $D = (D_{p,q})_{1\le q\le p\le n} \in \mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$, then for each p,q, $D_{p,q} = (D_{p,q} \cap E^+) \oplus (D_{p,q} \cap E^-)$ and $D^+ := (D_{p,q} \cap E^+)_{1\le q\le p\le n} \in \mathcal{D}_T^+$. The bijection is this map $D \mapsto D^+$.*

With the notation introduced in Remark 4.4, N acts simply transitively on $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$; $R = (\mathrm{Stab}(a) \cap \mathrm{Stab}(b))/N \simeq \prod_{k=1}^n \left(\mathrm{Stab}(\check{E}_k, \check{b}^k) \cap \mathrm{Stab}(\check{E}_k, \check{b}^k(\,\cdot\,,S\,\cdot))\right) = \prod_{k=1}^n \left(\mathrm{Stab}(\check{E}_k, \check{b}^k) \cap \mathrm{C}(S)\right) \simeq \prod_{k=1}^n \mathrm{GL}(\check{E}_k^+)$. The bijection $D \mapsto D^+$ commutes with the action of N.

Proof. It is a quick checking. Let us give only the key facts. Let $D = (D_{p,q})_{1\le q\le p\le n}$ be in $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$, then each $D_{p,q}$ is stable by S. Now $S_{|E^\pm} = \pm\lambda\,\mathrm{Id}_{E^\pm}$ so if $x \in D_{p,q}$ and $x = x^+ + x^-$ is the decomposition of x in (E^+, E^-), $\lambda(x^+ - x^-) = S(x) \in D_{p,q}$ so $x^\pm \in D_{p,q}$ and $D_{p,q} = (D_{p,q} \cap E^+) \oplus (D_{p,q} \cap E^-)$. Immediately, $D^\pm = (D_{p,q} \cap E^\pm)_{1\le q\le p\le n}$ is a decomposition of $E^\pm$ adapted to $T_{|E^\pm}$.

In the converse sense, let $D^+ = (D^+_{p,q})_{1\le q\le p\le n}$ be in $\mathcal{D}_T^+$ and let $D^{+*} = (D^{+*}_{p,q})_{1\le q\le p\le n}$ be the dual decomposition of the dual space E^{+*} of E^+, i.e. the decomposition given by: D^{+*} is the subspace of E^{+*} such that $\forall (p',q') \ne (p,q)$, $D^{+*}_{p,q}(D^+_{p',q'}) = \{0\}$, that is to say, $D^{+*}_{p,q} = (\oplus_{(p',q')\ne(p,q)} D_{p',q'})^\perp$. We set $\sharp : E^{+*} \to E^-$ the musical endomorphism associated to a; one checks then that $D^- := D^{+*\sharp} = ((D^{+*}_{p,q})^\sharp)_{1\le q\le p\le n}$ is a decomposition of E^- adapted to $T_{|E^-}$ and that $\widetilde{D} := (D^+_{p,q} \oplus D^-_{p,q})_{1\le q\le p\le n}$ is in $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$. This map $D^+ \mapsto \widetilde{D}$ is the converse map of $D \mapsto (D \cap E^+)$. It commutes with the action of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ so N, which acts simply transitively on $\mathcal{D}_T^+$ by the remark 4.4, acts simply stransitively on $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$. $\square$

On the same principle as in 3.c, thanks to Corollary 3.7, we now introduce "preferred" basis for a couple of forms (a, b), in the three cases of Theorem 4.3. We send back to 3.c for the notation: D_k, $\check{\beta}_k$, β_k ... Notice that, as T is *a-skew*-adjoint, and as β_k is a basis of D_k of the form $(T^{n-k}(\beta_{n,k}, \dots, T(\beta_{n,k}), \beta_{n,k})$, the general form of $\mathrm{Mat}_{\beta_k}(a_{|D_k})$ is:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & & \pm\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) \\ & & & & \iddots & \\ & & & -\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) & & \\ & & \mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) & & & \\ & -\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) & & & & \\ \mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) & & & & & \end{pmatrix}}_{k \text{ blocks}} .$$

Case a symmetric, non degenerate, b skew-symmetric, degenerate ($P = X$). If k is odd, we take an orthonormal basis $\check{\beta}_k$ of $(\check{E}_k, \check{b}_k)$; if k is even, a symplectic basis of $(\check{E}_k, \check{b}_k)$; in the former case, $d_k := \dim \check{E}_k$ and in the latter, $\dim \check{E}_k$ is even as $\check{b}_k$ is skew-symmetric, non degenerate, and $d_k := \frac{1}{2} \dim \check{E}_k$. Then:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & & I_{d_k} \\ & & & & \iddots & \\ & & & -I_{d_k} & & \\ & & I_{d_k} & & & \\ & -I_{d_k} & & & & \\ I_{d_k} & & & & & \end{pmatrix}}_{k \text{ blocks}} \quad \text{for } k \text{ odd},$$

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & & J_{d_k} \\ & & & & \iddots & \\ & & & -J_{d_k} & & \\ & & J_{d_k} & & & \\ & -J_{d_k} & & & & \\ J_{d_k} & & & & & \end{pmatrix}}_{k \text{ blocks}} \quad \text{for } k \text{ even, with } J_{d_k/2} = \begin{pmatrix} 0 & -I_{d_k/2} \\ I_{d_k/2} & 0 \end{pmatrix};$$

$$\mathrm{Mat}_{\beta_k}(B_{|D_k}) = \underbrace{\begin{pmatrix} 0 & I_{\delta_k} & & \\ & 0 & \ddots & \\ & & \ddots & I_{\delta_k} \\ & & & 0 \end{pmatrix}}_{k \text{ blocks}} \quad \begin{array}{l} \text{for all } k, \\ \text{with } \delta_k = d_k \text{ for } k \text{ odd} \\ \text{and } \delta_k = 2d_k \text{ for } k \text{ even.} \end{array}$$

Case a skew-symmetric, non degenerate, b symmetric, degenerate ($P = X$). The matrices are the same, except that the form of $\mathrm{Mat}_{\beta_k}(a_{|D_k})$ and of $\mathrm{Mat}_{\beta_k}(B_{|D_k})$in the cases "$k$ odd" and "k even" are swapped.

