Linear mod one transformations and the distribution of fractional parts $\{\xi(p/q)^n\}$

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Abstract. *For coprime integers $p > q \geq 2$ and for a positive real number $s < 1 - 1/p$, let $Z_{p/q}(s, s + 1/p)$ be the set of real numbers ξ such that $s \leq \{\xi(p/q)^n\} \leq s + 1/p$ holds for all non-negative integers n. Here, $\{\cdot\}$ denotes the fractional part. Flatto, Lagarias & Pollington showed that the set of s in $[0, 1 - 1/p]$ for which $Z_{p/q}(s, s+1/p)$ is empty is a dense set, and they deduced that $\limsup_{n\to\infty} \{\xi(p/q)^n\} - \liminf_{n\to\infty} \{\xi(p/q)^n\} \geq 1/p$ holds for all positive real numbers ξ. In the present work, we give further results on the sets $Z_{p/q}(s, s+1/p)$. For instance, we prove that they are empty for almost all s in $[0, 1 - 1/p]$.*

1. Introduction

It is well known (see *e.g.* [7], Chapter 1, Corollary 4.2) that for almost all real numbers $\theta \geq 1$ the sequence $\{\theta^n\}$ is uniformly distributed in $[0, 1]$. Here and in the sequel, $\{\cdot\}$ denotes the fractional part. However, very few results are known for specific values of θ, and the distribution of $\{(p/q)^n\}$ for coprime positive integers $p > q \geq 2$ remains an unsolved problem. Vijayaraghavan [10] showed that this sequence has infinitely many limit points, but we are unable to decide whether

$$\limsup_{n\to\infty} \left\{ \left(\frac{p}{q}\right)^n \right\} - \liminf_{n\to\infty} \left\{ \left(\frac{p}{q}\right)^n \right\} > \frac{1}{2}.$$

A striking progress has been recently made by Flatto, Lagarias & Polling-

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ton [5], who proved that, for all positive real numbers ξ, we have

$$\limsup_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\}-\liminf_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\}\geq\frac{1}{p}. \tag{1}$$

They were inspired by a paper of Mahler [8], who studied the hypothetical existence of so-called Z-numbers, i.e. positive real numbers ξ such that $0\leq\{\xi(3/2)^n\}<1/2$ for all integers $n\geq 0$. Extending this definition, Flatto *et al.* introduce, for an interval $[s,s+t[$ included in $[0,1[$, the set

$$Z_{p/q}(s,s+t):=\left\{\xi\in\mathbf{R}: s\leq\left\{\xi\left(\frac{p}{q}\right)^n\right\}<s+t \qquad \text{for all } n\geq 0\right\}.$$

To prove (1), they show that the set of s such that $Z_{p/q}(s,s+1/p)$ is empty is dense in $[0,1-1/p]$. Their argument uses Mahler's method, as explained in a preliminary work by Flatto [4] (for more bibliographical references, we refer the reader to [4] and [5]), and also relies on a careful study of contracting, linear transformations, which are very close to those investigated in [2] and [3].

The purpose of the present work is to show how the methods used in [2] and [3] apply to the transformations considered in [5], and to derive some interesting consequences. For instance, we prove that $Z_{p/q}(s,s+1/p)$ is empty for almost all (in the sense of Lebesgue measure) real numbers s in $[0,1-1/p]$ and we answer both questions posed at the end of [5].

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2. Statement of the results

Before stating our results, we have to introduce some notation, which will be used throughout the present paper. Let τ be a real number with $0\leq\tau<1$. For any integer k, set

$$\varepsilon_k(\tau)=[k\tau]-[(k-1)\tau],$$

where $[\cdot]$ denotes the integer part. The sequence $(\varepsilon_k(\tau))_{k\in\mathbf{Z}}$ only takes values 0 and 1 and, for τ irrational, it is usually called the characteristic Sturmian sequence associated to τ. For any non-zero rational a/b, with a and b coprime, the sequence $(\varepsilon_k(a/b))_{k\in\mathbf{Z}}$ is periodic with period b.

Our first result concerns the sets $Z_{p/q}(s,s+1/p)$ and complements a result of Flatto *et al.* who proved in [5] that the set of s in $[0,1-1/p]$ for which $Z_{p/q}(s,s+1/p)$ is empty is a dense set.

