On the convergents to algebraic numbers

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Abstract. *Let ξ be an irrational, algebraic number and denote by $(p_n/q_n)_{n\ge1}$ the sequence of its convergents. We give several results on the arithmetical properties and on the growth of the sequences $(p_n)_{n\ge1}$ and $(q_n)_{n\ge1}$. For coprime integers a and b with $a>b>1$, we study the length of the continued fraction expansions of the integral powers of the rational number a/b. Most of the results surveyed here are consequences of Roth's theorem or of one of its relatives.*

1. Introduction

The first result on the rational approximation of algebraic numbers goes back to 1844, when Liouville [19, 20] showed that an algebraic number of degree d cannot be approximated by rationals at an order greater than d. Liouville's theorem has been subsequently improved upon by Thue [44], Siegel [40], Dyson [12], Gelfond [15] and, finally, by Roth [36], who established that, like almost all real numbers (throughout the present paper, 'almost all' refers to the Lebesgue measure), the algebraic irrational numbers cannot be approximated by rationals at an order greater than 2.

Theorem (Roth, 1955). *Let θ be an algebraic real number. Let ε be a positive real number. Then there are only finitely many rational numbers p/q with $q\ge1$ such that*

$$0<\left|\theta-\frac{p}{q}\right|<\frac{1}{q^{2+\varepsilon}}\cdot \tag{1.1}$$

The same year, Davenport and Roth [11] gave a totally explicit estimate for the number of rational solutions to (1.1). Shortly afterwards, Ridout [34, 35] established two different generalizations of Roth's theorem which incorporate non-Archimedean valuations. In the sequel, for any prime number ℓ and any non-zero rational number x, we set $|x|_\ell := \ell^{-u}$, where u is the exponent of ℓ in the prime decomposition of x. Furthermore, we set $|0|_\ell = 0$. With this notation, the main result of [34] reads as follows.

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Theorem (Ridout, 1957). *Let S_1 and S_2 be disjoint finite sets of prime numbers. Let θ be a real algebraic number. Let ε be a positive real number. Then there are only finitely many rational numbers p/q with $q \geq 1$ such that*

$$\left|\theta - \frac{p}{q}\right| \cdot \prod_{\ell \in S_1} |p|_\ell \cdot \prod_{\ell \in S_2} |q|_\ell < \frac{1}{q^{2+\varepsilon}}. \tag{1.2}$$

A partial result towards an improvement of Roth's and Ridout's theorems has been obtained in 1958 by Cugiani [8] (see also [9, 10] and Appendix B of Mahler's book [26]). We refer the reader to Section 4 for a precise statement of a recent strengthening of this result, now called the Cugiani–Mahler theorem.

The purpose of this survey is to show different applications of Roth's theorem and its relatives to various questions on the sequence of convergents $(p_n/q_n)_{n\geq 1}$ to an algebraic number. Section 2 is devoted to the arithmetical properties of $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$. It includes an application of Ridout's theorem and one of Baker's theory of linear forms in logarithms. The rate of growth of the sequence $(q_n)_{n\geq 1}$ is investigated in Section 3 by means of a modern version of the theorem of Davenport and Roth that gives an explicit upper bound for the number of rational solutions to (1.1). Finally, Section 4 deals with the continued fraction expansions of the rational numbers $(a/b)^n$, with $1 < b < a$ and $n \geq 1$. It includes an application of Ridout's theorem and a new application of the Cugiani–Mahler theorem.

Notation. Throughout the present paper, ξ is an arbitrary irrational, real algebraic number and $(p_n/q_n)_{n\geq 1}$ denotes the sequence of its convergents. The constants implied by $\ll$ and $\gg$ depend at most on ξ, and we write $\ll_{\mathrm{eff}}$ and $\gg_{\mathrm{eff}}$ to emphasize that the implicit constant is effectively computable.

2. Arithmetical properties of convergents

In this section, for an integer x with $x \geq 2$, we denote by $P[x]$ its greatest prime factor and by $Q[x]$ its greatest square-free factor. Without mentioning it, we assume that the arguments of the function log (resp. $\log\log$, $\log\log\log, \ldots$) are greater than e (resp. e^e, $e^{e^e}, \ldots$).