Case a symmetric, non degenerate, b skew-symmetric, non degenerate ($P = (X - \lambda)(X + \lambda)$). In this case, for each k, $\check{E}_k = \check{E}_k^+ \oplus \check{E}_k^-$ with $S_{|\check{E}_k^\pm} = \pm\lambda \,\mathrm{Id}_{\check{E}_k^\pm}$ and $\check{b}_k$ is symmetric if k is odd, skew-symmetric if k is even. For all k, d_k is even. Remind that

$d_k = \dim \check{E}_k^+ = \frac{1}{2} \dim \check{E}_k$. We choose a basis $\check{\beta}_k$ of $\check{E}_k$, formed by a basis of $\check{E}_k^+$ and a basis of $\check{E}_k^-$ and in which:

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = L_{d_k} := \begin{pmatrix} 0 & I_{d_k} \\ I_{d_k} & 0 \end{pmatrix} \text{ for the odd } k,$$

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = J_{d_k} := \begin{pmatrix} 0 & -I_{d_k} \\ I_{d_k} & 0 \end{pmatrix} \text{ for the even } k,$$

$$\mathrm{Mat}_{\check{\beta}_k}(S) = \lambda I_{d_k,d_k} := \begin{pmatrix} \lambda I_{d_k} & 0 \\ 0 & -\lambda I_{d_k} \end{pmatrix} \text{ for all } k.$$

Then, in a basis β_k of D_k built, as in 3.c, after such a $\check{\beta}_k$, we get:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & & L_{d_k} \\ & & & & \iddots & \\ & & & -L_{d_k} & & \\ & & L_{d_k} & & & \\ & -L_{d_k} & & & & \\ L_{d_k} & & & & & \end{pmatrix}}_{k \text{ blocks}} \text{ for the odd } k,$$

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & & -J_{d_k} \\ & & & & \iddots & \\ & & & -J_{d_k} & & \\ & & J_{d_k} & & & \\ & -J_{d_k} & & & & \\ J_{d_k} & & & & & \end{pmatrix}}_{k \text{ blocks}} \text{ for the even } k,$$

$$\mathrm{Mat}_{\beta_k}(B_{|D_k}) = \underbrace{\begin{pmatrix} \lambda I_{d_k,d_k} & I_{2d_k} & & \\ & \lambda I_{d_k,d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \lambda I_{d_k,d_k} \end{pmatrix}}_{k \text{ blocks}} \text{ for all } k.$$

As at the end of the previous section follows from what precedes a characterization of the conjugaison class of a couple (a, b) under the action of $\mathrm{GL}(E)$.

4.6 Proposition *Let (a,b) be a couple of bilinear forms on a $\mathbb{K}$-vectorspace E, $\mathbb{K}$ algebraically closed, a symmetric and b skew-symmetric. We denote by B the endomorphism such that $b = a(\cdot\,, B\cdot)$ and by $\prod_{i=0}^{N} P_i^{n_i}$, with $P_0 = X$ and, for $i \geq 1$, $P_i = (X-\lambda_i)(X+\lambda_i)$ ($i \neq j \Rightarrow \lambda_i \neq \pm\lambda_j$), the minimal polynomial of B. Then, the couple (a,b) is characterized, up to conjugation by $\mathrm{GL}(E)$, by:*

- *the $(P_i, n_i)_{i=1}^{N}$,*

- *if $n_0 \neq 0$, the (even) dimensions, denoted above by $(2d_k)_{k=1,k \text{ even}}^{n_0}$, of the quotients $(\check{E}_k)_{k=1,k \text{ even}}^{n_0}$ (see Notation 2.2), the dimensions, denoted above by $(d_k)_{k=1,k \text{ odd}}^{n_0}$, of the quotients $(\check{E}_k)_{k=1,k \text{ odd}}^{n_0}$ and the signatures $(r_k, s_k)_{k=1,k \text{ odd}}^{n_0}$ of the corresponding forms $(\check{b}_k)_{k=1,k \text{ odd}}^{n_0}$ defined on the $(\check{E}_k)_{k=1,k \text{ odd}}^{n_0}$ ($r_k + s_k = d_k$),*

• *for $i \geq 1$, the (even) dimensions, denoted above by $(2d_k)_{k=1}^{n_i}$, of the quotients $(\check{E}_k)_{k=1}^{n_i}$, or equivalently the dimensions $(d_k)_{k=1}^{n_i}$, of the quotients $(\check{E}_k^+)_{k=1}^{n_i}$.*

Besides, if a is skew-symmetric and b symmetric, the same statement holds with "even" and "odd" swapped in the second point.

4.c The form of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$, when $\mathbb{K} = \mathbb{R}$

For the notation used here, we send back to Recall 3.9. As indicated in the Convention 4.2, we may suppose that the minimal polynomial of B is of the form Q^n with:

(i) Q irreducible and $Q(X) = Q(-X)$

(ii) or $Q = R(X)R(-X)$ with R irreducible and $R(X)$ and $R(-X)$ mutually prime.

In the case $\mathbb{K} = \mathbb{R}$, this gives four possible forms for Q. In case **(i)**, Q may be equal to X or to $(X-\lambda)(X-\overline{\lambda})$ with $\lambda \in \mathrm{i}\mathbb{R}^*$; in case **(ii)**, Q may be equal to $(X-\lambda)(X+\lambda)$ with $\lambda \in \mathbb{R}^*$ or to $(X-\lambda)(X-\overline{\lambda})(X+\lambda)(X+\overline{\lambda})$ with $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$. In turn, this gives five cases in the next theorem.

4.7 Theorem • *If a is symmetric and b skew-symmetric and degenerate (i.e. $P = X$), $\mathrm{O}(a) \cap \mathrm{Sp}(b) = R \ltimes N$ with $N = \{\gamma \in \mathrm{O}(a) \cap \mathrm{Sp}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{O}(a) \cap \mathrm{Sp}(b))/N \simeq \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1, k \text{ odd}}^n \mathrm{O}(r_k, s_k) \times \prod_{k=1, k \text{ even}}^n \mathrm{Sp}(2d_k, \mathbb{R})$ with (r_k, s_k) the signatures of the $\check{b}_k$ on $\check{E}_k$, k odd; $d_k := \dim \check{E}_k$ for k odd, $d_k := \frac{1}{2} \dim \check{E}_k$ for k even.*

