Linear mod one transformations
and the distribution of fractional parts \{\xi(p/q)^n\}

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Abstract. For coprime integers \(p > q \geq 2\) and for a
positive real number \(s < 1 - 1/p\), let \(Z_{p/q}(s, s + 1/p)\)
be the set of real numbers \(\xi\) such that \(s \leq \{\xi(p/q)^n\} \leq s + 1/p\) holds for all non-negative integers \(n\). Here, \(\{\cdot\}\)
denotes the fractional part. Flatto, Lagarias & Pollington
showed that the set of \(s\) in \([0, 1 - 1/p]\) for which
\(Z_{p/q}(s, s+1/p)\) is empty is a dense set, and they deduced
that \(\limsup_{n \to \infty} \{\xi(p/q)^n\} - \liminf_{n \to \infty} \{\xi(p/q)^n\} \geq 1/p\) holds for all positive real numbers \(\xi\). In the present
work, we give further results on the sets \(Z_{p/q}(s, s+1/p)\).
For instance, we prove that they are empty for almost
all \(s\) in \([0, 1 - 1/p]\).

1. Introduction

It is well known (see e.g. [7], Chapter 1, Corollary 4.2) that for almost
all real numbers \(\theta \geq 1\) the sequence \(\{\theta^n\}\) is uniformly distributed in \([0, 1]\). Here and in the sequel, \(\{\cdot\}\) denotes the fractional part. However, very few
results are known for specific values of \(\theta\), and the distribution of \(\{(p/q)^n\}\)
for coprime positive integers \(p > q \geq 2\) remains an unsolved problem.
Vijayaraghavan [10] showed that this sequence has infinitely many limit
points, but we are unable to decide whether

\[
\limsup_{n \to \infty} \left\{ \left( \frac{p}{q} \right)^n \right\} - \liminf_{n \to \infty} \left\{ \left( \frac{p}{q} \right)^n \right\} > \frac{1}{2}.
\]

A striking progress has been recently made by Flatto, Lagarias & Polling-

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ton [5], who proved that, for all positive real numbers $\xi$, we have

$$
\limsup_{n \to \infty} \left\{ \xi \left( \frac{p}{q} \right)^n \right\} - \liminf_{n \to \infty} \left\{ \xi \left( \frac{p}{q} \right)^n \right\} \geq \frac{1}{p}.
$$

They were inspired by a paper of Mahler [8], who studied the hypothetical existence of so-called $Z$-numbers, i.e. positive real numbers $\xi$ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for all integers $n \geq 0$. Extending this definition, Flatto et al. introduce, for an interval $[s, s+t]$ included in $[0,1]$, the set

$$
Z_{p/q}(s, s+t) := \left\{ \xi \in \mathbb{R} : s \leq \left\{ \xi \left( \frac{p}{q} \right)^n \right\} < s+t \text{ for all } n \geq 0 \right\}.
$$

To prove (1), they show that the set of $s$ such that $Z_{p/q}(s, s+1/p)$ is empty is dense in $[0,1-1/p]$. Their argument uses Mahler’s method, as explained in a preliminary work by Flatto [4] (for more bibliographical references, we refer the reader to [4] and [5]), and also relies on a careful study of contracting, linear transformations, which are very close to those investigated in [2] and [3].

The purpose of the present work is to show how the methods used in [2] and [3] apply to the transformations considered in [5], and to derive some interesting consequences. For instance, we prove that $Z_{p/q}(s, s+1/p)$ is empty for almost all (in the sense of Lebesgue measure) real numbers $s$ in $[0,1-1/p]$ and we answer both questions posed at the end of [5].

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### 2. Statement of the results

Before stating our results, we have to introduce some notation, which will be used throughout the present paper. Let $\tau$ be a real number with $0 \leq \tau < 1$. For any integer $k$, set

$$
\varepsilon_k(\tau) = \lfloor k \tau \rfloor - \lfloor (k-1) \tau \rfloor,
$$

where $[\cdot]$ denotes the integer part. The sequence $(\varepsilon_k(\tau))_{k \in \mathbb{Z}}$ only takes values 0 and 1 and, for $\tau$ irrational, it is usually called the characteristic Sturmian sequence associated to $\tau$. For any non-zero rational $a/b$, with $a$ and $b$ coprime, the sequence $(\varepsilon_k(a/b))_{k \in \mathbb{Z}}$ is periodic with period $b$.