Theorem 1. *Let $p > q \geq 2$ be coprime integers. Then the set $Z_{p/q}(s, s+1/p)$ is empty for a set of s of full Lebesgue measure in $[0, 1-1/p]$. More precisely, this set is empty when there exists a rational number a/b, with $b > a \geq 1$, such that*

$$\frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^b}{1 + \frac{q}{p} + \ldots + \left(\frac{q}{p}\right)^{b-1}} \leq \{(p-q)s\} \leq \frac{\sum_{k=1}^{b-2} \varepsilon_{-k}(a/b)\left(\frac{q}{p}\right)^k + \left(\frac{q}{p}\right)^{b-1}}{1 + \frac{q}{p} + \ldots + \left(\frac{q}{p}\right)^{b-1}}.$$

Further, if for some real number s in $[0, 1-1/p]$ the set $Z_{p/q}(s, s+1/p)$ is nonempty, then there exists an irrational number τ in $]0,1[$ such that

$$\{(p-q)s\} = \frac{p-q}{p} \sum_{k=1}^{\infty} \varepsilon_{-k}(\tau) \left(\frac{q}{p}\right)^k. \tag{2}$$

The proof of Theorem 1 is given in Section 3 and relies upon the main result of [2]. It also allows us to answer (in Theorem 3) a problem posed by Flatto *et al.* at the end of [5].

As an immediate application, we considerably improve Corollary 1.4a of [5].

Corollary 1. *The set $Z_{3/2}(s, s+1/3)$ is empty if*

$$s \in \{0\} \cup [8/57, 4/19] \cup [4/15, 2/5] \cup [26/57, 10/19] \cup \{2/3\}.$$

To prove Corollary 1, we check that the three intervals are given by Theorem 1 applied with the rationals 1/3, 1/2 and 2/3. Further, since for any irrational τ in $]0,1[$ there exists $k \geq 1$ and $\ell \geq 1$ such that $\varepsilon_{-k}(\tau) = 1$ and $\varepsilon_{-\ell}(\tau) = 0$, we get that $\frac{1}{3}\left(\sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)(2/3)^k\right) \neq 0, 2/3$, hence, by the last part of Theorem 1, the sets $Z_{3/2}(0, 1/3)$ and $Z_{3/2}(2/3, 1)$ are empty (a fact already proved in [5]).

We are not able to determine whether $Z_{p/q}(s, s+1/p)$ is empty for *all* values of s. However, we obtain some additional informations concerning the sets $Z_{p/q}(s, s+1/p)$ for exceptional values of s.

Theorem 2. *Let $p > q \geq 2$ be coprime integers. Let s in $[0, 1-1/p]$ satisfy (2) for some irrational τ. Then we have*

$$\mathrm{Card}\{\xi : 0 \leq \xi \leq x \text{ and } \xi \in Z_{p/q}(s, s+1/p)\} = O\big((\log_q x)^3\big).$$

Theorem 2 considerably improves Theorem 1.1 of [5] for $t = 1/p$, where the estimate $O(x^\gamma)$ is obtained, with $\gamma = \log_q \min\{2, p/q\}$. Its proof is given in Section 4, where we get strong conditions on $[\xi]$ for ξ belonging to $Z_{p/q}(s, s+1/p)$.

3. Proof of Theorem 1

In all what follows, for a map F and an integer $n \geq 0$, we denote by F^n the map $F \circ \ldots \circ F$, composed n times.

Keeping the notation of Part 3 of [5], we define for any real numbers $\beta > 1$ and $0 \leq \alpha < 1$ the map $f_{\beta,\alpha}$ by

$$f_{\beta,\alpha}(x) = \{\beta x + \alpha\} \quad \text{for } x \in [0, 1[.$$

We set

$$S_{\beta,\alpha} := \{x \in [0, 1[: 0 \leq f^n_{\beta,\alpha}(x) < 1/\beta \quad \text{for all } n \geq 0.\}.$$

Theorem 3.4 of [5] asserts that $S_{\beta,\alpha}$ is finite as soon as $0 \notin S_{\beta,\alpha}$, which is the case for a dense set of values of α in $[0, 1]$ (see Theorem 3.5 of [5]). The problem of the existence of values of α such that $S_{\beta,\alpha}$ is infinite is left open in [5]. Indeed, setting

$$E_\beta := \{\alpha \in [0, 1[: S_{\beta,\alpha} \text{ is an infinite set}\},$$

Flatto *et al.* conjecture that, for all $\beta > 1$, the set E_β is non empty and perfect, and has Lebesgue measure zero. The present section is concerned with the study of this problem, which we solve in Theorem 3 below.

