On the complexity of a putative counterexample to the p-adic Littlewood conjecture

Dmitry Badziahin, Yann Bugeaud, Manfred Einsiedler, & Dmitry Kleinbock

Abstract. *Let $\| \cdot \|$ denote the distance to the nearest integer and, for a prime number p, let $| \cdot |_p$ denote the p-adic absolute value. In 2004, de Mathan and Teulié asked whether $\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0$ holds for every badly approximable real number α and every prime number p. Among other results, we establish that, if the complexity of the sequence of partial quotients of a real number α grows too rapidly or too slowly, then their conjecture is true for the pair (α, p) with p an arbitrary prime.*

1. Introduction

A famous open problem in simultaneous Diophantine approximation is the Littlewood conjecture which claims that, for every given pair (α, β) of real numbers, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{1.1}$$

where $\| \cdot \|$ denotes the distance to the nearest integer. The first significant contribution on this question goes back to Cassels and Swinnerton-Dyer [7] who showed that (1.1) holds when α and β belong to the same cubic field. Despite some recent remarkable progress [20, 9] the Littlewood conjecture remains an open problem.

Let $\mathcal{D} = (d_k)_{k \geq 1}$ be a sequence of integers greater than or equal to 2. Set $e_0 = 1$ and, for any $n \geq 1$,

$$e_n = \prod_{1 \leq k \leq n} d_k.$$

For an integer q, set

$$w_{\mathcal{D}}(q) = \sup\{n \geq 0 : q \in e_n \mathbf{Z}\}$$

2000 *Mathematics Subject Classification :* 11J04; 11J61, 11J83, 37A35, 37A45, 37D40. Keywords: Diophantine approximation, Littlewood conjecture, complexity, continued fractions, measure rigidity.

D.B. was supported by grant EP/L005204/1 from EPSRC.

D.K. was supported in part by NSF grant DMS-1101320.

and

$$|q|_{\mathcal{D}} = 1/e_{w_{\mathcal{D}}(q)} = \inf\{1/e_n : q \in e_n \mathbf{Z}\}.$$

When $\mathcal{D}$ is the constant sequence equal to p, where p is a prime number, then $|\cdot|_{\mathcal{D}}$ is the usual p-adic value $|\cdot|_p$, normalized by $|p|_p = p^{-1}$. In analogy with the Littlewood conjecture, de Mathan and Teulié [16] proposed in 2004 the following conjecture.

Mixed Littlewood Conjecture. *For every real number α and every sequence $\mathcal{D}$ of integers greater than or equal to* 2, *we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0 \tag{1.2}$$

holds for every real number α.

Obviously, (1.2) holds if α is rational or has unbounded partial quotients. Thus, we only consider the case when α is an element of the set ***Bad*** of badly approximable numbers, where

$$\boldsymbol{Bad} = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

De Mathan and Teulié proved that (1.2) and even the stronger statement

$$\liminf_{q \to +\infty} q \cdot \log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} < +\infty \tag{1.3}$$

holds for every quadratic irrational α when the sequence $\mathcal{D}$ is bounded.

We highlight the particular case when $\mathcal{D}$ is the constant sequence equal to a prime number.

p-adic Littlewood Conjecture. *For every real number α and every prime number p, we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{1.4}$$

Einsiedler and Kleinbock [10] established that, for every given prime number p, the set of real numbers α such that the pair (α, p) does not satisfy (1.4) has zero Hausdorff dimension. They also explained how to modify their proof to get an analogous result when $\mathcal{D}$ is the constant sequence equal to $d \geq 2$ (not necessarily prime).

In an opposite direction, by means of a subtle Cantor-type construction, Badziahin and Velani [2] established that, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the set of real numbers α such that

$$\inf_{q \geq 3} q \cdot \log q \cdot \log\log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} > 0$$

has full Hausdorff dimension. Moreover, they showed that, for $\mathcal{D} = (2^{2^n})_{n \geq 1}$, the set of real numbers α such that

$$\inf_{q \geq 16} q \cdot \log\log q \cdot \log\log\log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} > 0$$

has full Hausdorff dimension.

Regarding explicit examples of real numbers α satisfying (1.4), it was proved in [5] that, if the sequence of partial quotients of the real number α contains arbitrarily long concatenations of a given finite block, then the pair (α, p) satisfies (1.4) for any prime number p. A precise statement is as follows.