Our first result concerns the sets $Z_{p/q}(s, s+1/p)$ and complements a result of Flatto et al. who proved in [5] that the set of $s$ in $[0,1-1/p]$ for which $Z_{p/q}(s, s+1/p)$ is empty is a dense set.
Theorem 1. Let \( p > q \geq 2 \) be coprime integers. Then the set \( Z_{p/q}(s, s + 1/p) \) is empty for a set of \( s \) of full Lebesgue measure in \([0, 1 - 1/p]\). More precisely, this set is empty when there exists a rational number \( a/b \), with \( b > a \geq 1 \), such that

\[
\sum_{k=1}^{b-2} \varepsilon_k(a/b) \left( \frac{q}{p} \right)^k + \left( \frac{q}{p} \right)^b \leq (p-q)s \leq \sum_{k=1}^{b-2} \varepsilon_k(a/b) \left( \frac{q}{p} \right)^k + \left( \frac{q}{p} \right)^{b-1}.
\]

Further, if for some real number \( s \) in \([0, 1 - 1/p]\) the set \( Z_{p/q}(s, s + 1/p) \) is nonempty, then there exists an irrational number \( \tau \) in \([0, 1]\) such that

\[
(p - q)s = \frac{p - q}{p} \sum_{k=1}^{\infty} \varepsilon_k(\tau) \left( \frac{q}{p} \right)^k.
\]  
(2)

The proof of Theorem 1 is given in Section 3 and relies upon the main result of [2]. It also allows us to answer (in Theorem 3) a problem posed by Flatto et al. at the end of [5].

As an immediate application, we considerably improve Corollary 1.4a of [5].

Corollary 1. The set \( Z_{3/2}(s, s + 1/3) \) is empty if

\[
s \in \{0\} \cup [8/57, 4/19] \cup [4/15, 2/5] \cup [26/57, 10/19] \cup \{2/3\}.
\]

To prove Corollary 1, we check that the three intervals are given by Theorem 1 applied with the rationals 1/3, 1/2 and 2/3. Further, since for any irrational \( \tau \) in \([0, 1]\) there exists \( k \geq 1 \) and \( \ell \geq 1 \) such that \( \varepsilon_k(\tau) = 1 \) and \( \varepsilon_{-\ell}(\tau) = 0 \), we get that \( \frac{1}{\ell} \left( \sum_{k=1}^{\infty} \varepsilon_k(\tau)(2/3)^k \right) \neq 0, 2/3 \), hence, by the last part of Theorem 1, the sets \( Z_{3/2}(0, 1/3) \) and \( Z_{3/2}(2/3, 1) \) are empty (a fact already proved in [5]).

We are not able to determine whether \( Z_{p/q}(s, s + 1/p) \) is empty for all values of \( s \). However, we obtain some additional informations concerning the sets \( Z_{p/q}(s, s + 1/p) \) for exceptional values of \( s \).

Theorem 2. Let \( p > q \geq 2 \) be coprime integers. Let \( s \) in \([0, 1 - 1/p]\) satisfy (2) for some irrational \( \tau \). Then we have

\[
\text{Card}\{\xi : 0 \leq \xi \leq x \text{ and } \xi \in Z_{p/q}(s, s + 1/p)\} = O((\log_q x)^3).
\]

Theorem 2 considerably improves Theorem 1.1 of [5] for \( t = 1/p \), where the estimate \( O(x^\gamma) \) is obtained, with \( \gamma = \log_q \min\{2, p/q\} \). Its
proof is given in Section 4, where we get strong conditions on $[\xi]$ for $\xi$ belonging to $Z_{p/q}(s, s + 1/p)$.

3. Proof of Theorem 1

In all what follows, for a map $F$ and an integer $n \geq 0$, we denote by $F^n$ the map $F \circ \ldots \circ F$, composed $n$ times.