In the rest of this section, f stands for $f_{\beta,\alpha}$.

Lemma 1. *Assume that there exists an integer $N \geq 1$ with $f^N(0) = 0$ and such that $f^k(0) \notin \{0\} \cup [1/\beta, 1[$ for any integer $1 \leq k \leq N-1$. Then $S_{\beta,\alpha}$ is the finite set $\{0, f(0), \ldots, f^{N-1}(0)\}$.*

Proof. We use the notation of Lemmas 3.1, 3.2 and 3.3 of [5], and we only point out which slight changes should be made in their proofs to get our lemma. A continuity argument shows that α is the right endpoint of the interval $I_{N-1} = [\cdot, \alpha[$. It follows that R_{-1} is empty, whence all the R_k's, $k \geq 0$, are empty. Arguing as in [5], we denote by p_n the left endpoint of L_n for $n \geq 0$, and we set

$$q_k := \lim_{j \to \infty} p_{k+jN}, \quad 0 \leq k \leq N-1.$$

Each q_k coincides with a right endpoint of some I_j, $0 \leq j \leq N-1$. It follows that $S_{\beta,\alpha} = \{q_0, \ldots, q_{N-1}\} = \{0, f(0), \ldots, f^{N-1}(0)\}$, as claimed. □

Lemma 2. *Let $\beta > 1$. Then, for each α in $[0, 1[$, the set $S_{\beta,\alpha}$ is infinite if, and only if, $f^N_{\beta,\alpha}(0) \notin \{0\} \cup [1/\beta, 1[$ for all integers $N \geq 1$.*

Proof. The 'only if' parts follows from Lemma 1 and Theorem 3.4 of [5], which states that $f^N(0) \geq 1/\beta$ implies that $S_{\beta,\alpha}$ is a finite set. To prove the "if" part, we assume that

$$0 < f^N(0) < 1/\beta \quad \text{for all integers } N \geq 1, \tag{3}$$

and we show by contradiction that the $f^k(0)$'s, $k \geq 1$, are distinct. To this end, assume that $1 \leq k < \ell$ are minimal with $f^k(0) = f^\ell(0)$. Then there is an integer j with $0 \leq j \leq [\beta]$ such that

$$f^{k-1}(0) = f^{\ell-1}(0) + \frac{j}{\beta}.$$

By (3), we must have $j = 0$, which, by minimality of k, yields $k = 1$. It follows that $f^{\ell-1}(0) = 0$, a contradiction with (3). □

Lemma 3. *Let $\beta > 1$ and put $\gamma = 1/\beta$. Set $J^1_1(\gamma) = [\gamma, 1[$ and, for coprime integers $b > a \geq 1$,*

$$J^a_b(\gamma) = \left[\frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(a/b)\gamma^k + \gamma^b}{1 + \gamma + \ldots + \gamma^{b-1}}, \frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(a/b)\gamma^k + \gamma^{b-1}}{1 + \gamma + \ldots + \gamma^{b-1}}\right].$$

Then, the intervals $J^a_b(\gamma)$ are disjoint for different choices of coprime integers $b > a \geq 1$. Further, the set $S_{\beta,\alpha}$ is finite if, and only if, there exist coprime integers $b \geq a \geq 1$ such that $\alpha \in J^a_b(\gamma)$. Moreover, $S_{\beta,\alpha}$ is empty if, and only if, α is the left endpoint of some $J^a_b(\gamma)$, and otherwise $S_{\beta,\alpha}$ has exactly b elements, which are cyclically permuted under the action of f, if α is in $J^a_b(\gamma)$, but is not its left endpoint. Finally, $S_{\beta,\alpha}$ is infinite if, and only if, there exists some irrational number τ in $]0, 1[$ such that

$$\alpha = (1 - \gamma) \sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\gamma^k.$$

The proof of Lemma 3 depends heavily on results obtained in [2] (see also [3]) concerning the function

$$T_{\gamma,\alpha} : x \to \{\gamma x + \alpha\},$$

where $0 < \gamma \leq 1$ and $0 \leq \alpha < 1$ are real numbers such that $\gamma + \alpha > 1$. The map $T_{\gamma,\alpha}$ is piecewise linear, contracting and is continuous except on the left at $\theta := (1 - \alpha)/\gamma$. For $n \geq 1$, we put

$$T^n_{\gamma,\alpha}(1) = T^{n-1}_{\gamma,\alpha}(\gamma + \alpha - 1) = T^{n-1}_{\gamma,\alpha}\big(\lim_{x \to 1_-} T_{\gamma,\alpha}(x)\big). \tag{4}$$

We have obtained in [2] a precise description of the dynamic of $T_{\gamma,\alpha}$.