Before considering algebraic numbers, we mention that Erdős and Mahler [13] proved that for almost all real numbers θ with $0 \leq \theta \leq 1$ we have

$$P[s_n] \geq \exp\left\{\frac{\log s_n}{20 \log\log s_n}\right\}, \tag{2.1}$$

for every sufficiently large n, where s_n denotes the denominator of the nth convergent to θ. With the same notation, Shorey and Srinivasan [43] proved that for any positive real number δ, for almost all real numbers θ with $0 \leq \theta \leq 1$ we have

$$Q[s_n] \geq s_n (\log s_n)^{-1-\delta}, \tag{2.2}$$

for every sufficiently large n. Perhaps, (2.1) and (2.2) hold for any irrational algebraic θ, but we are very far away from being able to confirm this.

In the sequel of this section, ξ is a real, irrational, algebraic number, and $(p_n/q_n)_{n\geq 1}$ denotes the sequence of its convergents. Using his p-adic version of the Thue–Siegel theorem, Mahler [23] proved that the greatest prime factor of $p_n q_n$ tends to infinity with n. He also established that the greatest prime factor of p_n (and also that of q_n) is unbounded. Subsequently, by working out a p-adic version of a result of Dyson [12] on rational approximation to algebraic numbers, Mahler [24] showed that, when ξ is either quadratic or cubic, then both $P[p_n]$ and $P[q_n]$ tend to infinity. Ridout's theorem [34] allows one to extend the latter result of Mahler.

Theorem R. *For every irrational, real algebraic number, both $P[p_n]$ and $P[q_n]$ tend to infinity with n.*

Proof. We know from the theory of continued fractions that $|\xi - p_n/q_n| < q_n^{-2}$ for $n \geq 1$. If there exist an infinite sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ and an integer P such that $P[q_{n_j}] < P$ for $j \geq 1$, then we get a contradiction from Ridout's theorem with $\varepsilon = 1$ by taking for S_1 the empty set and for S_2 the set of prime numbers less than P. This shows that $P[q_n]$ tends to infinity with n and a similar proof yields the same conclusion for $P[p_n]$. □

Theorem R is ineffective, and it would be very desirable to get an effective estimate for the growth of $P[p_n q_n]$, $P[p_n]$, and $P[q_n]$.

Using Baker's theory, Shorey [41] established a quantitative form of Mahler's result.

Theorem S. *For every irrational, real algebraic number we have*

$$P[p_n q_n] \gg_{\mathrm{eff}} \log\log q_n.$$

It turns out that it is possible to slightly improve upon Theorem S, by using Matveev's recent estimate for linear forms in logarithms. We denote by $\lfloor x \rfloor$ the integer part of the real number x.

Theorem 1. *For every irrational, real algebraic number ξ we have*

$$P[q\lfloor q\xi \rfloor] \gg_{\mathrm{eff}} \log\log q \cdot \frac{\log\log\log q}{\log\log\log\log q}.$$

Proof. We follow Shorey's proof. Without any loss of generality, we assume that q is large enough. Set $p = \lfloor q\xi \rfloor$ and observe that $0 < |\xi - p/q| \leq 1/q$. To shorten the notation, for any positive integer j, we write $\log_j$ for the jth iterate of the function log. Assume that

$$P[pq] \leq \delta \, \log_2 q \, \frac{\log_3 q}{\log_4 q}$$

is satisfied for any δ with $0 < \delta < 1$. We will arrive at a contradiction for a certain value of δ depending only on ξ. Let m be the number of distinct prime factors of pq. By the Prime Number Theorem, we have

$$m \leq 2\,\delta \, \frac{\log_2 q}{\log_4 q}.$$

Denoting by $\ell_1, \ell_2, \ldots$ the increasing sequence of all the prime numbers, there exist positive integers $i_1 < \ldots < i_k$ with $k \leq m$ and non-zero integers $a_{i_1}, \ldots, a_{i_k}$ such that

$$\Lambda := |\xi \ell_{i_1}^{a_{i_1}} \ldots \ell_{i_k}^{a_{i_k}} - 1| \ll_{\mathrm{eff}} 1/q. \tag{2.3}$$

By assumption, Λ is non-zero. Check that

$$\ell_{i_j} \leq \ell_{i_k} \leq \delta(\log_2 q / \log_4 q) \log_3 q, \quad |a_{i_j}| \ll_{\mathrm{eff}} \log q \quad (j = 1, \ldots, k).$$