Keeping the notation of Part 3 of [5], we define for any real numbers $\beta > 1$ and $0 \leq \alpha < 1$ the map $f_{\beta, \alpha}$ by

$$f_{\beta, \alpha}(x) = \{\beta x + \alpha\} \text{ for } x \in [0, 1].$$

We set

$$S_{\beta, \alpha} := \{x \in [0, 1]: 0 \leq f^n_{\beta, \alpha}(x) < 1/\beta \text{ for all } n \geq 0\}.$$ 

Theorem 3.4 of [5] asserts that $S_{\beta, \alpha}$ is finite as soon as $0 \notin S_{\beta, \alpha}$, which is the case for a dense set of values of $\alpha$ in $[0, 1]$ (see Theorem 3.5 of [5]). The problem of the existence of values of $\alpha$ such that $S_{\beta, \alpha}$ is infinite is left open in [5]. Indeed, setting

$$E_\beta := \{\alpha \in [0, 1]: S_{\beta, \alpha} \text{ is an infinite set}\},$$

Flatto et al. conjecture that, for all $\beta > 1$, the set $E_\beta$ is non empty and perfect, and has Lebesgue measure zero. The present section is concerned with the study of this problem, which we solve in Theorem 3 below.

In the rest of this section, $f$ stands for $f_{\beta, \alpha}$.

**Lemma 1.** Assume that there exists an integer $N \geq 1$ with $f^N(0) = 0$ and such that $f^k(0) \notin \{0\} \cup [1/\beta, 1]$ for any integer $1 \leq k \leq N - 1$. Then $S_{\beta, \alpha}$ is the finite set $\{0, f(0), \ldots, f^{N-1}(0)\}$.

**Proof.** We use the notation of Lemmas 3.1, 3.2 and 3.3 of [5], and we only point out which slight changes should be made in their proofs to get our lemma. A continuity argument shows that $\alpha$ is the right endpoint of the interval $I_{N-1} = [\cdot, \alpha]$. It follows that $R_{-1}$ is empty, whence all the $R_k$’s, $k \geq 0$, are empty. Arguing as in [5], we denote by $p_n$ the left endpoint of $L_n$ for $n \geq 0$, and we set

$$q_k := \lim_{j \to \infty} p_{k + jN}, \quad 0 \leq k \leq N - 1.$$ 

Each $q_k$ coincides with a right endpoint of some $I_j$, $0 \leq j \leq N - 1$. It follows that $S_{\beta, \alpha} = \{q_0, \ldots, q_{N-1}\} = \{0, f(0), \ldots, f^{N-1}(0)\}$, as claimed.

\[ \square \]
Lemma 2. Let $\beta > 1$. Then, for each $\alpha$ in $[0, 1[$, the set $S_{\beta, \alpha}$ is infinite if, and only if, $f^N_{\beta, \alpha}(0) \notin \{0\} \cup [1/\beta, 1[$ for all integers $N \geq 1$.

Proof. The ‘only if’ parts follows from Lemma 1 and Theorem 3.4 of [5], which states that $f^N(0) \geq 1/\beta$ implies that $S_{\beta, \alpha}$ is a finite set. To prove the “if” part, we assume that

$$0 < f^N(0) < 1/\beta \quad \text{for all integers } N \geq 1,$$

and we show by contradiction that the $f^k(0)$’s, $k \geq 1$, are distinct. To this end, assume that $1 \leq k < \ell$ are minimal with $f^k(0) = f^\ell(0)$. Then there is an integer $j$ with $0 \leq j \leq \lfloor \beta \rfloor$ such that

$$f^{k-1}(0) = f^{\ell-1}(0) + \frac{j}{\beta}.$$

By (3), we must have $j = 0$, which, by minimality of $k$, yields $k = 1$. It follows that $f^{\ell-1}(0) = 0$, a contradiction with (3).

Lemma 3. Let $\beta > 1$ and put $\gamma = 1/\beta$. Set $J^1_{\beta}(\gamma) = [\gamma, 1[$ and, for coprime integers $b > a \geq 1$,

$$J^a_{b}(\gamma) = \left[ \frac{\sum_{k=1}^{b-1} \varepsilon_k(a/b) \gamma^k + \gamma^b}{1 + \gamma + \cdots + \gamma^{b-1}}, \frac{\sum_{k=1}^{b-1} \varepsilon_k(a/b) \gamma^k + \gamma^{b-1}}{1 + \gamma + \cdots + \gamma^{b-1}} \right].$$