Proposition 1. *Let α and γ be real numbers with $0 < \gamma < 1$ and $0 \leq \alpha < 1$. Let a and b be coprime integers with $b \geq a \geq 1$ and define the interval $I_b^a(\gamma)$ by*

$$I_b^a(\gamma) = \left[\frac{P_b^a(\gamma)}{1+\gamma+...+\gamma^{b-1}}, \frac{P_b^a(\gamma)+\gamma^{b-1}-\gamma^b}{1+\gamma+...+\gamma^{b-1}}\right],$$

where P_b^a is the polynomial

$$P_b^a(\gamma) = \sum_{k=0}^{b-1} \epsilon_{-k}(a/b)\gamma^k.$$

Then the map $T_{\gamma,\alpha}$ has an attractive, periodic orbit with the same dynamic as the rotation $T_{1,a/b}$ if, and only if, $\alpha \in I_b^a(\gamma)$.

Proof. This is Théorème 1.1 of [3]. Observe that, in the case $b = a = 1$ of the proposition, the attractive orbit of $T_{\gamma,\alpha}$ is equal to $\{0\}$ when $\gamma+\alpha \leq 1$. □

Remark 1. Let γ be a real number with $0 < \gamma < 1$. It follows from Proposition 1 that the intervals $I_b^a(\gamma)$ are disjoint for different choices of a, b. Indeed, for distinct rational numbers a/b and a'/b', the rotations $T_{1,a/b}$ and $T_{1,a'/b'}$ have different dynamics.

Remark 2. In [2] and [3], we gave two different proofs of Theorem 1: one dynamical (see [2]) and one algebraic (see [3]). The dynamical proof rests on the study of the position of the critical point $\theta := (1-\alpha)/\gamma$ of $T_{\gamma,\alpha}$, which lies in $[0, 1[$, since $\gamma + \alpha > 1$. We assumed that $\theta \notin T_{\gamma,\alpha}^n([0, 1[)$ for some integer $n \geq 1$, and we set

$$b := \inf\{n \mid \theta \notin T_{\gamma,\alpha}^n([0, 1[)\} + 1.$$

Since θ is the only discontinuity of $T_{\gamma,\alpha}$, it is easy to see that for any $n \geq b$ the set $T_{\gamma,\alpha}^n([0, 1[)$ is the union of b disjoint intervals, whose lengths tend to zero when k goes to infinity. To give a more precise result, it is convenient to introduce some notation.

Notation. *For any real numbers $x < y$ in $[0, 1]$, we write $\langle x, y\rangle$ for the closed interval $[x, y]$ if $y < 1$ and $x > 0$, we set $\langle x, 1\rangle$ for $\{0\} \cup [x, 1[$ and $\langle 0, x\rangle$ for $[0, x]$. Moreover, for any increasing function f on $]x, 1[$, we write $f(\langle x, 1\rangle)$ for $\langle f(x), f(1)\rangle$.*

With these notations, we have

$$T_{\gamma,\alpha}^{b-1}([0, 1[) = [0, 1[\setminus \bigcup_{k=1}^{b-1} \langle T_{\gamma,\alpha}^k(1), T_{\gamma,\alpha}^k(0)\rangle$$

and $\theta \in \langle T_{\gamma,\alpha}^{b-1}(1), T_{\gamma,\alpha}^{b-1}(0)\rangle$. As shown in [2], the critical point θ is in $\langle T_{\gamma,\alpha}^{b-1}(1), T_{\gamma,\alpha}^{b-1}(0)\rangle$ if, and only if, there exists a positive integer $a < b$, coprime with b, such that α is in $I_b^a(\gamma)$.

We have now all the tools to prove Lemma 3.