It then follows from Matveev's theorem [27] that there exists an effectively computable constant c_1, depending only on ξ, such that

$$\log \Lambda > -c_1^m (\log_3 q)^m (\log_2 q). \tag{2.4}$$

We then infer from (2.3) and (2.4) that

$$\log_2 q \ll_{\mathrm{eff}} m \log_4 q + \log_3 q \ll_{\mathrm{eff}} \delta \log_2 q + \log_3 q.$$

Selecting δ small enough, we get a contradiction. $\square$

Corollary 1. *For every irrational, real algebraic number we have*

$$P[p_n q_n] \gg_{\mathrm{eff}} \log \log q_n \cdot \frac{\log \log \log q_n}{\log \log \log \log q_n}.$$

Corollary 1 does not give any effective lower bound for $P[q_n]$, nor for $P[p_n]$. When ξ is a quadratic surd, then its continued fraction expansion is ultimately periodic and the sequences $(p_n)_{n \geq 1}$ and $(q_n)_{n \geq 1}$ are binary recurring sequences. Using again Baker's theory, we get in this case the effective lower bounds

$$P[p_n] \gg_{\mathrm{eff}} (\log p_n)^{1/3}, \qquad P[q_n] \gg_{\mathrm{eff}} (\log q_n)^{1/3},$$
$$Q[p_n] \gg_{\mathrm{eff}} \frac{(\log \log p_n)^2}{\log \log \log p_n}, \qquad Q[q_n] \gg_{\mathrm{eff}} \frac{(\log \log q_n)^2}{\log \log \log q_n},$$

as proved by Győry, Mignotte, and Shorey [17].

Problem 1. *To give an effective lower bound for $P[q_n]$ (resp. for $P[p_n]$) when ξ is an algebraic number of degree at least three.*

To conclude this section, let us mention that Erdős and Mahler [13] established that $P[q_{n-1} q_n q_{n+1}]$ tends to infinity with n. However, their result is not effective. Using Baker's theory of linear forms in logarithms, Shorey [42] proved that

$$P[q_{n-1} q_n q_{n+1}] \gg_{\mathrm{eff}} \log \log q_n$$

and

$$\log Q[q_{n-1} q_n q_{n+1}] \gg_{\mathrm{eff}} \log \log q_n.$$

3. On the growth of the denominators of convergents

Let ξ be a real, irrational, algebraic number, and denote by $(p_n/q_n)_{n\geq 1}$ the sequence of its convergents. It immediately follows from the theory of continued fractions that the rate of increase of $(q_n)_{n\geq 1}$ is at least exponential. Our purpose in the present section is to estimate it from above. Recall that Lévy [18] established in 1936 that, for almost all real numbers θ, we have

$$\frac{\log s_n}{n} \underset{n\to+\infty}{\longrightarrow} \frac{\pi^2}{12\log 2},$$

where s_n denotes the denominator of the nth convergent to θ.

It is well known that, when ξ is quadratic, then the sequence $(q_n^{1/n})_{n\geq 1}$ is bounded and, even, converges. One generally believes that $(q_n^{1/n})_{n\geq 1}$ also remains bounded when the degree of ξ is greater than two. However, we seem to be very far away from a proof (or a disproof).

The first general upper estimate for the rate of increase of $(q_n)_{n\geq 1}$ follows from Liouville's theorem, which easily yields that

$$\log\log q_n \ll n.$$

A slight sharpening, namely the estimate

$$\log\log q_n = o(n),$$

can be deduced from Roth's theorem.

In Roth's joint work with Davenport [11], some steps from [36] are made totally explicit in order to get an explicit upper estimate for the cardinality $\mathcal{N}(\theta,\varepsilon)$ of the set of rational solutions to (1.1). This enabled Davenport and Roth [11] to prove that

$$\log\log q_n \ll \frac{n}{\sqrt{\log n}}, \tag{3.1}$$

see also Mignotte [30].

A much better upper bound for $\mathcal{N}(\theta,\varepsilon)$ was established by Bombieri and van der Poorten [3] (see also Luckhardt [22], who used his result to improve upon (3.1), and Locher [21]) and subsequently slightly refined by Evertse [14]. Before stating a result extracted from the end of Section 6 of [14], we recall that the Mahler measure of an algebraic number equals the leading coefficient of its minimal polynomial over the integers times the product of the moduli of its complex conjugates of modulus at least 1.