Then, the intervals $J^a_{b}(\gamma)$ are disjoint for different choices of coprime integers $b > a \geq 1$. Further, the set $S_{\beta, \alpha}$ is finite if, and only if, there exist coprime integers $b > a \geq 1$ such that $\alpha \in J^a_{b}(\gamma)$. Moreover, $S_{\beta, \alpha}$ is empty if, and only if, $\alpha$ is the left endpoint of some $J^a_{b}(\gamma)$, and otherwise $S_{\beta, \alpha}$ has exactly $b$ elements, which are cyclically permuted under the action of $f$, if $\alpha$ is in $J^a_{b}(\gamma)$, but is not its left endpoint. Finally, $S_{\beta, \alpha}$ is infinite if, and only if, there exists some irrational number $\tau$ in $[0, 1[$ such that

$$\alpha = (1 - \gamma) \sum_{k=1}^{\infty} \varepsilon_k(\tau) \gamma^k.$$

The proof of Lemma 3 depends heavily on results obtained in [2] (see also [3]) concerning the function

$$T_{\gamma, \alpha} : x \rightarrow \{\gamma x + \alpha\},$$

where $0 < \gamma \leq 1$ and $0 \leq \alpha < 1$ are real numbers such that $\gamma + \alpha > 1$. The map $T_{\gamma, \alpha}$ is piecewise linear, contracting and is continuous except on the left at $\theta := (1 - \alpha)/\gamma$. For $n \geq 1$, we put

$$T^n_{\gamma, \alpha}(1) = T^{n-1}_{\gamma, \alpha}(\gamma + \alpha - 1) = T^{n-1}_{\gamma, \alpha} \left( \lim_{x \to 1-} T_{\gamma, \alpha}(x) \right).$$

We have obtained in [2] a precise description of the dynamic of $T_{\gamma, \alpha}$.
\textbf{Proposition 1.} Let $\alpha$ and $\gamma$ be real numbers with $0 < \gamma < 1$ and $0 \leq \alpha < 1$. Let $a$ and $b$ be coprime integers with $b \geq a \geq 1$ and define the interval $I_b^a(\gamma)$ by

$$I_b^a(\gamma) = \left[ \frac{P_b^a(\gamma)}{1 + \gamma + \ldots + \gamma^{b-1}}, \frac{P_b^a(\gamma) + \gamma^{b-1} - \gamma^b}{1 + \gamma + \ldots + \gamma^{b-1}} \right],$$

where $P_b^a$ is the polynomial

$$P_b^a(\gamma) = \sum_{k=0}^{b-1} \epsilon^{-k}(a/b)\gamma^k.$$

Then the map $T_{\gamma,\alpha}$ has an attractive, periodic orbit with the same dynamics as the rotation $T_{1,a/b}$ if, and only if, $\alpha \in I_b^a(\gamma)$.

\textbf{Proof.} This is Théorème 1.1 of [3]. Observe that, in the case $b = a = 1$ of the proposition, the attractive orbit of $T_{\gamma,\alpha}$ is equal to \{0\} when $\gamma + \alpha \leq 1$. \hfill \Box

\textbf{Remark 1.} Let $\gamma$ be a real number with $0 < \gamma < 1$. It follows from Proposition 1 that the intervals $I_b^a(\gamma)$ are disjoint for different choices of $a$, $b$. Indeed, for distinct rational numbers $a/b$ and $a'/b'$, the rotations $T_{1,a/b}$ and $T_{1,a'/b'}$ have different dynamics.

\textbf{Remark 2.} In [2] and [3], we gave two different proofs of Theorem 1: one dynamical (see [2]) and one algebraic (see [3]). The dynamical proof rests on the study of the position of the critical point $\theta := (1 - \alpha)/\gamma$ of $T_{\gamma,\alpha}$, which lies in $[0, 1]$, since $\gamma + \alpha > 1$. We assumed that $\theta \notin T_{\gamma,\alpha}^n([0, 1])$ for some integer $n \geq 1$, and we set

$$b := \inf\{n \mid \theta \notin T_{\gamma,\alpha}^n([0, 1])\} + 1.$$

Since $\theta$ is the only discontinuity of $T_{\gamma,\alpha}$, it is easy to see that for any $n \geq b$ the set $T_{\gamma,\alpha}^n([0, 1])$ is the union of $b$ disjoint intervals, whose lengths tend to zero when $k$ goes to infinity. To give a more precise result, it is convenient to introduce some notation.

\textbf{Notation.} For any real numbers $x < y$ in $[0, 1]$, we write $\langle x, y \rangle$ for the closed interval $[x, y]$ if $y < 1$ and $x > 0$, we set $\langle x, 1 \rangle$ for $\{0\} \cup [x, 1]$ and $\langle 0, x \rangle$ for $[0, x]$. Moreover, for any increasing function $f$ on $[x, 1]$, we write $f(\langle x, 1 \rangle)$ for $\langle f(x), f(1) \rangle$.

