On U_m-numbers with small transcendence measure

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1. Introduction

In 1932 and 1939, Mahler and Koksma, respectively, introduced two classifications of complex numbers according to their approximation properties by algebraic numbers. We refer the reader to the books of Schneider [8] and of Baker [1] for details, including a proof that both classifications are (in some sense) equivalent. In the present note, we restrict our attention to the class of U^*-numbers (in the terminology of Koksma), which coincides with that of U-numbers (in the terminology of Mahler). Let ξ be a complex number and let $n \geq 1$ be an integer. By definition, $w_n^*(\xi)$ denotes the supremum of the real numbers w for which there exist infinitely many algebraic numbers α of degree at most n satisfying $0 < |\xi - \alpha| < \mathrm{H}(\alpha)^{-w-1}$. Here, $\mathrm{H}(\alpha)$ denotes the naïve height of α, that is the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbf{Z}$. The number ξ is a U-number if, and only if, there exists an integer $n \geq 1$ such that $w_n^*(\xi) = +\infty$. If m denotes the least integer with this property, then, according to LeVeque [7], ξ is called a U_m-number. We observe that the set of real U_1-numbers coincides with that of Liouville numbers.

LeVeque [7] proved that, regardless of the integer $m \geq 1$, the set of real U_m-numbers is non-empty. More precisely, he constructed explicit examples of real U_m-numbers ξ as m-roots of some Liouville numbers. Using a method of Güting [6, Satz 10], it is not difficult to prove that these particular numbers satisfy

$$w_n^*(\xi) \leq mn + m - 1, \qquad \text{for any } 1 \leq n < m. \tag{1}$$

The purpose of the present note is to present a new, effective method for constructing real U_m-numbers ξ with a much better control on $w_n^*(\xi)$ than that given by estimate (1).

2. New result

Our main result asserts the existence of U_m-numbers ξ with relatively small values of $w_n^*(\xi)$ for $1 \leq n < m$.

Theorem. *Let $m > 1$ be an integer. There are infinitely many U_m-numbers ξ with*

$$w_n^*(\xi) \leq m + n - 1, \qquad \textit{for any } \ 1 \leq n < m. \tag{2}$$

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To prove the Theorem, following an idea of Alniaçik & Avci, we construct fast converging sequences of real algebraic integers of degree m, whose limits turn out to be U_m-numbers with the required properties. The key ingredient is Proposition 1 below asserting that algebraic numbers are not too badly approximable by algebraic integers of same degree. It would have been possible to replace Proposition 1 by Corollaire 2 of [3] (notice that the assumption "ξ n'est pas un entier algébrique de degré n" in that statement is in fact not needed), but the exponent of $\mathrm{H}(\alpha)$ in (2) would then be $-n-\varepsilon$ (where ε is some positive real number), and, more important, the constant κ in (3) would then be ineffective. Indeed, the proof of Corollaire 2 of [3] depends ultimately on the celebrated Schmidt's Subspace Theorem, while Proposition 1 below is effective.

It is expected that there exist U_m-numbers ξ with $w_n^*(\xi) = n$ for any integer n with $1 \le n < m$. This is true for $m = 2$, as can be easily shown by constructing suitable continued fraction expansions. Our Theorem provides a small step towards the resolution of that problem for $m \ge 3$.

3. Auxiliary results

The proof of the Theorem relies on a new result of Bugeaud [2] about the approximation of algebraic numbers by algebraic integers of same degree.

Proposition 1. *Let $n \ge 2$ be an integer and ξ be a real algebraic number of degree n. There exist a constant $\kappa(\xi) > 1$ and infinitely many algebraic integers α of degree n such that*

$$0 < |\xi - \alpha| < \kappa(\xi)\, \mathrm{H}(\alpha)^{-n}. \tag{3}$$

The proof of Proposition 1 rests on a method developed by Davenport & Schmidt [4] and recently reworked by Bugeaud & Teulié [3] and by Teulié [9]. It is sharp inasmuch as algebraic numbers are not well approximable by algebraic numbers of same degree: this is a consequence of the so-called Liouville's inequality, that we recall below in the form proved by Güting [5, Theorem 6].

Lemma 1. *Let α_1 and α_2 be two distinct algebraic numbers of degree n_1 and n_2, respectively, and of height at most H_1 and H_2, respectively. If they are not conjugate, then we have*

$$|\alpha_1 - \alpha_2| \ge \min\{n_1, n_2\}^{-1}\, (n_1+1)^{-n_2}\, (n_2+1)^{-n_1}\, H_1^{-n_2}\, H_2^{-n_1}.$$

In addition to Lemma 1, we use several times the following easy consequence of the triangle inequality.