Theorem E. *Let θ be an algebraic number of degree d with $0<\theta<1$. Let ε be a positive real number with $\varepsilon<1/5$. The inequality*

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$$

has at most

$$\mathcal{N}_1(\theta,\varepsilon) := 2\cdot 10^7\,\varepsilon^{-3}(\log\varepsilon^{-1})^2\,(\log 4d)\,(\log\log 4d)$$

rational solutions p/q *with* $q \ge \max\{4^{2/\varepsilon}, M(\theta)\}$.

Theorem 4 of Mueller and Schmidt [31] implies that, regarding the dependence on d, Theorem E is best possible up to, maybe, the factor $(\log\log 4d)$.

In [1], Theorem E is used to strengthen (3.1). We give below the proof of the following (very) slightly refined version of Theorem 4.1 of [1].

Theorem 2. *Let* ξ *be an arbitrary irrational, real algebraic number of degree* d *and let* $(p_n/q_n)_{n\ge1}$ *denote the sequence of its convergents. Then we have*

$$\log\log q_n \le 4\cdot 10^7\,n^{2/3}\,(\log n)^{2/3}\,(\log\log n)\,(\log 4d)^{1/3}\,(\log\log 4d)^2, \tag{3.2}$$

for all positive n *with*

$$n \ge \max\{60, 4\log M(\xi), d\}.$$

We point out that Theorem 2 is fully effective, although Roth's theorem is not. The constant $4\cdot 10^7$ in (3.2) can be replaced by a smaller one: our aim was to state a fully explicit upper bound and we made no effort for lowering this constant.

Clearly, inequality (3.2) holds with q_n replaced by the nth partial quotient of ξ. This strongly improves upon a result of Wolfskill [46] valid only for cubic irrationals.

Proof of Theorem 2. We follow the proof of Theorem 4.1 from [1] and we make every step explicit. The basic idea is to introduce more parameters in the proof of Theorem 3 of [11]. Recall that we have

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}, \tag{3.3}$$

for $n \ge 1$.

Let N be an integer with $N \ge \max\{60, 4\log M(\xi), d\}$. Let h be the smallest positive integer with $h \ge 6$ and $q_h \ge \max\{16^{N^{1/3}}, M(\xi)\}$. Since $q_h \ge 2^{h/2}$, we have

$$h \le \max\{8N^{1/3}, 3\log M(\xi)\}.$$

Put $\mathcal{S}_0 = \{h, h+1, \ldots, N\}$. Let $k \ge 3$ be an integer and $\varepsilon_1, \ldots, \varepsilon_k$ be real numbers with

$$0 < N^{-1/3} < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_k < 1,$$

that will be selected later on. For $j = 1, \ldots, k$, let $\mathcal{S}_j$ denote the set of positive integers n such that $h \le n \le N$ and $q_{n+1} > q_n^{1+\varepsilon_j}$. Observe that $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \ldots \supset \mathcal{S}_k$. It follows from (3.3) that, for n in $\mathcal{S}_j$, the convergent p_n/q_n gives a solution to

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon_j}}.$$

By our choice of h, the cardinality of $\mathcal{S}_j$ is at most $\mathcal{N}_1(\xi, \varepsilon_j)$.

Write

$$\mathcal{S}_0 = (\mathcal{S}_0 \setminus \mathcal{S}_1) \cup (\mathcal{S}_1 \setminus \mathcal{S}_2) \cup \ldots \cup (\mathcal{S}_{k-1} \setminus \mathcal{S}_k) \cup \mathcal{S}_k.$$

Let j be an integer with $1 \le j \le k$. The cardinality of $\mathcal{S}_0 \setminus \mathcal{S}_1$ is obviously bounded by N and, for $j \ge 2$, the cardinality of $\mathcal{S}_{j-1} \setminus \mathcal{S}_j$ is at most $\mathcal{N}_1(\xi, \varepsilon_{j-1})$. Furthermore, for every n in $\mathcal{S}_{j-1} \setminus \mathcal{S}_j$, we get

$$\frac{\log q_{n+1}}{\log q_n} \le 1 + \varepsilon_j.$$

Recall that d denotes the degree of ξ. The Liouville inequality as stated by Waldschmidt [45], p.84, asserts that

$$\left| \xi - \frac{p}{q} \right| \ge \frac{1}{M(\xi) \cdot (2q)^d}, \tag{3.4}$$

for every rational number p/q. Consequently, we infer from (3.3) that

$$\frac{\log q_{n+1}}{\log q_n} \le 2d$$

holds if $q_n \ge M(\xi)$ and $n \ge h$.