With these notations, we have

$$T_{\gamma,\alpha}^{b-1}([0, 1]) = [0, 1] \setminus \bigcup_{k=1}^{b-1} \langle T_{\gamma,\alpha}^k(1), T_{\gamma,\alpha}^k(0) \rangle.$$
and $\theta \in \langle T_{\gamma,\alpha}^{-1}(1), T_{\gamma,\alpha}^{-1}(0) \rangle$. As shown in [2], the critical point $\theta$ is in
$\langle T_{\gamma,\alpha}^{-1}(1), T_{\gamma,\alpha}^{-1}(0) \rangle$ if, and only if, there exists a positive integer $a < b$, coprime with $b$, such that $\alpha$ is in $I_{b}^{a}(\gamma)$.

We have now all the tools to prove Lemma 3.

Proof of Lemma 3. Let $\beta > 1$ and $0 \leq \alpha < 1$. Put $\gamma = 1/\beta$. We readily verify that the conclusion of the lemma holds when $\alpha$ is in $J_{1}^{1}(\gamma)$, by Lemma 2. Thus, we assume now that $0 \leq \alpha < \gamma$. We recall that $f$ stands for $f_{\beta,\alpha}$. We observe that $f$ is a bijection from $[0, 1/\beta]$ onto $[0, 1]$, and we denote by $g := g_{\beta,\alpha}$ the inverse of this restriction, i.e. for $x \in [0, 1]$,

$$g_{\beta,\alpha}(x) = \begin{cases} \frac{1}{\beta} x + \frac{1-\alpha}{\beta}, & 0 \leq x < \alpha, \\ \frac{1}{\beta} x - \frac{\alpha}{\beta}, & \alpha \leq x < 1. \end{cases} \tag{5}$$

Thus $g$ is piecewise linear, contracting and continuous, except on the left at $\alpha$.

The maps $T_{\gamma,1-\alpha}$ and $g_{1/\gamma,\alpha}$ are closely related. Namely, for any real number $x$ in $[0, 1]$, we have

$$\{T_{\gamma,1-\alpha}(x) + \alpha\} = g_{1/\gamma,\alpha}(\{x + \alpha\}) = \gamma x. \tag{6}$$

Assume now that the set $S_{\beta,\alpha}$ is finite. In view of Lemma 2, there exists a positive integer $N$ such that $f_{\beta,\alpha}^{N}(0) \in \langle 1/\beta, 1 \rangle$. Denote by $b$ the smallest positive integer with this property. Then we have $0 \in g_{b}^{b}(\langle \gamma, 1 \rangle)$, or, equivalently, $\alpha \in g_{b}^{-1}(\langle \gamma, 1 \rangle)$. Since $\alpha < \gamma$, we have $b \geq 2$. According to Lemma 3.1 of [5], the sets $g_{b}^{k}(\langle \gamma, 1 \rangle)$ for $0 \leq k \leq b-1$ are nonempty intervals. It follows from (6) and (4) that

$$g_{b}^{-1}(\langle \gamma, 1 \rangle) = \langle T_{\gamma,1-\alpha}^{-1}(\gamma - \alpha) + \alpha, T_{\gamma,1-\alpha}^{-1}(1 - \alpha) + \alpha \rangle$$

$$= \langle T_{\gamma,1-\alpha}^{-1}(1) + \alpha, T_{\gamma,1-\alpha}^{-1}(0) + \alpha \rangle.$$ 

Consequently, $\alpha$ belongs to $g_{b}^{-1}(\langle \gamma, 1 \rangle)$ if, and only if, $0$ is in $\langle T_{\gamma,1-\alpha}^{-1}(1) + \alpha, T_{\gamma,1-\alpha}^{-1}(0) + \alpha \rangle$, which, since $\alpha < \gamma$, is equivalent to say that $\alpha/\gamma$ belongs to $\langle T_{\gamma,1-\alpha}^{-1}(1), T_{\gamma,1-\alpha}^{-1}(0) \rangle$. Setting $u := 1 - \alpha$, we have shown that

$$\alpha \in g_{b}^{-1}(\langle \gamma, 1 \rangle)$$

if, and only if,

$$\frac{1 - u}{\gamma} \in \langle T_{\gamma,u}^{-1}(1), T_{\gamma,u}^{-1}(0) \rangle. \tag{7}$$
Notice that the condition \( \alpha < \gamma \) is equivalent to \( \gamma + u > 1 \). According to the remark following Proposition 1, we know that (7) holds if, and only if, there exists a positive integer \( a < b \), coprime with \( b \), such that

\[
u \in I_a^b(\gamma).\]