• *If a is skew-symmetric and b symmetric and degenerate (i.e. $P = X$), $\mathrm{Sp}(a) \cap \mathrm{O}(b) = R \ltimes N$ with $N = \{\gamma \in \mathrm{Sp}(a) \cap \mathrm{O}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; N acts simply transitively on $\mathcal{D}_{(a,T)}$ and is nilpotent, $R = (\mathrm{Sp}(a) \cap \mathrm{O}(b))/N \simeq \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \simeq \prod_{k=1, k \text{ odd}}^n \mathrm{Sp}(2d_k, \mathbb{R}) \times \prod_{k=1, k \text{ even}}^n \mathrm{O}(r_k, s_k)$ with (r_k, s_k) the signatures of the $\check{b}_k$ on $\check{E}_k$, k even; $d_k := \frac{1}{2} \dim \check{E}_k$ for k odd, $d_k := \dim \check{E}_k$ for k even.*

• *If both forms are non degenerate and if $P = (X-\lambda)(X+\lambda)$, $\lambda \in \mathbb{R}^*$, then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ acts on $E^+ = \mathrm{Ker}(S - \lambda \,\mathrm{Id})$. This representation of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ in $\mathrm{GL}(E^+)$ is faithful and its image is the commutant $\mathrm{C}(T_{|E^+}) \subset \mathrm{GL}(E^+)$ of $T_{|E^+}$.*

• *If both forms are non degenerate and if $P = (X-\lambda)(X-\overline{\lambda})$, $\lambda \in \mathrm{i}\mathbb{R}^*$, then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ acts on $E' = \mathrm{Ker}(S^{\mathbb{C}} - \lambda \,\mathrm{Id}_{E^{\mathbb{C}}}) \subset E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$, which is endowed with the non degenerate sesquilinear form $\underline{a} : z_1, z_2 \mapsto= a^{\mathbb{C}}(z_1, \overline{z_2})$. We denote by $\mathcal{D}'_{(\underline{a},T)}$ the set of the decompositions of E' adapted to $(\underline{a}, T)$ and by $\check{\underline{b}}_k$ the non degenerate hermitian form defined on the quotient $\check{E}'_k$ by $\check{\underline{b}}_k = \underline{a}(\,\cdot\,, T^{k-1}\cdot\,)$ for the k such that $a(\,\cdot\,, T^{k-1}\cdot\,)$ is symmetric and by $\check{\underline{b}}_k = -\mathrm{i}\underline{a}(\,\cdot\,, T^{k-1}\cdot\,)$ for the k such that $a(\,\cdot\,, T^{k-1}\cdot\,)$ is skew-symmetric. Then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b) = R \ltimes N$ with $N = \{\gamma \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)\,;\, \gamma$ acts trivially on the $\check{E}_k$, $k \leq n\}$; the representation of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ in $\mathrm{GL}_{\mathbb{C}}(E')$ is faithful; N acts simply transitively on $\mathcal{D}'_{(\underline{a},T)}$ and is nilpotent, $R = (\mathrm{Stab}(a) \cap \mathrm{Stab}(b))/N \simeq \prod_{k=1}^n \mathrm{Stab}(\check{E}'_k, \check{\underline{b}}_k) \simeq \prod_{k=1}^n \mathrm{U}(r_k, s_k)$ with (r_k, s_k) the signature of $\check{\underline{b}}_k$ defined on $\check{E}'_k$.*

• *If both forms are non degenerate and if $P = (X-\lambda)(X-\overline{\lambda})(X+\lambda)(X+\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$, $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ acts faithfully on $E^+ = \mathrm{Ker}(S^{\mathbb{C}} - \lambda \,\mathrm{Id})$; the image of this representation is the commutant $\mathrm{C}(T_{|E^+}) \subset \mathrm{GL}_{\mathbb{C}}(E^+)$ of $T_{|E^+}$.*

4.8 Remark In the third case of Theorem 4.7, $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is isomorphic to the commutant of a the real endomorphism $T_{|E^+}$; In the fifth case of Theorem 4.7, $\mathrm{Stab}(a) \cap$

$\mathrm{Stab}(b)$ is isomorphic to the commutant of a the complex endomorphism $T_{|E^+}$. Proposition 2.6 gives the structure of such groups (see also remark 4.4).

Proof of the theorem. The first three cases behave as when $\mathbb{K}$ is algebraically closed.

For the fourth case, we have to show that $\rho : \mathrm{Stab}(a) \cap \mathrm{Stab}(b) \to (\mathrm{U}(E', \underline{a}) \cap \mathrm{C}(T_{|E'}))$ defined by $\rho(\gamma) = \gamma_{|E'}$ is bijective. We have $E = E' \oplus \overline{E'}$; by Lemma 4.1, E' and $\overline{E'}$ are a-totally isotropic, so $\underline{a} : z_1, z_2 \mapsto a(z_1, \overline{z_2})$ is non degenerate on E' thus $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ preserves E' and $\underline{a}$ and commutes with $T_{|E'}$. So ρ maps $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ in $\mathrm{U}(E', \underline{a}) \cap \mathrm{C}(T_{|E'})$. Conversely, if $\gamma \in \mathrm{U}(E', \underline{a}) \cap \mathrm{C}(T_{|E'})$, $\gamma = \widetilde{\gamma}_{|E'}$ where $\widetilde{\gamma}$ is defined by $\widetilde{\gamma}_{|E'} = \gamma$ and, on $\overline{E'}$, $\widetilde{\gamma}(z) = \overline{\widetilde{\gamma}(\overline{z})}$; besides $\widetilde{\gamma}$ is the only antecedent of γ by ρ, which is hence bijective.

In the fifth case, let us set $E^{\pm} = \mathrm{Ker}(S^{\mathbb{C}} \mp \lambda \,\mathrm{Id}_{E^{\mathbb{C}}}) \subset E^{\mathbb{C}}$, as $\lambda \notin \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $E^{\mathbb{C}} = E^+ \oplus E^- \oplus \overline{E^+} \oplus \overline{E^-}$. By Lemma 4.1, a is non degenerate on $E^+ \oplus E^-$ and $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ stabilizes this subspace. As the action of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ on $E^{\mathbb{C}}$ commutes with $z \mapsto \overline{z}$, the restriction $\mathrm{Stab}(a) \cap \mathrm{Stab}(b) \ni \gamma \mapsto \gamma_{|E^+ \oplus E^-}$ is faithfull. Let us denote by $\flat$ and $\sharp$ both converse musical isomorphisms $E^- \rightleftarrows (E^+)^*$. Then, as it was shown in the proof of Theorem 4.3, $\gamma \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ if and only if $\gamma_{E^+ \oplus E^-}$ is of the form $(\gamma^+, \gamma^-) \in \mathrm{GL}(E^+) \times \mathrm{GL}(^-)$ with

- $\gamma^+ \in \mathrm{C}(T_{|E^+})$,
- $\gamma^- = \sharp \circ {}^t(\gamma^+)^{-1} \circ \flat$.