Combining these estimates with the fact that $\mathcal{S}_k$ has at most $\mathcal{N}_1(\xi, \varepsilon_k)$ elements, we obtain that

$$\begin{aligned} \frac{\log q_N}{\log q_h} &= \frac{\log q_N}{\log q_{N-1}} \times \frac{\log q_{N-1}}{\log q_{N-2}} \times \ldots \times \frac{\log q_{h+1}}{\log q_h} \\ &\le (1 + \varepsilon_1)^N \prod_{j=2}^{k} (1 + \varepsilon_j)^{\mathcal{N}_1(\xi, \varepsilon_{j-1})} \, (2d)^{\mathcal{N}_1(\xi, \varepsilon_k)}. \end{aligned}$$

Taking the logarithm and using the fact that $\log(1+u) < u$ for any positive real number u, we get

$$\log\log q_N - \log\log q_h \le N\,\varepsilon_1 + \sum_{j=2}^{k} \varepsilon_j \, \mathcal{N}_1(\xi, \varepsilon_{j-1}) + \mathcal{N}_1(\xi, \varepsilon_k) \cdot (\log 2d). \tag{3.5}$$

We now select $\varepsilon_1, \ldots, \varepsilon_k$. For $j = 1, \ldots, k$, set

$$\varepsilon_j = N^{-(3^k - 3^{j-1})/(3^{k+1}-1)} \, (\log N)^{2/3} \, (\log 4d)^{(3^{k-j}+1)/3^{k+1-j}}.$$

We check that $0 < N^{-1/3} < \varepsilon_1 < \ldots < \varepsilon_k < 1$, and we easily infer from (3.5) and Theorem E that

$$\begin{aligned} \log\log q_N - \log\log q_h \le\ & 2k10^7 \, N^{2/3} \, N^{2/3^k} \, (\log N)^{2/3} \, (\log 4d)^{1/3} \, (\log\log 4d) \\ & + N^{2/3} \, N^{2/3^k} \, (\log N)^{2/3} \, (\log 4d)^{(3^{k-1}+1)/3^k}. \end{aligned} \tag{3.6}$$

Choosing for k the smallest integer greater than $(\log\log N) \cdot (\log^+ \log\log 4d)$, we get from (3.6) that

$$\log \frac{\log q_N}{\log q_h} \le 3 \cdot 10^7 \, N^{2/3} \, (\log N)^{2/3} \, (\log\log N) \, (\log 4d)^{1/3} \, (\log\log 4d) \, (\log^+ \log\log 4d).$$

Here, the function $\log^+$ is defined on the set of positive real numbers by setting $\log^+ x = \max\{\log x, 1\}$.

Our choice of h implies that $q_{h-1} < M(\xi) + 16^{N^{1/3}}$. Combined with (3.4), this gives

$$\log\log q_h \leq \log(4d) + 2\log N + 2\log\log M(\xi).$$

Since $M(\xi) \leq 2^N$ and $d \leq N$, we get

$$\log\log q_N \leq 4 \cdot 10^7\, N^{2/3}\, (\log N)^{2/3}\, (\log\log N)\, (\log 4d)^{1/3}\, (\log\log 4d)^2.$$

This concludes the proof. □

As a consequence of our theorem, we get an estimate for the maximal growth of the sequence of denominators of convergents to algebraic numbers of bounded degree.

Corollary 2. *Let θ be an irrational, real number and let $(r_n/s_n)_{n\geq 1}$ denote the sequence of its convergents. Let $d \geq 2$ be an integer. If*

$$\limsup_{n\to+\infty} \frac{\log\log s_n}{n^{2/3}(\log n)^{2/3}\,(\log\log n)\,(\log 4d)^{1/3}\,(\log\log 4d)^2} > 4 \cdot 10^7,$$

then θ is transcendental or algebraic of degree greater than d.

Corollary 2 improves the Corollaire of Mignotte [30].