(8)

As \( u = 1 - \alpha \), Proposition 1 implies that (8) can be rewritten as

\[
\sum_{k=0}^{b-1} (1 - \varepsilon_k(a/b))\gamma^k + \gamma^b - \gamma^{b-1} \leq \alpha \leq \sum_{k=0}^{b-1} (1 - \varepsilon_k(a/b))\gamma^k
\]

(9)

\[
1 + \gamma + \ldots + \gamma^{b-1}
\]

Since \( \varepsilon_k(1 - a/b) = 1 - \varepsilon_k(a/b) \) for any integer \( k \) not multiple of \( b \) and not congruent to one modulo \( b \) (to see this, it suffices to note that \( \lfloor ja/b \rfloor = \lfloor ja/b \rfloor - 1 \) if \( b \) does not divide \( j \)), (9) becomes

\[
\sum_{k=1}^{b-1} \varepsilon_k(1 - a/b)\gamma^k + \gamma^b \leq \alpha \leq \sum_{k=1}^{b-1} \varepsilon_k(1 - a/b)\gamma^k + \gamma^{b-1}
\]

which proves that the set \( S_{\beta,\alpha} \) is finite if, and only if, there exist coprime integers \( b \geq a \geq 1 \) such that \( \alpha \in J_a^b(\gamma) \).

Further, for \( \alpha \in J_a^b(\gamma) \), a direct calculation shows that \( S_{\beta,\alpha} \) is empty if \( \alpha \) is the left endpoint of \( J_a^b(\gamma) \) and has exactly \( b \) elements otherwise.

Moreover, we infer from Lemma 2 and (8) that \( S_{\beta,\alpha} \) is infinite if, and only if,

\[
1 - \alpha \in [0, 1[ \setminus \bigcup_{1 \leq a \leq b \atop (a,b)=1} I_a^b(\gamma),
\]

that is (see [2, Théorème 2](*) or [3, page 207]) if, and only if, there exists some irrational number \( \tau \) in \([0,1[\) such that

\[
1 - \alpha = (1 - \gamma) \sum_{k=0}^{\infty} \varepsilon_{-k}(\tau)\gamma^k,
\]

and the last assertion of the lemma follows since \( \varepsilon_0(\tau) = 1 \) and \( \varepsilon_{-k}(\tau) = 1 - \varepsilon_{-k}(1 - \tau) \) for any integer \( k \geq 1 \).

Finally, the fact that the intervals \( J_a^b(\gamma) \) are disjoint follows from (8), (9), and Remark 1.

\[
\square
\]

**Theorem 3.** For any real number \( \beta > 1 \), the set

\[
E_\beta := \{ \alpha \in [0,1[ : S_{\beta,\alpha} \text{ is an infinite set} \}
\]

\[
= [0,1[ \setminus \bigcup_{1 \leq a \leq b \atop (a,b)=1} J_a^b(\gamma)
\]

(*) There is a misprint in the statement of [2, Théorème 2]: one should read \((S(\alpha))_{-k}\) instead of \((S(\alpha))_k\).
has measure zero, is uncountable and is not closed.

Proof. Denote by \( \mu \) the Lebesgue measure. It follows from Lemma 3 that

\[
\mu(E_\beta) = 1 - \sum_{b=1}^{\infty} \frac{\varphi(b)(\gamma^{b-1} - \gamma^b)}{1 + \gamma + \ldots + \gamma^{b-1}},
\]

where \( \varphi \) is Euler totient function, i.e. \( \varphi(b) \) counts the number of integers \( a, 1 \leq a \leq b \), which are coprime with \( b \). Since

\[
\frac{\gamma^{b-1} - \gamma^b}{1 + \ldots + \gamma^{b-1}} = (1 - \gamma)^2 \frac{\gamma^{b-1}}{1 - \gamma^b}, \quad \text{for } b \geq 1,
\]

and

\[
\sum_{b=1}^{\infty} \varphi(b) \frac{\gamma^{b-1}}{1 - \gamma^b} = \frac{1}{(1 - \gamma)^2},
\]

we infer from Theorem 309 of [6] that \( \mu(E_\beta) = 0 \).