So the restriction $\mathrm{Stab}(a) \cap \mathrm{Stab}(b) \ni \gamma \mapsto \gamma_{|E^+}$ is faithful and maps $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ onto $\mathrm{C}(T_{|E^+})$. $\square$

As is the case $\mathbb{K}$ algebraically closed, we can detail the action of $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ in the case where a and b are non degenerate. The proof, straightforward and very similar to that of Corollary 4.5, is left to the reader.

4.9 Corollary • *In the third case of Theorem 4.7, the set* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *of the decompositions of* E *simultaneously adapted to* (a, T) *and to* $(a(\,\cdot\,, S\cdot\,), T)$ *is non empty; it is canonically in bijection with the set* $\mathcal{D}^+_T$ *of the decompositions of* E^+*, adapted to* T*; the bijection is the same as the one given in Corollary 4.5.* N *acts simply transitively on* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$*;* $R = (\mathrm{Stab}(a) \cap \mathrm{Stab}(b))/N \simeq \prod_{k=1}^n \left(\mathrm{Stab}(\check{E}_k, \check{b}^k) \cap \mathrm{Stab}(\check{E}_k, \check{b}^k(\,\cdot\,, S\,\cdot))\right) = \prod_{k=1}^n \left(\mathrm{Stab}(\check{E}_k, \check{b}^k) \cap \mathrm{C}(S)\right) \simeq \prod_{k=1}^n \mathrm{GL}(\check{E}^+_k)$.

• *In the fourth case of Theorem 4.7, the set* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *of the decompositions of* E *simultaneously adapted to* (a, T) *and to* $(a(\,\cdot\,, S\cdot\,), T)$ *is non empty; it is canonically in bijection with the set* $\mathcal{D}'_{(\underline{a},T)}$ *of the decompositions of* $E' = \mathrm{Ker}(S^{\mathbb{C}} - \lambda \,\mathrm{Id}_{E^{\mathbb{C}}})$*, adapted to* $(\underline{a}, T)$*. If* $D \in \mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *and* $D^{\mathbb{C}}$ *is the complexification of* D*, the bijection is given by* $D \mapsto D^{\mathbb{C}} \cap E'$*. The converse map is* $D' \mapsto \widetilde{D} = (\Re(D'_{p,q} \oplus \overline{D'_{p,q}})_{1 \le q \le p \le n}$*.* N *acts simply transitively on* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *and* $R \simeq \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \cap \mathrm{Stab}(\check{E}_k, \check{b}_k(\,\cdot\,, S\cdot)) = \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \cap \mathrm{C}(S) \simeq \prod_{k=1}^n \mathrm{U}(\check{E}'_k, \underline{\check{b}}_k)$.

• *In the fifth case of Theorem 4.7, the set* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *of the decompositions of* E *simultaneously adapted to* (a, T) *and to* $(a(\,\cdot\,, S\cdot\,), T)$ *is non empty; it is canonically in bijection with the set* $\mathcal{D}^+_T$ *of the decompositions of* $E^+ = \mathrm{Ker}(S^{\mathbb{C}} - \lambda \,\mathrm{Id}_{E^{\mathbb{C}}})$*, adapted to* $T_{|E^+}$*. If* $D \in \mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *and* $D^{\mathbb{C}}$ *is the complexification of* D*, the bijection is given by* $D \mapsto D^{\mathbb{C}} \cap E^+$*. The converse map is* $D^+ \mapsto \widetilde{D} = (\Re(D^+_{p,q} \oplus D^-_{p,q} \oplus \overline{D^+_{p,q}} \oplus \overline{D^-_{p,q}})_{1 \le q \le p \le n}$ *with* $(D^-_{p,q})_{1 \le q \le p \le n} = ((D^+_{p,q})_{1 \le q \le p \le n})^{*\sharp}$*.* N *acts simply transitively on* $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\cdot),T)}$ *and* $R \simeq \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \cap \mathrm{Stab}(\check{E}_k, \check{b}_k(\,\cdot\,, S\cdot)) = \prod_{k=1}^n \mathrm{Stab}(\check{E}_k, \check{b}_k) \cap \mathrm{C}(S) \simeq \prod_{k=1}^n \mathrm{GL}_{\mathbb{C}}(\check{E}^+_k)$.

We finally introduce the preferred basis for the couple (a, B), thanks to Corollary 3.7. We send back to 3.c for the notation: D_k, $\check{\beta}_k$, β_k ...

Cases a skew-symmetric, b symmetric, degenerate or a skew-symmetric, b symmetric, degenerate $(P = X)$ or a symmetric, b skew-symmetric, non degenerate and $P = (X - \lambda)(X + \lambda)$, $\lambda \in \mathbb{R}^*$. The obtained matrices are nearly the same as in the case where $\mathbb{K}$ is algebraically closed, so see pp. 15 sq. The two only changes are in the case $P = X$, a skew-symmetric, non degenerate, b symmetric, where, for the odd k:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & I_{r_r,s_k} \\ & & & \iddots & \\ & & -I_{r_r,s_k} & & \\ & I_{r_r,s_k} & & & \\ -I_{r_r,s_k} & & & & \\ I_{r_r,s_k} & & & & \end{pmatrix}}_{k \text{ blocks}},$$

and in the case $P = X$, a skew-symmetric, non degenerate, b symmetric, where a similar change, a replacement of the I_{d_k} by I_{r_k,s_k}, occurs for the even k.