4. On the continued fractions of powers of rational numbers

Let a and b be coprime integers with $1 < b < a$. Let ε be a positive real number. Applying Ridout's theorem, Mahler [25] proved that

$$\left\| \left(\frac{a}{b}\right)^n \right\| \geq 2^{-\varepsilon n}$$

holds for every sufficiently large integer n. Here, $\|\cdot\|$ denotes the distance to the nearest integer. This implies that the first partial quotient of $\|(a/b)^n\|$ is less than $2^{\varepsilon n}$ when n is sufficiently large. As we shall see below, Ridout's theorem also yields some information on the other partial quotients of $\|(a/b)^n\|$.

A rational number r has exactly two continued fraction expansions. These are $[r]$ and $[r-1;1]$ if r is an integer and, otherwise, one of them has the form $[a_0; a_1, \ldots, a_{n-1}, a_n]$ with $a_n \geq 2$, and the other one is $[a_0; a_1, \ldots, a_{n-1}, a_n - 1, 1]$. In the sequel, we denote by $\mathcal{L}(r)$ the length of the shortest continued fraction equal to r.

In 1973, Mendès France [28] asked whether

$$\sup_{n\geq 1} \mathcal{L}\big((a/b)^n\big) = +\infty$$

holds for all coprime integers a and b with $1 < b < a$. In a series of notes, Choquet [5] gave an affirmative answer to this question. Independently, Pourchet [33] applied Ridout's theorem to obtain a stronger statement, quoted below.

Theorem P. *For all coprime integers a and b with $1 < b < a$, for any positive real number ε, the partial quotients of $\|(a/b)^n\|$ are all less than $2^{\varepsilon n}$ when n is sufficiently large. In particular, we have*

$$\lim_{n\to+\infty} \mathcal{L}\big((a/b)^n\big) = +\infty.$$

Pourchet never published his result. Although some details of the proof have been given by van der Poorten [32] (see also [7] and [47], Ex. II.6), we include below a proof of Theorem P. The function field analogue has been solved by Grisel [16].

Notice that the trivial upper bound

$$\mathcal{L}\big((a/b)^n\big) \leq 3n \log b$$

is valid for all positive integers a, b, n with $1 < b < a$. Theorem P does not provide any information on the speed of growth of $\mathcal{L}((a/b)^n)$. This is due to the ineffectiveness of Ridout's theorem. However, it turns out that the use of another strengthening of Roth's theorem, namely the Cugiani–Mahler theorem, allows us to get some additional information.

Theorem 3. *For all coprime integers a and b with $1 < b < a$, there exist a positive constant C and arbitrarily large integers n such that*

$$\mathcal{L}\big((a/b)^n\big) > C\left(\frac{\log n}{\log\log n}\right)^{1/4}.$$

We will use the following version of the Cugiani–Mahler theorem, that we extract from Bombieri and Gubler [2] (see also Bombieri and van der Poorten [3]).

Theorem BG. *Let S be a finite set of prime numbers. Let θ be a real algebraic number of degree d. For any positive real number t set*

$$f(t) = 7\,(\log 4d)^{1/2}\left(\frac{\log\log(t+\log 4)}{\log(t+\log 4)}\right)^{1/4}.$$

Let $(r_j/s_j)_{j\geq 1}$ be the sequence of rational solutions, written in reduced form, to

$$\left|\theta - \frac{r}{s}\right| \cdot \prod_{\ell\in S} |rs|_\ell \leq \frac{1}{s^{2+f(\log s)}}$$

ordered such that $1 \leq s_1 < s_2 < \ldots$ Then either the sequence $(r_j/s_j)_{j\geq 1}$ is finite or

$$\limsup_{j\to+\infty} \frac{\log r_{j+1}}{\log r_j} = +\infty. \tag{4.1}$$

Proof. This follows from Theorem 6.5.10 of [2]. □

Proof of Theorem P. Let p/q be a solution to

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| < \frac{1}{q^2},$$

with p and q coprime. Let ε be a positive real number. Let S_1 (resp. S_2) be the set of prime divisors of a (resp. b). By Ridout's theorem, there exists a positive real number $C(\varepsilon)$ such that $C(\varepsilon) \le 1/4$ and

$$\left| 1 - \frac{qa^n}{pb^n} \right| \cdot a^{-n} \cdot b^{-n} \ge 2\, C(\varepsilon) p^{-2} b^{-2n} a^{-\varepsilon n}, \tag{4.2}$$

since $pb^n \le a^{2n}$. The integers qa^n and pb^n may not be coprime, but this does not matter since the integers p and q occurring in the statement of Ridout's theorem are not assumed to be coprime. If