Further, the last assertion of Lemma 3 combined with Théorème 2 of [2] implies that \( E_\beta \) is uncountable. Moreover, if the irrational number \( \tau \) tends to a rational number, then \((1 - \gamma) \sum_{k=1}^{\infty} \varepsilon_{-k}(\tau) \gamma^k \) tends to an endpoint of some interval \( J^a_b(\gamma) \). Hence, \( E_\beta \) is not closed. \( \square \)

Remark 3. An interesting open problem would be to determine the Hausdorff dimension of the sets \( E_\beta \).

To complete the proof of Theorem 1, we recall a crucial result of [5].

Proposition 2. Let \( p > q \geq 2 \) be coprime integers, and let \( s \) in \([0,1-1/p]\). If the set \( S_{p/q}\{\langle p-q \rangle s \} \) is finite, then \( Z_{p/q}(s, s + 1/p) \) is empty.

Proof. This is Theorem 3.2 of [5]. \( \square \)

Proof of Theorem 1. This statement easily follows from Lemma 3 and Theorem 3, combined with Proposition 2. \( \square \)

4. Proof of Theorem 2

We now investigate the behaviour of \( f := f_{\beta, \alpha} \) when \( \alpha \) is in \( E_\beta \). Recall that \( g_{\beta, \alpha} \) is defined in (5). The following lemmas answer a question posed by Flatto et al. at the end of [5].

Lemma 4. Let \( \beta > 1 \) and \( \alpha \in E_\beta \). For \( n \geq 0 \), put

\[
I_n := g^n_{\beta, \alpha}([1/\beta, 1[).
\]
Then we have
\[ S_{\beta, \alpha} := [0, 1] \setminus \bigcup_{n \geq 0} I_n. \]

In particular, \( S_{\beta, \alpha} \) has measure zero.

Proof. Arguing as in Lemma 3.1 of [5], we use that \( f^n(0) \notin [1/\beta, 1] \) for all \( n \geq 0 \) to deduce that the \( I_n \)'s, \( n \geq 0 \), are mutually disjoint. If \( x \in [0, 1] \) is in some \( I_n \) with \( n \geq 0 \), we get that \( f^n(x) \in [1/\beta, 1] \), whence \( x \notin S_{\beta, \alpha} \).

Otherwise, it is clear that \( x \in S_{\beta, \alpha} \). Further,
\[
\mu\left( \bigcup_{n \geq 0} I_n \right) = \left(1 - \frac{1}{\beta}\right) \sum_{n \geq 0} \frac{1}{\beta^n} = 0,
\]
as claimed. \( \square \)

As in [5], we associate to \( f = f_{\beta, \alpha} \) the natural symbolic dynamics, which assigns to each \( x \) in \( [0, 1] \) the integer
\[ S_f(x) = [\beta x + \alpha], \]
and we call the sequence
\[ a_n := S_f(f^n(x)), \quad n \geq 0, \]
the \( f \)-expansion of \( x \). If \( x \) is in \( S_{\beta, \alpha} \), then \( 0 \leq f^n_{\beta, \alpha}(x) < 1/\beta \) for all \( n \geq 0 \), and its \( f \)-expansion is uniquely composed of 0's and 1's.

**Lemma 5.** Let \( \beta > 1 \) and \( \alpha \in E_{\beta} \). Let \( \tau \) in \( ]0, 1[ \) be defined by
\[
\alpha = \left(1 - \frac{1}{\beta}\right) \sum_{k=1}^{\infty} \frac{\varepsilon_{-k}(\tau)}{\beta^k}.
\]
The set \( S_{\beta, \alpha} \) is uncountable, not closed and, for any \( x \) in \( S_{\beta, \alpha} \), there exists \( 0 \leq \eta < 1 \) such that the \( f \)-expansion of \( x \) is the Sturmian sequence \( (a_n)_{n \geq 0} \) given by
\[ a_n = [(n + 1)\tau + \eta] - [n\tau + \eta]. \]

Proof. We point out that \( \tau \) is irrational. We first show that \( g \) and the (irrational) rotation\n\[ R_{1-\tau} : x \mapsto \{x + 1 - \tau\}, \quad x \in [0, 1], \]
are semi-conjugate.
We claim that the intervals $I_n$, $n \geq 0$, are ordered as the sequence \((\{n(1 - \tau)\})_{n \geq 0}\). To see this, for any $\beta' > 1$, we define
\[
\Psi_{\beta'}(\tau) = \left(1 - \frac{1}{\beta'}\right) \sum_{k=1}^{\infty} \frac{\varepsilon_{-k}(\tau)}{\beta'^k},
\]
and we observe that, for any $n \geq 0$, the function
\[
\beta' \mapsto \inf\left( g_{\beta', \Psi_{\beta'}(\tau)}([1/\beta', 1]) \right)
\]
is continuous on $]1, +\infty[$ and tends to $\{n(1 - \tau)\}$ when $\beta'$ tends to 1.