Proof of Lemma 3. Let $\beta > 1$ and $0 \le \alpha < 1$. Put $\gamma = 1/\beta$. We readily verify that the conclusion of the lemma holds when α is in $J_1^1(\gamma)$, by Lemma 2. Thus, we assume now that $0 \le \alpha < \gamma$. We recall that f stands for $f_{\beta,\alpha}$. We observe that f is a bijection from $[0, 1/\beta[$ onto $[0, 1[$, and we denote by $g := g_{\beta,\alpha}$ the inverse of this restriction, i.e. for $x \in [0, 1[$,

$$g_{\beta,\alpha}(x) = \begin{cases} \frac{1}{\beta}x + \frac{1-\alpha}{\beta}, & 0 \le x < \alpha, \\ \\ \frac{1}{\beta}x - \frac{\alpha}{\beta}, & \alpha \le x < 1. \end{cases} \tag{5}$$

Thus g is piecewise linear, contracting and continuous, except on the left at α.

The maps $T_{\gamma,1-\alpha}$ and $g_{1/\gamma,\alpha}$ are closely related. Namely, for any real number x in $[0, 1[$, we have

$$\{T_{\gamma,1-\alpha}(x) + \alpha\} = g_{1/\gamma,\alpha}\big(\{x + \alpha\}\big) = \gamma x. \tag{6}$$

Assume now that the set $S_{\beta,\alpha}$ is finite. In view of Lemma 2, there exists a positive integer N such that $f_{\beta,\alpha}^N(0) \in \langle 1/\beta, 1\rangle$. Denote by b the smallest positive integer with this property. Then we have $0 \in g^b(\langle\gamma, 1\rangle)$, or, equivalently, $\alpha \in g^{b-1}(\langle\gamma, 1\rangle)$. Since $\alpha < \gamma$, we have $b \ge 2$. According to Lemma 3.1 of [5], the sets $g^k(\langle\gamma, 1\rangle)$ for $0 \le k \le b-1$ are nonempty intervals. It follows from (6) and (4) that

$$\begin{aligned} g^{b-1}(\langle\gamma, 1\rangle) &= \langle T_{\gamma,1-\alpha}^{b-1}(\gamma - \alpha) + \alpha, T_{\gamma,1-\alpha}^{b-1}(1 - \alpha) + \alpha\rangle \\ &= \langle T_{\gamma,1-\alpha}^{b}(1) + \alpha, T_{\gamma,1-\alpha}^{b}(0) + \alpha\rangle. \end{aligned}$$

Consequently, α belongs to $g^{b-1}(\langle\gamma, 1\rangle)$ if, and only if, 0 is in $\langle T_{\gamma,1-\alpha}^b(1) + \alpha, T_{\gamma,1-\alpha}^b(0) + \alpha\rangle$, which, since $\alpha < \gamma$, is equivalent to say that α/γ belongs to $\langle T_{\gamma,1-\alpha}^{b-1}(1), T_{\gamma,1-\alpha}^{b-1}(0)\rangle$. Setting $u := 1 - \alpha$, we have shown that

$$\alpha \in g^{b-1}(\langle\gamma, 1\rangle)$$

if, and only if,

$$\frac{1-u}{\gamma} \in \langle T_{\gamma,u}^{b-1}(1), T_{\gamma,u}^{b-1}(0)\rangle. \tag{7}$$

Notice that the condition $\alpha < \gamma$ is equivalent to $\gamma + u > 1$. According to the remark following Proposition 1, we know that (7) holds if, and only if, there exists a positive integer $a < b$, coprime with b, such that

$$u \in I_b^a(\gamma). \tag{8}$$

As $u = 1 - \alpha$, Proposition 1 implies that (8) can be rewritten as

$$\frac{\sum_{k=0}^{b-1}(1 - \varepsilon_{-k}(a/b))\gamma^k + \gamma^b - \gamma^{b-1}}{1 + \gamma + ... + \gamma^{b-1}} \le \alpha \le \frac{\sum_{k=0}^{b-1}(1 - \varepsilon_{-k}(a/b))\gamma^k}{1 + \gamma + ... + \gamma^{b-1}}. \tag{9}$$

Since $\varepsilon_k(1 - a/b) = 1 - \varepsilon_k(a/b)$ for any integer k not multiple of b and not congruent to one modulo b (to see this, it suffices to note that $[-ja/b] = -[ja/b] - 1$ if b does not divide j), (9) becomes

$$\frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(1 - a/b)\gamma^k + \gamma^b}{1 + \gamma + ... + \gamma^{b-1}} \le \alpha \le \frac{\sum_{k=1}^{b-1} \varepsilon_{-k}(1 - a/b)\gamma^k + \gamma^{b-1}}{1 + \gamma + ... + \gamma^{b-1}},$$

which proves that the set $S_{\beta,\alpha}$ is finite if, and only if, there exist coprime integers $b \ge a \ge 1$ such that $\alpha \in J_b^a(\gamma)$.