Lemma 2. *Let $(\alpha_1, \alpha_2, \beta)$ be a triple of real numbers. Let H, N and c be positive real numbers such that*

$$0 < |\alpha_1 - \alpha_2| < N^{-1} \quad \text{and} \quad |\alpha_1 - \beta| \ge c^{-1}\, H^{-1}.$$

Then, for any positive real number δ, we have

$$|\alpha_2 - \beta| \ge (c+\delta)^{-1}\, H^{-1} \quad \textit{as soon as} \quad H \le c^{-1}\delta(c+\delta)^{-1} N.$$

Proof : Assuming that $H < c^{-1}\delta(c+\delta)^{-1}N$, we have

$$\begin{aligned}|\alpha_2 - \beta| &\geq |\alpha_1 - \beta| - |\alpha_1 - \alpha_2| \\ &\geq c^{-1}\, H^{-1} - N^{-1} \\ &\geq H^{-1}\big(c^{-1} - c^{-1}\delta(c+\delta)^{-1}\big) = H^{-1}(c+\delta)^{-1},\end{aligned}$$

as claimed. □

4. Proof of the Theorem

Let $0 < \varepsilon < 1/2$ and $m \geq 2$ be an integer. We construct by induction a rapidly converging sequence of real algebraic integers of degree m, whose limit is a U_m-number with (2).

For $1 \leq t \leq m-1$, we denote by β_t a real algebraic number of degree t.

Let ξ_1 be a real algebraic number of degree m. By Lemma 1, we have

$$|\xi_1 - \beta_t| \geq c_1^{-1}\mathrm{H}(\xi_1)^{-t}\,\mathrm{H}(\beta_t)^{-m}, \tag{4}$$

where $c_1 = (m+1)^{2m}$. By Proposition 1, there are infinitely many real algebraic integers ξ_2 of degree m with

$$0 < |\xi_1 - \xi_2| < \kappa(\xi_1)\,\mathrm{H}(\xi_2)^{-m}. \tag{5}$$

We infer from (4), (5) and Lemma 2 applied to the triple (ξ_1, ξ_2, β_t) with $H = \mathrm{H}(\beta_t)^m$, $N = \kappa(\xi_1)^{-1}\,\mathrm{H}(\xi_2)^m$, $c = c_1\mathrm{H}(\xi_1)^m$ and $\delta = 1/2$ that

$$|\xi_2 - \beta_t| > (c_2 + 1/2)^{-1}\mathrm{H}(\beta_t)^{-m}, \tag{6}$$

with $c_2 = c_1\mathrm{H}(\xi_1)^m$, provided that

$$\mathrm{H}(\beta_t) \leq (c_2(2c_2+1))^{-1/m}\,\kappa(\xi_1)^{-1/m}\,\mathrm{H}(\xi_2) := A.$$

Further, if $\mathrm{H}(\beta_t) > A$, we apply Lemma 1 to get

$$|\xi_2 - \beta_t| > c_1^{-1}\mathrm{H}(\xi_2)^{-t}\,\mathrm{H}(\beta_t)^{-m}. \tag{7}$$

We infer from (7), $\varepsilon < 1$ and $\mathrm{H}(\beta_t) > A$ that

$$\begin{aligned}|\xi_2 - \beta_t| &> c_1^{-1}\big(c_2(2c_2+1)\kappa(\xi_1)\big)^{-t/m}\,\mathrm{H}(\beta_t)^{-m-t} \\ &> \mathrm{H}(\beta_t)^{-m-t}\big(\log 3\mathrm{H}(\beta_t)\big)^{-1}\end{aligned} \tag{8}$$

holds if $\mathrm{H}(\beta_t)$ is large enough, say if $\mathrm{H}(\beta_t) > c_3(m, \xi_1)$. We choose ξ_2 such that we have $\mathrm{H}(\xi_2) > \mathrm{H}(\xi_1)$ and $A > c_3(m, \xi_1)$. Then, by (6) and (8), we see that

$$|\xi_2 - \beta_t| > (c_2 + 1/2)^{-1}\mathrm{H}(\beta_t)^{-m-t}\big(\log 3\mathrm{H}(\beta_t)\big)^{-1}$$

holds for any algebraic number β_t.

Assume now that we have constructed $\xi_2, \ldots, \xi_n$ real algebraic integers of degree m such that

$$|\xi_{i+1} - \beta_t| > \left(c_2 + \frac{1}{2} + \ldots + \frac{1}{2^i}\right)^{-1} \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t)\right)^{-1}, \qquad \text{for any } \beta_t,$$

$$|\xi_i - \xi_{i+1}| < \kappa(\xi_i)\, \mathrm{H}(\xi_{i+1})^{-m} \quad \text{and} \quad \mathrm{H}(\xi_{i+1}) > \kappa(\xi_i)\, \mathrm{H}(\xi_i)^i$$

for $1 \le i \le n-1$.