Case a symmetric, non degenerate, b skew-symmetric, non degenerate and $P = (X - \lambda)(X - \overline{\lambda}))$, $\lambda \in \mathrm{i}\mathbb{R}^*$. In this case, for each k, $\check{E}_k^{\mathbb{C}} = \check{E}'_k \oplus \overline{\check{E}'_k}$ with $S_{|\check{E}'_k} = \lambda \,\mathrm{Id}_{\check{E}'_k}$ and $S_{|\overline{\check{E}'_k}} = \overline{\lambda}\,\mathrm{Id}_{\overline{\check{E}'_k}}$. We choose a basis $(z_i)_{i=1}^{d_k}$ of $\check{E}'_k$, $\underline{\check{b}}_k$-pseudo-orthonormal, i.e. in which $\mathrm{Mat}_{(z_i)_{i=1}^{d_k}}(\underline{\check{b}}_k) = I_{r_k,s_k}$. We then set $\check{\beta}_k = ((x_j)_{j=1}^{d_k}, (y_j)_{j=1}^{d_k})$ with: $\forall j$, $x_j = \frac{1}{\sqrt{2}}(z_j + \overline{z_j})$ and $y_j = \frac{1}{\mathrm{i}\sqrt{2}}(-z_j + \overline{z_j})$. In $\check{\beta}_k$, one checks that:

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = I_{r_k,s_k,r_k,s_k} := \begin{pmatrix} I_{r_k,s_k} & 0 \\ 0 & I_{r_k,s_k} \end{pmatrix} \text{ if } k \text{ is odd i.e. if } \check{b}_k \text{ is symmetric,}$$

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = J_{r_k,s_k} := \begin{pmatrix} 0 & -I_{r_k,s_k} \\ I_{r_k,s_k} & 0 \end{pmatrix} \text{ if } k \text{ is even i.e. if } \check{b}_k \text{ is skew-symmetric,}$$

$$\mathrm{Mat}_{\check{\beta}_k}(S) = \Lambda_{d_k} := \begin{pmatrix} 0 & -\nu I_{d_k} \\ \nu I_{d_k} & 0 \end{pmatrix} = \nu J_{d_k} \text{ with } \lambda = \mathrm{i}\nu,\ \nu \in \mathbb{R}^*, \text{ for all } k.$$

Then, in a basis β_k of D_k built, as in 3.c, after such a $\check{\beta}_k$, we get:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & +I_{r_k,s_k,r_k,s_k} \\ & & & \iddots & \\ & & -I_{r_k,s_k,r_k,s_k} & & \\ & +I_{r_k,s_k,r_k,s_k} & & & \\ -I_{r_k,s_k,r_k,s_k} & & & & \\ +I_{r_k,s_k,r_k,s_k} & & & & \end{pmatrix}}_{k \text{ blocks}} \text{ for the odd } k,$$

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & -J_{r_k,s_k} \\ & & & \iddots & \\ & & -J_{r_k,s_k} & & \\ & +J_{r_k,s_k} & & & \\ -J_{r_k,s_k} & & & & \\ +J_{r_k,s_k} & & & & \end{pmatrix}}_{k \text{ blocks}} \text{ for the even } k,$$

$$\mathrm{Mat}_{\beta_k}(B_{|D_k}) = \underbrace{\begin{pmatrix} \Lambda_{d_k} & I_{2d_k} & & \\ & \Lambda_{d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \Lambda_{d_k} \end{pmatrix}}_{k \text{ blocks}} \quad \text{for all } k, \text{ with } \Lambda_{d_k} = \nu J_{d_k} := \begin{pmatrix} 0 & -\nu I_{d_k} \\ \nu I_{d_k} & 0 \end{pmatrix}.$$

Case a symmetric, non degenerate, b skew-symmetric, non degenerate and $P = (X-\lambda)(X-\overline{\lambda})(X+\lambda)(X+\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$. In this case, for each k, $\check{E}_k^{\mathbb{C}} = \check{E}_k^+ \oplus \overline{\check{E}_k^+} \oplus \check{E}_k^- \oplus \overline{\check{E}_k^-}$ with $S_{|\check{E}_k^\pm} = \pm\lambda\,\mathrm{Id}_{\check{E}_k^\pm}$ and $S_{|\overline{\check{E}_k^\pm}} = \pm\overline{\lambda}\,\mathrm{Id}_{\overline{\check{E}_k^\pm}}$. After Lemma 4.1, each $\check{E}_k^\pm$ or $\overline{\check{E}_k^\pm}$ is $\check{b}_k$-totally degenerate and $\check{E}_k^+ \oplus \check{E}_k^-$ and $\overline{\check{E}_k^-} \oplus \overline{\check{E}_k^-}$ are each $\check{b}_k$-non degenerate and orthogonal to each other. We choose any basis $(z_j^+)_{j=1}^{d_k}$ of $\check{E}_k^+$, and set $(z_j^-)_{j=1}^{d_k}$ the basis of $\check{E}_k^-$ such that $\check{b}_k(z_i^-, z_j^+) = \delta_{i,j}$ (so $\check{b}_k(z_i^+, z_j^-) = \delta_{i,j}$ if k is such that $\check{b}_k$ is symmetric, else $\check{b}_k(z_i^+, z_j^-) = -\delta_{i,j}$). We finally set:

$$\check{\beta}_k = ((x_j^+)_{j=1}^d, (y_j^+)_{j=1}^d, (x_j^-)_{j=1}^d, (y_j^-)_{j=1}^d),\ \forall j,\ x_j^\pm = \frac{1}{\sqrt{2}}(z_j^\pm + \overline{z_j^\pm}),\ y_j^\pm = \frac{1}{\mathrm{i}\sqrt{2}}(-z_j^\pm + \overline{z_j^\pm}).$$

One checks then that:

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = L_{d_k,d_k} := \begin{pmatrix} 0 & I_{d_k,d_k} \\ I_{d_k,d_k} & 0 \end{pmatrix} \text{ if } k \text{ is odd i.e. if } \check{b}_k \text{ is symmetric,}$$

$$\mathrm{Mat}_{\check{\beta}_k}(\check{b}_k) = J_{d_k,d_k} := \begin{pmatrix} 0 & -I_{d_k,d_k} \\ I_{d_k,d_k} & 0 \end{pmatrix} \text{ if } k \text{ is even i.e. if } \check{b}_k \text{ is skew-symmetric,}$$

$$\mathrm{Mat}_{\check{\beta}_k}(S) = \begin{pmatrix} \Lambda_{d_k} & 0 \\ 0 & -\Lambda_{d_k} \end{pmatrix} \text{ with } \Lambda_{d_k} := \begin{pmatrix} \mu I_{d_k} & -\nu I_{d_k} \\ \nu I_{d_k} & \mu I_{d_k} \end{pmatrix} \text{ where } \lambda = \mu + \mathrm{i}\nu,\ \mu,\nu \in \mathbb{R}^*.$$

Then, in a basis β_k of D_k built, as in 3.c, after such a $\check{\beta}_k$, we get:

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & +L_{d_k,d_k} \\ & & & \iddots & \\ & & -L_{d_k,d_k} & & \\ & +L_{d_k,d_k} & & & \\ -L_{d_k,d_k} & & & & \\ +L_{d_k,d_k} & & & & \end{pmatrix}}_{k \text{ blocks}} \quad \text{for the odd } k,$$

$$\mathrm{Mat}_{\beta_k}(a_{|D_k}) = \underbrace{\begin{pmatrix} & & & & -J_{d_k,d_k} \\ & & & \iddots & \\ & & -J_{d_k,d_k} & & \\ & +J_{d_k,d_k} & & & \\ -J_{d_k,d_k} & & & & \\ +J_{d_k,d_k} & & & & \end{pmatrix}}_{k \text{ blocks}} \quad \text{for the even } k,$$

$$\mathrm{Mat}_{\beta_k}(B_{|D_k}) = \underbrace{\begin{pmatrix} \widetilde{\widetilde{\Lambda}}_{d_k} & I_{2d_k} & & \\ & \widetilde{\widetilde{\Lambda}}_{d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \widetilde{\widetilde{\Lambda}}_{d_k} \end{pmatrix}}_{k \text{ blocks}} \quad \text{for all } k, \text{ with } \widetilde{\widetilde{\Lambda}}_{d_k} = \begin{pmatrix} \Lambda_{d_k} & 0 \\ 0 & \Lambda_{d_k} \end{pmatrix}.$$

As usual follows from what precedes a characterization of the conjugaison class of a couple (a, b) under the action of $\mathrm{GL}(E)$.

4.10 Proposition *Let (a, b) be a couple of bilinear forms on a real vectorspace E, a symmetric and b skew-symmetric. We denote by B the endomorphism such that $b = a(\,\cdot\,, B\,\cdot\,)$ and by $\prod_{i=0}^{N} P_i^{n_i}$ the minimal polynomial of B; by Lemma 4.1, $P_0 = X$ (with possibly $n_0 = 0$) and, for $i \geq 1$, $P_i = (X - \lambda_i)(X + \lambda_i)$, $\lambda_i \in \mathbb{R}^*$ or $P_i = (X - \lambda_i)(X - \overline{\lambda_i})$, $\lambda_i \in \mathrm{i}\mathbb{R}^*$ or $P_i = (X - \lambda_i)(X - \overline{\lambda_i})(X + \lambda_i)(X + \overline{\lambda_i})$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$. Then, the couple (a, b) is characterized, up to conjugation by $\mathrm{GL}(E)$, by:*

- *the $(P_i, n_i)_{i=1}^{N}$,*
- *if $n_0 \neq 0$, the (even) dimensions, denoted above by $(2d_k)_{k=1,k\text{ even}}^{n_0}$, of the quotients $(\check{E}_k)_{k=1,k\text{ even}}^{n_0}$ (see Notation 2.2), the dimensions, denoted above by $(d_k)_{k=1,k\text{ odd}}^{n_0}$, of the quotients $(\check{E}_k)_{k=1,k\text{ odd}}^{n_0}$ and the signatures $(r_k, s_k)_{k=1,k\text{ odd}}^{n_0}$ of the corresponding forms $(\check{b}_k)_{k=1,k\text{ odd}}^{n_0}$ defined on the $(\check{E}_k)_{k=1,k\text{ odd}}^{n_0}$ $(r_k + s_k = d_k)$,*
- *for each Q_i of the form $(X - \lambda_i)(X + \lambda_i)$, $\lambda_i \in \mathbb{R}^*$, the (even) dimensions $(2d_k)_{k=1}^{n_i}$ of the quotients $(\check{E}_k)_{k=1}^{n_i}$ (see Notation 2.2), or equivalently the dimensions $(d_k)_{k=1}^{n_i}$, of the quotients $(\check{E}_k^{+})_{k=1}^{n_i}$.*
- *for each Q_i of the form $(X - \lambda_i)(X - \overline{\lambda_i})$, $\lambda_i \in \mathrm{i}\mathbb{R}^*$, the dimensions $(2d_k)_{k=1}^{n_i}$ of the quotients $(\check{E}_k)_{k=1}^{n_i}$, or equivalently the complex dimensions $(d_k)_{k=1}^{n_i}$ of the quotients $(\check{E}_k')_{k=1}^{n_i}$, and the signatures $(r_k, s_k)_{k=1}^{n_i}$ of the hermitian forms $\underline{\check{b}}_k$ defined on them $(r_k + s_k = d_k)$,*
- *for each Q_i of the form $(X - \lambda_i)(X - \overline{\lambda_i})(X + \lambda_i)(X + \overline{\lambda_i})$, $\lambda_i \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$, the (multiple of four) dimensions $(4d_k)_{k=1}^{n_i}$ of the quotients $(\check{E}_k)_{k=1}^{n_i}$, or equivalently the complex dimensions $(d_k)_{k=1}^{n_i}$ of the quotients $(\check{E}_k^{+})_{k=1}^{n_i}$.*

Besides, if a is skew-symmetric and b symmetric, the same statement holds with "even" and "odd" swapped in the second point.

5 Some last general remarks

5.1 Remark This work is based on the decomposition $B = S + T$ of B in its semi-simple and nilpotent parts. So, similar results could be obtained by the same way for any perfect base field $\mathbb{K}$, as such a decomposition $B = S + T$ exists over such base fields.

5.2 Remark One may distinguish the cases where $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is simply isomorphic to the commutant of some nilpotent endomorphism T from those where $\mathrm{Stab}(a) \cap \mathrm{Stab}(b) = R \ltimes N$, N acting simply transitively on some $\mathcal{D}_{(a,T)}$ or $\mathcal{D}_{(a,T)} \cap \mathcal{D}_{(a(\,\cdot\,,S\,\cdot\,),T)}$. When $\mathbb{K} = \mathbb{R}$, the first case happens when one of a, b is symmetric, the other skew-symmetric and:

- $P = (X - \lambda)(X + \lambda)$, $\lambda \in \mathbb{R}^*$ (then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is isomorphic to the commutant of a real nilpotent endomorphism),
- $P = (X - \lambda)(X - \overline{\lambda})(X + \lambda)(X + \overline{\lambda})$, $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathrm{i}\mathbb{R})$ (then $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ is isomorphic to the commutant of a complex nilpotent endomorphism).

The second case consists of all other cases.

5.3 Remark J. Milnor proposed an autonomous proof of the following fact: in finite dimension, if two real bilinear symmetric forms have no common isotropic vector, they are simultaneously diagonalizable. This result can also be seen as a corollary of the simultaneous matricial reduction of two real bilinear symmetric forms given in 3.c. Indeed if a and b have no common isotropic vector, a admits no B-stable isotropic subspace, so $d_k = 0$ for $k > 1$ —see the matrices—; the $\check{E}_k$ are $\{0\}$ except $\check{E}_1$. The normal form given in 3.c is then a simultaneous diagonalization.