Theorem BDM. *Let $\alpha = [a_0; a_1, a_2, \ldots]$ be in **Bad**. Let $T \geq 1$ be an integer and $b_1, \ldots, b_T$ be positive integers. If there exist two sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ of positive integers with $(h_k)_{k\geq 1}$ being unbounded and*

$$a_{m_k+j+nT} = b_j, \quad \text{for every } j = 1, \ldots, T \text{ and every } n = 0, \ldots, h_k - 1,$$

then the pair (α, p) satisfies (1.4) for any prime number p.

The main purposes of the present note is to give new combinatorial conditions ensuring that a real number satisfies the p-adic (Theorem 2.1) and the mixed (Corollary 2.4) Littlewood conjectures and to study the complexity of the continued fraction expansion of a putative counterexample to (1.2) or (1.4); see Theorem 2.1 and Corollary 2.4 below. Furthermore, in Section 3 we make a connection between the mixed Littlewood conjecture and a problem on the evolution of the sequence of the Lagrange constants of the multiples of a given real number. Proofs of our results are given in Sections 4 to 6.

Throughout the paper, we assume that the reader is familiar with the classical results from the theory of continued fractions.

2. New results on the mixed and the p-adic Littlewood conjectures

To present our results, we adopt a point of view from combinatorics on words. We look at the continued fraction expansion of a given real number α as an infinite word.

For an infinite word $\mathbf{w} = w_1 w_2 \ldots$ written on a finite alphabet and for an integer $n \geq 1$, we denote by $p(n, \mathbf{w})$ the number of distinct blocks of n consecutive letters occurring in $\mathbf{w}$, that is,

$$p(n, \mathbf{w}) := \mathrm{Card}\{w_{\ell+1} \ldots w_{\ell+n} : \ell \geq 0\}.$$

The function $n \mapsto p(n, \mathbf{w})$ is called the *complexity function* of $\mathbf{w}$. For a badly approximable real number $\alpha = [a_0; a_1, a_2, \ldots]$, we set

$$p(n, \alpha) := p(n, a_1 a_2 \ldots), \quad n \geq 1,$$

and we call $n \mapsto p(n, \alpha)$ the *complexity function* of α. Observe that, for all positive integers n, n', we have

$$p(n + n', \alpha) \leq p(n, \alpha) \cdot p(n', \alpha),$$

thus, the sequence $(\log p(n, \alpha))_{n\geq 1}$ is subadditive and the sequence $((\log p(n, \alpha))/n)_{n\geq 1}$ converges.

In the present paper we show that if the real number α is a counterexample to the p-adic Littlewood conjecture, then its complexity function $n \mapsto p(n, \alpha)$ can neither increase too slowly nor too rapidly as n tends to infinity.

2.1. High complexity case

For a positive integer K, set

$$\boldsymbol{Bad}_K := \{\alpha = [a_0; a_1, a_2 \ldots] \; : \; a_i \leq K, i \geq 1\}$$

and observe that the set of badly approximable numbers is the union over all positive integers K of the sets $\boldsymbol{Bad}_K$. It immediately follows from the definition of the complexity function $n \mapsto p(n, \alpha)$ that, for every α in $\boldsymbol{Bad}_K$ and every $n \geq 1$, we have

$$p(n, \alpha) \leq K^n.$$

Consequently, the complexity function of the continued fraction of any number α in $\boldsymbol{Bad}$ grows at most exponentially fast. Our first result shows that a putative counterexample to the p-adic Littlewood conjecture must satisfy a much more restrictive condition.

Theorem 2.1. *Let α be a real number satisfying*

$$\lim_{n \to \infty} \frac{\log p(n, \alpha)}{n} > 0. \tag{2.1}$$

Then, for every prime number p, we have

$$\inf_{q \geq 1} q \cdot ||q\alpha|| \cdot |q|_p = 0.$$

In other words the complexity of the continued fraction expansion of every potential counterexample to the p-adic Littlewood conjecture must grow subexponentially.

Our proof relies on a p-adic generalisation of the measure classification result in [14] (provided by [11]), the connection between such dynamical results and the Diophantine approximation problem as was used before in [9, 10], and the observation that one counterexample actually gives rise to many more counterexamples (see Proposition 4.1).