We set $\Phi(0) = 0$ and, for $n \geq 1$, $\Phi(I_n) = \{n(1 - \tau)\}$. The map $\Phi$ is monotone and, by Lemma 4, is defined on a dense subset of $[0, 1[$, thus, we can extend it by continuity to $[0, 1]$. Consequently, the set $S_{\beta, \alpha}$ is uncountable and not closed. For all $y \in [0, 1[$, we have
\[
\Phi \circ g_{\beta, \alpha}(y) = R_{1 - \tau} \circ \Phi(y),
\]
hence,
\[
R_{\tau} \circ \Phi(z) = \Phi \circ f_{\beta, \alpha}(z),
\]
for $z \in [0, 1/\beta]$. Since $\Phi(0) = 0$ and $f((1 - \alpha)/\beta) = 0$, we get from (10) that $\Phi((1 - \alpha)/\beta) = 1 - \tau$. It follows that
\[
0 \leq z < \frac{1 - \alpha}{\beta} \quad \text{if and only if} \quad 0 \leq \Phi(z) < 1 - \tau.
\]
By induction, (10) yields for any integer $n \geq 1$ that
\[
0 \leq f^n(z) < \frac{1 - \alpha}{\beta} \quad \text{if and only if} \quad 0 \leq R^n_{\tau}(\Phi(z)) < 1 - \tau. \tag{11}
\]

Let $x \in S_{\beta, \alpha}$ and denote by $(a_n)_{n \geq 0}$ its $f$-expansion. It follows from (11) that $a_n = 0$ if, and only if, $0 \leq R^n_{\tau}(\Phi(z)) < 1 - \tau$. Hence, we get that
\[
a_n = [(n + 1)\tau + \Phi(z)] - [n\tau + \Phi(z)],
\]
and the proof of Lemma 5 is complete. \qed

We need to recall an important result of [5]. For the definition of $T$-expansion, we refer the reader to [5].
Proposition 3. Let \( p > q \geq 2 \) be coprime integers. Then a positive real number \( \xi \) is in \( Z_{p/q}(s, s + 1/p) \) if, and only if, both conditions (C1) and (C2) hold, with

(C1) \[
0 \leq f^n \left( q(\{\xi\} - s) \right) < q/p \text{ for all } n \geq 0,
\]

and (C2): the \( T \)-expansion \( (a_n) \) of \( [\xi] \) and the \( f \)-expansion \( (b_n) \) of \( q(\{\xi\} - s) \) are related by

\[
\sigma(a_n) = b_n \text{ for all } n \geq 0,
\]

where \( \sigma \) is the permutation of \( \{0, 1, \ldots, q - 1\} \) given by

\[
\sigma(i) \equiv -pi - [(p - q)s] \pmod q.
\]

Further, the set \( Z_{p/q}(s, s + 1/p) \) contains at most one element in each unit interval \([m, m + 1]\), where \( m \) is a non-negative integer.

Proof. This follows from Proposition 2.1 and Theorem 1.1 of [5].

Proof of Theorem 2. Let \( \xi \) be in \( Z_{p/q}(s, s + 1/p) \). By Proposition 3 and Lemma 5, the \( T \)-expansion of \( [\xi] \) is an infinite Sturmian word. It has been shown by Mignosi [9] (see [1] for an alternative proof) that, for any integer \( m \geq 1 \), there are \( O(m^3) \) Sturmian words of length \( m \) (recall that any subword of a Sturmian sequence is called a Sturmian word). Since Lemma 2.2 of [5] asserts that the first \( m \) terms of the \( T \)-expansion of an integer \( g \) are uniquely determined by \( g \) modulo \( q^m \), we conclude that at most \( O(m^3) \) integers less than \( q^m \) may have a Sturmian \( T \)-expansion, and Theorem 2 is proved.

References


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