5.4 Remark Let us suppose that a and b are both symmetric or both skew-symmetric and that $\deg P = 1$; besides we denote $\mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ by Ω; Ω acts on each $E/\operatorname{Ker} T^k = E^k$, $k \leq n$, we denote by Ω^k the image of Ω by this representation in $\mathrm{GL}(E^k)$. We remind that b^k is the form $a(\,\cdot\,, T^k\,\cdot\,)$ defined on E^k. It follows easily from Proposition 3.6 that:

5.5 Corollary (of Proposition 3.6) *For all $k \leq n$:*

$$\begin{aligned}\Omega^k &= \mathrm{Stab}(b^k) \cap \mathrm{Stab}(b^k(\,\cdot\,, T\,\cdot\,)) \\ &= \{\gamma \in \mathrm{Stab}(b^k);\ \operatorname{Ker} T^{k+1}/\operatorname{Ker} T^k \textit{ is stable by } \gamma \\ &\qquad\qquad \textit{and the action of } \gamma \textit{ modulo } \operatorname{Ker} T^{k+1} \textit{ is in } \Omega^{k+1}\}\end{aligned}$$

In turn, this corollary enables, by a downward induction on k, to detail the structure of Ω more precisely than the decomposition $\Omega = R \ltimes N$ given here. This is done in [1] chapter 2 §II.7. The results are clumbersome so we do not detail them here; they can be established by the reader if needed. A similar work can be done in the general case a, b reflexive.

6 Appendix: table of the obtained matrices

6.1 Proposition *Let (a, b) be a couple of reflexive forms on a finite dimensional real vector space E. We suppose that a is non degenerate and set $b = a(\,\cdot\,, B\cdot\,)$. Then, with respect to a and b, E splits in a sum $E = \overset{\perp}{\bigoplus\limits_{P \in \mathcal{P}}} E_P$ with $\mathcal{P}$ a finite subset of $\mathbb{R}[X]$ such that, on each E_P, the minimal polynomial of B is P. Moreover:*

- *the polynomials of $\mathcal{P}$ are of the types given in table 2,*

- *there is a basis β of each E_P in which the following matrices are block-diagonal: $\mathrm{Mat}_\beta(a_{|E_P}) = \mathrm{diag}(A_1, \ldots, A_n)$ and $\mathrm{Mat}_\beta(B_{|E_P}) = \mathrm{diag}(B_1, \ldots, B_n)$, with the blocks A_k and B_k given, for each possible type of polynomial P, in table 2. The principle of this simultaneous reduction is due to Weierstraß [8].*

- $I_p \in \mathrm{M}_p(\mathbb{R})$ is the identity matrix of order p,
- $I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} \in \mathrm{M}_{r+s}(\mathbb{R})$,
- $I_{r,s,t,u} = \begin{pmatrix} I_{r,s} & 0 \\ 0 & I_{t,u} \end{pmatrix} \in \mathrm{M}_{r+s+t+u}(\mathbb{R})$,
- $J_d = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix} \in \mathrm{M}_{2d}(\mathbb{R})$,
- $J_{r,s} = \begin{pmatrix} 0 & -I_{r,s} \\ I_{r,s} & 0 \end{pmatrix} \in \mathrm{M}_{2(r+s)}(\mathbb{R})$,
- $L_d = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix} \in \mathrm{M}_{2d}(\mathbb{R})$,

(Continued on the next page)

TABLE 1 – *Notation of the employed matrices.*

- $L_{r,s} = \begin{pmatrix} 0 & I_{r,s} \\ I_{r,s} & 0 \end{pmatrix} \in \mathrm{M}_{2(r+s)}(\mathbb{R})$,
- If $\lambda = \mu + i\nu \in \mathbb{C}$, $(\mu, \nu) \in \mathbb{R}^2$, one sets $\Lambda_p = \mu I_{2p} + \nu J_p = \begin{pmatrix} \mu I_p & -\nu I_p \\ \nu I_p & \mu I_p \end{pmatrix} \in \mathrm{M}_{2p}(\mathbb{R})$. Similarly, $\overline{\Lambda}_p$ is the matrix associated to $\overline{\lambda} = \mu - i\nu$.
- If Λ_p is as above: $*\ \widetilde{\Lambda}_p = \begin{pmatrix} \Lambda_p & 0 \\ 0 & \overline{\Lambda}_p \end{pmatrix} \in \mathrm{M}_{4p}(\mathbb{R})$

  $*\ \widetilde{\widetilde{\Lambda}}_p = \begin{pmatrix} \Lambda_p & 0 \\ 0 & -\Lambda_p \end{pmatrix} \in \mathrm{M}_{4p}(\mathbb{R})$.

TABLE 1 – *Notation of the employed matrices.*

In the following table, the matrices A_k and B_k have k blocks, as indicated in the first line ; d_k may be any integer.

a and b symmetric		
$\boxed{P = X - \lambda,\ \lambda \in \mathbb{R}}$	$A_k = \underbrace{\begin{pmatrix} & & I_{r_k,s_k} \\ & \iddots & \\ I_{r_k,s_k} & & \end{pmatrix}}_{k \text{ blocks}}$	$B_k = \underbrace{\begin{pmatrix} \lambda I_{d_k} & I_{d_k} & & \\ & \lambda I_{d_k} & \ddots & \\ & & \ddots & I_{d_k} \\ & & & \lambda I_{d_k} \end{pmatrix}}_{k \text{ blocks}}$
with $r_k + s_k = d_k$.		
$\boxed{P=(X-\lambda)(X-\overline{\lambda}),\ \lambda \in \mathbb{C} \setminus \mathbb{R}}$	$A_k = \begin{pmatrix} & & L_{d_k} \\ & \iddots & \\ L_{d_k} & & \end{pmatrix}$	$B_k = \begin{pmatrix} \Lambda_{d_k} & I_{2d_k} & & \\ & \Lambda_{d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \Lambda_{d_k} \end{pmatrix}$
with Λ_{d_k} a matrix corresponding to one of the eigenvalues λ of P, see table 1.		
a and b skew-symmetric		
$\boxed{P = X - \lambda}$	$A_k = \begin{pmatrix} & & J_{d_k} \\ & \iddots & \\ J_{d_k} & & \end{pmatrix}$	$B_k = \begin{pmatrix} \lambda I_{2d_k} & I_{2d_k} & & \\ & \lambda I_{2d_k} & \ddots & \\ & & \ddots & I_{2d_k} \\ & & & \lambda I_{2d_k} \end{pmatrix}$
$\boxed{P=(X-\lambda)(X-\overline{\lambda}),\ \lambda \in \mathbb{C} \setminus \mathbb{R}}$	$A_k = \begin{pmatrix} & & J_{2d_k} \\ & \iddots & \\ J_{2d_k} & & \end{pmatrix}$	$B_k = \begin{pmatrix} \widetilde{\Lambda}_{d_k} & I_{4d_k} & & \\ & \widetilde{\Lambda}_{d_k} & \ddots & \\ & & \ddots & I_{4d_k} \\ & & & \widetilde{\Lambda}_{d_k} \end{pmatrix}$
with $\widetilde{\Lambda}_{d_k}$ a matrix corresponding to one of the eigenvalues λ of P, see table 1.		
a symmetric, b skew-symmetric		
		(Continued on the next page)