2.2. Low complexity case

A well-known result of Morse and Hedlund [17, 18] asserts that $p(n, \mathbf{w}) \geq n + 1$ for $n \geq 1$, unless $\mathbf{w}$ is ultimately periodic (in which case there exists a constant C such that $p(n, \mathbf{w}) \leq C$ for $n \geq 1$). Infinite words $\mathbf{w}$ satisfying $p(n, \mathbf{w}) = n+1$ for every $n \geq 1$ do exist and are called *Sturmian words*. In the present paper we show that if α is a counterexample to the p-adic (or, even, to the mixed) Littlewood conjecture, then the lower bound for the complexity function of α must be stronger than this estimate. Before stating our result we give a classical definition (see e.g. [1]).

Definition 2.2. *An infinite word $\mathbf{w}$ is recurrent if every finite block occurring in $\mathbf{w}$ occurs infinitely often.*

Classical examples of recurrent infinite words include periodic words, Sturmian words, the Thue–Morse word, etc.

Theorem 2.3. *Let $(a_k)_{k\geq 1}$ be a sequence of positive integers. If there exists an integer $m \geq 0$ such that the infinite word $a_{m+1}a_{m+2}\ldots$ is recurrent, then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0.$$

As a particular case, Theorem 2.3 asserts that (1.2) holds for every quadratic number α and every sequence $\mathcal{D}$ of integers greater than or equal to 2, including unbounded sequences (unlike in [16], where $\mathcal{D}$ is assumed to be bounded). Unlike in [16], our proof does not use p-adic analysis.

Theorem 2.3 implies a non-trivial lower bound for the complexity function of the continued fraction expansion of a putative counterexample to (1.2).

Corollary 2.4. *Let $\mathcal{D}$ be a sequence of integers greater than or equal to 2 and α be a real number such that the pair $(\alpha, \mathcal{D})$ is a counterexample to the mixed Littlewood conjecture (i.e., does not satisfy (1.2)). Then, the complexity function of α satisfies*

$$\lim_{n\to+\infty} p(n, \alpha) - n = +\infty.$$

The next corollary highlights a special family of infinite recurrent words. A finite word $w_1 \ldots w_n$ is called a *palindrome* if $w_{n+1-i} = w_i$ for $i = 1, \ldots, n$.

Corollary 2.5. *Let $(a_k)_{k\geq 1}$ be a sequence of positive integers. If there exists an increasing sequence $(n_j)_{j\geq 1}$ of positive integers such that $a_1 \ldots a_{n_j}$ is a palindrome for $j \geq 1$, then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0.$$

As shown in Section 6, our approach allows us to give an alternative proof to (1.3) when α is quadratic irrational and $\mathcal{D}$ is bounded. Furthermore, we are able to quantify Theorem 2.3 for a special class of recurrent words.

Definition 2.6. *We say that an infinite word* $\mathbf{w}$ *is linearly recurrent if there exists $C > 1$ such that the distance between two consecutive occurrences of any finite block W occurring in* $\mathbf{w}$ *is bounded by C times the length of W.*

We obtain the following quantitative result.

Theorem 2.7. *Let $(a_k)_{k\geq 1}$ be a bounded sequence of positive integers. If there exists an integer $m \geq 0$ such that the infinite word $a_{m+1}a_{m+2}\ldots$ is linearly recurrent, then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies*

$$\liminf_{q\to+\infty} q \cdot (\log\log q)^{1/2} \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} < +\infty.$$

2.3. Comparison with the Littlewood conjecture

According to Section 5 of [16], the initial motivation of the introduction of the mixed Littlewood conjecture was the study of a problem quite close to the Littlewood conjecture, but seemingly a little simpler, with the hope to find new ideas suggesting a possible approach towards the resolution of the Littlewood conjecture itself.

We are not aware of any relationship between both conjectures. For instance, a real number α being given, we do not know any connection between the two statements '(1.2) holds for every sequence $\mathcal{D}$' and '(1.1) holds for every real number β'.

The interested reader is directed to [4] for a survey of recent results and developments on and around the Littlewood conjecture and its mixed analogue. He will notice that the state-of-the-art regarding the Littlewood and the p-adic Littlewood conjectures is essentially, but not exactly, the same.