Further, for α in $J_b^a(\gamma)$, a direct calculation shows that $S_{\beta,\alpha}$ is empty if α is the left endpoint of $J_b^a(\gamma)$ and has exactly b elements otherwise.

Moreover, we infer from Lemma 2 and (8) that $S_{\beta,\alpha}$ is infinite if, and only if,

$$1 - \alpha \in [0, 1[\setminus \bigcup_{\substack{1 \le a \le b \\ (a,b)=1}} I_b^a(\gamma),$$

that is (see [2, Théorème 2](*) or [3, page 207]) if, and only if, there exists some irrational number τ in $]0, 1[$ such that

$$1 - \alpha = (1 - \gamma) \sum_{k=0}^{\infty} \varepsilon_{-k}(\tau)\gamma^k,$$

and the last assertion of the lemma follows since $\varepsilon_0(\tau) = 1$ and $\varepsilon_{-k}(\tau) = 1 - \varepsilon_{-k}(1 - \tau)$ for any integer $k \ge 1$.

Finally, the fact that the intervals $J_b^a(\gamma)$ are disjoint follows from (8), (9), and Remark 1. □

Theorem 3. *For any real number $\beta > 1$, the set*

$$E_\beta := \{\alpha \in [0, 1[: S_{\beta,\alpha} \text{ is an infinite set}\} = [0, 1[\setminus \bigcup_{\substack{1 \le a \le b \\ (a,b)=1}} J_b^a(\gamma)$$

(*) There is a misprint in the statement of [2, Théorème 2]: one should read $(S(\alpha))_{-k}$ instead of $(S(\alpha))_k$.

has measure zero, is uncountable and is not closed.

Proof. Denote by μ the Lebesgue measure. It follows from Lemma 3 that

$$\mu(E_\beta) = 1 - \sum_{b=1}^{\infty} \frac{\varphi(b)(\gamma^{b-1} - \gamma^b)}{1 + \gamma + \ldots + \gamma^{b-1}},$$

where φ is Euler totient function, i.e. $\varphi(b)$ counts the number of integers a, $1 \le a \le b$, which are coprime with b. Since

$$\frac{\gamma^{b-1} - \gamma^b}{1 + \ldots + \gamma^{b-1}} = (1-\gamma)^2 \, \frac{\gamma^{b-1}}{1-\gamma^b}, \quad \text{for } b \ge 1,$$

and

$$\sum_{b=1}^{\infty} \varphi(b) \, \frac{\gamma^{b-1}}{1-\gamma^b} = \frac{1}{(1-\gamma)^2},$$

we infer from Theorem 309 of [6] that $\mu(E_\beta) = 0$.

Further, the last assertion of Lemma 3 combined with Théorème 2 of [2] implies that E_β is uncountable. Moreover, if the irrational number τ tends to a rational number, then $(1-\gamma)\sum_{k=1}^{\infty} \varepsilon_{-k}(\tau)\gamma^k$ tends to an endpoint of some interval $J_b^a(\gamma)$. Hence, E_β is not closed. □

Remark 3. An interesting open problem would be to determine the Hausdorff dimension of the sets E_β.

To complete the proof of Theorem 1, we recall a crucial result of [5].

Proposition 2. *Let $p > q \ge 2$ be coprime integers, and let s in $[0, 1-1/p]$. If the set $S_{p/q,\{(p-q)s\}}$ is finite, then $Z_{p/q}(s, s+1/p)$ is empty.*

Proof. This is Theorem 3.2 of [5]. □

Proof of Theorem 1. This statement easily follows from Lemma 3 and Theorem 3, combined with Proposition 2. □

4. Proof of Theorem 2

We now investigate the behaviour of $f := f_{\beta,\alpha}$ when α is in E_β. Recall that $g_{\beta,\alpha}$ is defined in (5). The following lemmas answer a question posed by Flatto *et al.* at the end of [5].