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| \le \frac{1}{2}, \tag{4.3}$$

then $pb^n \le 2qa^n$, and it follows from (4.2) that

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| \ge C(\varepsilon)\, q^{-2}\, a^{-\varepsilon n}. \tag{4.4}$$

Since (4.4) also holds if (4.3) is not satisfied, it implies that the partial quotients of $(a/b)^n$ are all less than $a^{\varepsilon n}/C(\varepsilon)$, thus less than $a^{2\varepsilon n}$ if n is sufficiently large. Consequently, we get that $\mathcal{L}((a/b)^n) \ge 1/(2\varepsilon)$ for n large enough. This proves Theorem P. $\square$

Proof of Theorem 3. Let S be the set of prime divisors of ab. We may assume that the (ordered) sequence $(r_j/s_j)_{j\ge1}$ of rational solutions (written in their lowest form) to

$$\left| 1 - \frac{r}{s} \right| \cdot \prod_{\ell \in S} |rs|_\ell \le \frac{1}{s^{2+f(\log s)}}$$

is infinite. By (4.1), there exist arbitrarily large integers j and n such that $r_{j+1} > a^{2n}$ and $r_j < a^{n/2}$. By Theorem P, if n is sufficiently large, then there exists a convergent p'_n/q'_n to $(a/b)^n$ with $a^{n/3} < p'_n < a^{n/2}$. Any convergent p/q to $(a/b)^n$ with $p \le p'_n$ is a convergent to p'_n/q'_n. Write $qa^n/(pb^n) = r/s$ with r and s positive and coprime. Write $rs = tt'$, where t' is the largest integer coprime with ab. Since $a^{n/2} \le r \le a^{2n}$, the rational number r/s does not belong to the sequence $(r_j/s_j)_{j\ge1}$, thus

$$\left| 1 - \frac{qa^n}{pb^n} \right| \cdot t^{-1} \ge s^{-2-f(\log s)},$$

that is,

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| \ge \frac{p}{q}\, t\, s^{-2-f(\log s)}. \tag{4.5}$$

We may assume that (4.3) holds (which is the case if $q \geq 2$) and, since $t' \leq pq$, we infer from (4.5) that

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| \geq \frac{1}{2q^2}\, s^{-f(\log s)}.$$

Since $s \geq a^{cn}$ for some positive real number c, there is a positive constant κ, depending only on a, such that

$$\left| \left(\frac{a}{b}\right)^n - \frac{p}{q} \right| \geq q^{-2}\, a^{-\kappa n (\log\log n)^{1/4} (\log n)^{-1/4}}. \tag{4.6}$$

Observe that (4.6) remains true when (4.3) does not hold. Then, we get that every partial quotient of p'_n/q'_n is less than $a^{\kappa n (\log\log n)^{1/4} (\log n)^{-1/4}}$. Consequently, the length of the continued fraction expansion of p'_n/q'_n is at least equal to some constant times $(\log n)^{1/4} \cdot (\log\log n)^{-1/4}$. Since p'_n/q'_n is a convergent to $(a/b)^n$, this is also a lower bound for the length of the continued fraction expansion of $(a/b)^n$. □

Theorem 3 is a small step towards the resolution of the following question.

Problem 2. *To give an effective lower bound for* $\mathcal{L}\big((a/b)^n\big)$.

Theorem P has been extended by Corvaja and Zannier [7] to quotients of power sums. Recall that the continued fraction expansion of a real number θ is eventually periodic if, and only if, θ is a quadratic surd. Mendès France [29] asked whether for every real quadratic irrational ξ and every positive M, there exist integers n such that the length of the period of the continued fraction expansion of ξ^n exceeds M. This question was completely solved by Corvaja and Zannier [6]. Furthermore, results on the length of the period of the continued fraction for values of the square root of power sums have been given by Bugeaud and Luca [4] and by Scremin [39].

The key tool for the proofs of the results from [4, 6, 7, 39] is a powerful, deep generalization of Roth's theorem, namely the Schmidt Subspace Theorem [38] (and, more precisely, its non-Archimedean extension, worked out by Schlickewei [37]). However, it does not seem to us that the Subspace Theorem and its relatives could be of some help for improving upon Theorems 1 to 3 from the present paper.

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