By Proposition 1, there exists a real algebraic number ξ_{n+1} of degree m satisfying

$$\mathrm{H}(\xi_{n+1}) > \kappa(\xi_n)\, \mathrm{H}(\xi_n)^n \quad \text{and} \quad |\xi_n - \xi_{n+1}| < \kappa(\xi_n)\, \mathrm{H}(\xi_{n+1})^{-m}. \tag{9}$$

By assumption, we have

$$|\xi_n - \beta_t| \ge c_4^{-1}\, \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t)\right)^{-1},$$

with $c_4 = c_2 + 2^{-1} + \ldots + 2^{-n+1}$, and we infer from Lemma 2 applied to the triple $(\xi_n, \xi_{n+1}, \beta_t)$ with $\delta = 2^{-n}$ that

$$|\xi_{n+1} - \beta_t| \ge (c_4 + 2^{-n})^{-1}\, \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t)\right)^{-1}, \tag{10}$$

holds provided that

$$\mathrm{H}(\beta_t)^{m+t} \left(\log 3\mathrm{H}(\beta_t)\right) \le c_4^{-1}\, (2^n c_4 + 1)^{-1}\, \kappa(\xi_n)^{-1}\, \mathrm{H}(\xi_{n+1})^m. \tag{11}$$

By Lemma 1, we have

$$|\xi_n - \beta_t| \ge c_1^{-1} \mathrm{H}(\xi_n)^{-t} \mathrm{H}(\beta_t)^{-m},$$

and we apply Lemma 2 to the triple $(\xi_n, \xi_{n+1}, \beta_t)$ with the parameters $H = \mathrm{H}(\beta_t)^m$, $N = \kappa(\xi_n)^{-1}\, \mathrm{H}(\xi_{n+1})^m$, $c = c_1 \mathrm{H}(\xi_n)^t$ and $\delta = \mathrm{H}(\xi_n)^t$ to get

$$|\xi_{n+1} - \beta_t| > (c_1 + 1)^{-1} \mathrm{H}(\xi_n)^{-t} \mathrm{H}(\beta_t)^{-m}$$

provided that

$$\mathrm{H}(\beta_t)^m \le c_1^{-1}\, (c_1 + 1)^{-1}\, \kappa(\xi_n)^{-1}\, \mathrm{H}(\xi_n)^{-t}\, \mathrm{H}(\xi_{n+1})^m := B. \tag{12}$$

Hence, we have

$$|\xi_{n+1} - \beta_t| > \mathrm{H}(\beta_t)^{-m-1} \tag{13}$$

as soon as β_t satisfies

$$(c_1 + 1)\mathrm{H}(\xi_n)^t \le \mathrm{H}(\beta_t) \le B^{1/m}. \tag{14}$$

By choosing $\mathrm{H}(\xi_{n+1})$ large enough in terms of m and ξ_n, (11) is satisfied for all algebraic numbers β_t with $\mathrm{H}(\beta_t) < (c_1 + 1)\mathrm{H}(\xi_n)^t$. Thus, by (10), (13) and (14), we have

$$|\xi_{n+1} - \beta_t| > (c_4 + 2^{-n})^{-1} \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t)\right)^{-1} \tag{15}$$

when $\mathrm{H}(\beta_t) \le B^{1/m}$.

Finally, assuming that $\mathrm{H}(\beta_t) > B^{1/m}$, we use

$$|\xi_{n+1} - \beta_t| > c_1^{-1} \mathrm{H}(\xi_{n+1})^{-t} \mathrm{H}(\beta_t)^{-m},$$

given by Lemma 1, to get from (12) that, for $\mathrm{H}(\xi_{n+1})$ large enough in terms of m and ξ_n, we have

$$\begin{aligned} |\xi_{n+1} - \beta_t| &> c_1^{-1} \left(c_1(c_1+1)\kappa(\xi_n)\mathrm{H}(\xi_n)^t \right)^{-t/m} \mathrm{H}(\beta_t)^{-m-t} \\ &> \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t) \right)^{-1}. \end{aligned} \tag{16}$$

The desired conclusion follows, namely that, for any algebraic number β_t, we have

$$|\xi_{n+1} - \beta_t| > \left(c_2 + \frac{1}{2} + \ldots + \frac{1}{2^n} \right)^{-1} \mathrm{H}(\beta_t)^{-m-t} \left(\log 3\mathrm{H}(\beta_t) \right)^{-1}.$$

Further, we easily see from (9) that, for any $2 \le i < j$, we have

$$|\xi_j - \xi_i| \le 2\, \mathrm{H}(\xi_i)^{-i}, \tag{17}$$

for any algebraic number β_t. Thus, the sequence $(\xi_j)_{j \ge 1}$ is a Cauchy sequence, and we denote its limit by ξ. Letting j tend to infinity in (17), we get

$$|\xi - \xi_i| \le 2\, \mathrm{H}(\xi_i)^{-i},$$

whence $w_m^*(\xi) = +\infty$. However, by (15) and (16), we get that $w_t^*(\xi) \le m + t - 1$, for any $1 \le t \le m - 1$.

□

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