Lemma 4. *Let $\beta > 1$ and $\alpha \in E_\beta$. For $n \ge 0$, put*

$$I_n := g_{\beta,\alpha}^n([1/\beta, 1[).$$

Then we have

$$S_{\beta,\alpha} := [0,1[\setminus \bigcup_{n\geq 0} I_n.$$

In particular, $S_{\beta,\alpha}$ *has measure zero.*

Proof. Arguing as in Lemma 3.1 of [5], we use that $f^n(0) \notin [1/\beta, 1[$ for all $n \geq 0$ to deduce that the I_n's, $n \geq 0$, are mutually disjoints. If $x \in [0,1[$ is in some I_n with $n \geq 0$, we get that $f^n(x) \in [1/\beta, 1[$, whence $x \notin S_{\beta,\alpha}$. Otherwise, it is clear that $x \in S_{\beta,\alpha}$. Further,

$$\mu\Big(\bigcup_{n\geq 0} I_n\Big) = \left(1 - \frac{1}{\beta}\right) \sum_{n\geq 0} \frac{1}{\beta^n} = 0,$$

as claimed. □

As in [5], we associate to $f = f_{\beta,\alpha}$ the natural symbolic dynamics, which assigns to each x in $[0,1[$ the integer

$$S_f(x) = [\beta x + \alpha],$$

and we call the sequence

$$a_n := S_f(f^n(x)), \;\; n \geq 0,$$

the f-expansion of x. If x is in $S_{\beta,\alpha}$, then $0 \leq f^n_{\beta,\alpha}(x) < 1/\beta$ for all $n \geq 0$, and its f-expansion is uniquely composed of 0's and 1's.

Lemma 5. *Let* $\beta > 1$ *and* $\alpha \in E_\beta$*. Let* τ *in* $]0,1[$ *be defined by*

$$\alpha = \left(1 - \frac{1}{\beta}\right) \sum_{k=1}^{\infty} \frac{\varepsilon_{-k}(\tau)}{\beta^k}.$$

The set $S_{\beta,\alpha}$ *is uncountable, not closed and, for any* x *in* $S_{\beta,\alpha}$*, there exists* $0 \leq \eta < 1$ *such that the* f*-expansion of* x *is the Sturmian sequence* $(a_n)_{n\geq 0}$ *given by*

$$a_n = [(n+1)\tau + \eta] - [n\tau + \eta].$$

Proof. We point out that τ is irrational. We first show that g and the (irrational) rotation

$$R_{1-\tau} : x \mapsto \{x + 1 - \tau\}, \quad x \in [0,1[,$$

are semi-conjugate.

We claim that the intervals I_n, $n \geq 0$, are ordered as the sequence $\big(\{n(1-\tau)\}\big)_{n\geq 0}$. To see this, for any $\beta' > 1$, we define

$$\Psi_{\beta'}(\tau) = \left(1 - \frac{1}{\beta'}\right) \sum_{k=1}^{\infty} \frac{\varepsilon_{-k}(\tau)}{\beta'^k},$$

and we observe that, for any $n \geq 0$, the function

$$\beta' \longmapsto \inf\big(g^n_{\beta', \Psi_{\beta'}(\tau)}([1/\beta', 1[)\big)$$

is continuous on $]1, +\infty[$ and tends to $\{n(1-\tau)\}$ when β' tends to 1.

We set $\Phi(0) = 0$ and, for $n \geq 1$, $\Phi(I_n) = \{n(1-\tau)\}$. The map Φ is monotone and, by Lemma 4, is defined on a dense subset of $[0, 1[$, thus, we can extend it by continuity to $[0, 1[$. Consequently, the set $S_{\beta,\alpha}$ is uncountable and not closed. For all $y \in [0, 1[$, we have

$$\Phi \circ g_{\beta,\alpha}(y) = R_{1-\tau} \circ \Phi(y),$$

hence,

$$R_\tau \circ \Phi(z) = \Phi \circ f_{\beta,\alpha}(z), \tag{10}$$

for $z \in [0, 1/\beta]$. Since $\Phi(0) = 0$ and $f\big((1-\alpha)/\beta\big) = 0$, we get from (10) that $\Phi\big((1-\alpha)/\beta\big) = 1 - \tau$. It follows that

$$0 \leq z < \frac{1-\alpha}{\beta} \quad \text{if and only if} \quad 0 \leq \Phi(z) < 1 - \tau.$$