TABLE 2 – *Different possible forms for $P \in \mathcal{P}$; form of the A_k and B_k in each case.*

$$\boxed{\begin{array}{c}P=X\\(b\text{ degenerate})\end{array}}\qquad \text{For } k \text{ odd: } A_k = \begin{pmatrix} & & & +I_{r_k,s_k}\\ & & \iddots & \\ & +I_{r_k,s_k} & & \\ -I_{r_k,s_k} & & & \\ +I_{r_k,s_k} & & & \end{pmatrix}, \text{ for } k \text{ even:}$$

$$A_k = \begin{pmatrix} & & & -J_{d_k}\\ & & \iddots & \\ & +J_{d_k} & & \\ -J_{d_k} & & & \\ +J_{d_k} & & & \end{pmatrix} \text{ and for all } k\text{: } B_k = \begin{pmatrix} 0 & I_{\delta_k} & & \\ & 0 & \ddots & \\ & & \ddots & I_{\delta_k}\\ & & & 0 \end{pmatrix}$$

with $r_k + s_k = d_k$ and, for k odd, $\delta_k = d_k$ and for k even, $\delta_k = 2d_k$.

$$\boxed{\begin{array}{c}P=(X-\lambda)(X+\lambda)\\\lambda\in\mathbb{R}^*\end{array}}\qquad \text{For } k \text{ odd: } A_k = \begin{pmatrix} & & & +L_{d_k}\\ & & \iddots & \\ & +L_{d_k} & & \\ -L_{d_k} & & & \\ +L_{d_k} & & & \end{pmatrix}, \text{ for } k \text{ even:}$$

$$A_k = \begin{pmatrix} & & & -J_{d_k}\\ & & \iddots & \\ & +J_{d_k} & & \\ -J_{d_k} & & & \\ +J_{d_k} & & & \end{pmatrix} \text{ and for all } k\text{: } B_k = \begin{pmatrix} \lambda I_{d_k,d_k} & I_{2d_k} & & \\ & \lambda I_{d_k,d_k} & \ddots & \\ & & \ddots & I_{2d_k}\\ & & & \lambda I_{d_k,d_k} \end{pmatrix}$$

$$\boxed{\begin{array}{c}P=(X-\lambda)(X-\overline{\lambda}),\\\lambda\in \mathrm{i}\mathbb{R}^*\end{array}}\qquad \text{For } k \text{ odd: } A_k = \begin{pmatrix} & & & +I_{r_k,s_k,r_k,s_k}\\ & & \iddots & \\ & +I_{r_k,s_k,r_k,s_k} & & \\ -I_{r_k,s_k,r_k,s_k} & & & \\ +I_{r_k,s_k,r_k,s_k} & & & \end{pmatrix}, \text{ for } k \text{ even:}$$

$$A_k = \begin{pmatrix} & & & -J_{r_k,s_k}\\ & & \iddots & \\ & +J_{r_k,s_k} & & \\ -J_{r_k,s_k} & & & \\ +J_{r_k,s_k} & & & \end{pmatrix} \text{ and for all } k\text{: } B_k = \begin{pmatrix} \Lambda_{d_k} & I_{2d_k} & & \\ & \Lambda_{d_k} & \ddots & \\ & & \ddots & I_{2d_k}\\ & & & \Lambda_{d_k} \end{pmatrix}$$

with $r_k + s_k = d_k$ and with Λ_{d_k} a matrix corresponding to one of the eigenvalues $\lambda \in \mathrm{i}\mathbb{R}^*$ of P, see table 1.

$$\boxed{\begin{array}{c}P=(X-\lambda)(X-\overline{\lambda})\\(X+\lambda)(X+\overline{\lambda}),\\\lambda\in\mathbb{C}\setminus(\mathbb{R}\cup \mathrm{i}\mathbb{R}).\end{array}}\qquad \text{For } k \text{ odd: } A_k = \begin{pmatrix} & & & +L_{d_k,d_k}\\ & & \iddots & \\ & +L_{d_k,d_k} & & \\ -L_{d_k,d_k} & & & \\ +L_{d_k,d_k} & & & \end{pmatrix}, \text{ for } k \text{ even:}$$

(Continued on the next page)

TABLE 2 – *Different possible forms for $P \in \mathcal{P}$; form of the A_k and B_k in each case.*

$A_k = \begin{pmatrix} & & & & -J_{d_k,d_k} \\ & & & \iddots & \\ & & +J_{d_k,d_k} & & \\ & -J_{d_k,d_k} & & & \\ +J_{d_k,d_k} & & & & \end{pmatrix}$ and for all k: $B_k = \begin{pmatrix} \overset{\approx}{\Lambda}_{d_k} & I_{4d_k} & & \\ & \overset{\approx}{\Lambda}_{d_k} & \ddots & \\ & & \ddots & I_{4d_k} \\ & & & \overset{\approx}{\Lambda}_{d_k} \end{pmatrix}$ with $\overset{\approx}{\Lambda}_{d_k}$ a matrix corresponding to one of the eigenvalues λ of P, see table 1.
a skew-symmetric, b symmetric and $P = X$ (b degenerate)
Same blocks A_k and B_k as in the case a symmetric, b skew-symmetric, $P = X$, swapping the forms of A_k and B_k for k odd and even.

TABLE 2 – *Different possible forms for $P \in \mathcal{P}$; form of the A_k and B_k in each case.*

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