By induction, (10) yields for any integer $n \geq 1$ that

$$0 \leq f^n(z) < \frac{1-\alpha}{\beta} \quad \text{if and only if} \quad 0 \leq R^n_\tau\big(\Phi(z)\big) < 1 - \tau. \tag{11}$$

Let $x \in S_{\beta,\alpha}$ and denote by $(a_n)_{n\geq 0}$ its f-expansion. It follows from (11) that $a_n = 0$ if, and only if, $0 \leq R^n_\tau\big(\Phi(z)\big) < 1 - \tau$. Hence, we get that

$$a_n = [(n+1)\tau + \Phi(z)] - [n\tau + \Phi(z)],$$

and the proof of Lemma 5 is complete. □

We need to recall an important result of [5]. For the definition of T-expansion, we refer the reader to [5].

Proposition 3. *Let $p > q \geq 2$ be coprime integers. Then a positive real number ξ is in $Z_{p/q}(s, s+1/p)$ if, and only if, both conditions $(C1)$ and $(C2)$ hold, with*

$$(C1) \qquad 0 \leq f^n\big(q(\{\xi\} - s)\big) < q/p \ \text{ for all } n \geq 0,$$

and $(C2)$: the T-expansion (a_n) of $[\xi]$ and the f-expansion (b_n) of $q(\{\xi\}-s)$ are related by

$$\sigma(a_n) = b_n \ \text{ for all } n \geq 0,$$

where σ is the permutation of $\{0, 1, \ldots, q-1\}$ given by

$$\sigma(i) \equiv -pi - [(p-q)s] \ (\text{mod } q).$$

Further, the set $Z_{p/q}(s, s+1/p)$ contains at most one element in each unit interval $[m, m+1[$, where m is a non-negative integer.

Proof. This follows from Proposition 2.1 and Theorem 1.1 of [5]. □

Proof of Theorem 2. Let ξ be in $Z_{p/q}(s, s+1/p)$. By Proposition 3 and Lemma 5, the T-expansion of $[\xi]$ is an infinite Sturmian word. It has been shown by Mignosi [9] (see [1] for an alternative proof) that, for any integer $m \geq 1$, there are $O(m^3)$ Sturmian words of length m (recall that any subword of a Sturmian sequence is called a Sturmian word). Since Lemma 2.2 of [5] asserts that the first m terms of the T-expansion of an integer g are uniquely determined by g modulo q^m, we conclude that at most $O(m^3)$ integers less than q^m may have a Sturmian T-expansion, and Theorem 2 is proved. □

References

[1] J. Berstel and M. Pocchiola, A geometric proof of the enumeration formula for sturmian words, Int. J. Algebra and Computation 3 (1993), 349–355.

[2] Y. Bugeaud, *Dynamique de certaines applications contractantes linéaires par morceaux sur* [0, 1[, C. R. Acad. Sci. Paris Sér. I 317 (1993), 575–578.

[3] Y. Bugeaud et J.-P. Conze, *Calcul de la dynamique de transformations linéaires contractantes mod 1 et arbre de Farey*, Acta Arith. 88 (1999), 201–218.

[4] L. Flatto, *Z-numbers and β-transformations*, in: Symbolic Dynamics and Its Applications, P. Walters (ed.), Contemp. Math. 135 (1992), Amer. Math. Soc., Providence, R. I., 181–202.

[5] L. Flatto, J. C. Lagarias and A. D. Pollington, *On the range of fractional parts* $\{\xi(p/q)^n\}$, Acta Arith. 70 (1995), 125–147.

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, fifth ed., Clarendon Press, 1979.

[7] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley–Interscience, 1974.

[8] K. Mahler, *An unsolved problem on the powers of* 3/2, J. Austral. Math. Soc. 8 (1968), 313–321.

[9] F. Mignosi, *On the number of factors of Sturmian words*, Theoret. Comput. Sci. 82 (1991), 71–84.

[10] T. Vijayaraghavan, *On the fractional parts of the powers of a number, I*, J. London Math. Soc. 15 (1940), 159–160.

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