For instance, Theorem 5 in [15] asserts that for every real number α with (2.1), we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0,$$

for every real number β. This is the exact analogue to Theorem 2.1 above. However, the low complexity case remains very mysterious for the Littlewood conjecture, since we even do not know whether or not it holds for the pair $(\sqrt{2}, \sqrt{3})$.

3. On the Lagrange constants of the multiples of a real number

Our main motivation was the study of the p-adic and the mixed Littlewood conjectures. However, the proofs of Theorems 2.3 and 2.8 actually give us much stronger results on the behaviour of the Lagrange constants of the multiples of certain real numbers.

Definition 3.1. *The Lagrange constant $c(\alpha)$ of an irrational real number α is the quantity*

$$c(\alpha) := \liminf_{q \to +\infty} q \cdot ||q\alpha||.$$

Clearly, α is in ***Bad*** if and only if $c(\alpha) > 0$. A classical theorem of Hurwitz (see [19, 3]) asserts that $c(\alpha) \leq 1/\sqrt{5}$ for every irrational real number α.

For any positive integer n and any badly approximable number α we have

$$\frac{c(\alpha)}{n} \leq c(n\alpha) \leq nc(\alpha). \tag{3.1}$$

To see this, note that

$$\left|n\alpha - \frac{np}{q}\right| = n\left|\alpha - \frac{p}{q}\right|$$

and

$$\left|\alpha - \frac{p}{nq}\right| = \frac{1}{n}\left|n\alpha - \frac{np}{nq}\right|.$$

The first general result on the behaviour of the sequence $(c(n\alpha))_{n \geq 1}$ is Theorem 1.11 of Einsiedler, Fishman, and Shapira [8], reproduced below.

Theorem EFS. *Every badly approximable real number α satisfies*

$$\inf_{n\geq 1} c(n\alpha) = 0.$$

Theorem EFS motivates the following question.

Problem 3.2. *Prove or disprove that every badly approximable real number α satisfy*

$$\lim_{n\to+\infty} c(n\alpha) = 0. \tag{3.2}$$

There is a clear connection between Problem 3.2 and the mixed Littlewood conjecture. Indeed, if α satisfies (3.2) and if $\mathcal{D}$ is as in Section 1, then, keeping the notation from this section, for every $\varepsilon > 0$, there exists a positive integer n such that $c(e_n\alpha) < \varepsilon$. Consequently, there are arbitrarily large integers q with the property that

$$q \cdot ||qe_n\alpha|| < \varepsilon$$

thus,

$$qe_n \cdot ||qe_n\alpha|| \cdot |qe_n|_{\mathcal{D}} < \varepsilon,$$

since $|qe_n|_{\mathcal{D}} \leq 1/e_n$. This proves that (1.2) holds for the pair $(\alpha, \mathcal{D})$.

Our proof of Theorem 2.3 actually gives the following stronger result.

Theorem 3.3. *Let $(a_k)_{k\geq 1}$ be a sequence of positive integers. If there exists an integer $m \geq 0$ such that the infinite word $a_{m+1}a_{m+2}\ldots$ is recurrent, then the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies (3.2) and, moreover,*

$$c(n\alpha) \leq \frac{8q_m^2}{n}, \quad \text{for } n \geq 1,$$

where q_m denotes the denominator of the rational number $[0; a_1, \ldots, a_m]$.

In view of the left-hand inequality of (3.1), the conclusion of Theorem 3.3 is nearly best possible.

Using the same arguments as for the proof of Corollary 2.4, we establish that the complexity function of a real number which does not satisfy (3.2) cannot be too small.

Corollary 3.4. *Let α be a real number such that*

$$\sup_{n\geq 1} n\, c(n\alpha) = +\infty.$$

Then, the complexity function of α satisfies

$$\lim_{n\to+\infty} p(n, \alpha) - n = +\infty.$$

4. High complexity case

We follow the interpretation of the p-adic Littlewood conjecture used by Einsiedler and Kleinbock in [10] and consider the following more general problem:

Generalized p-adic Littlewood Conjecture. *For every prime number p and for every pair $(u, v) \in \mathbf{R}_{>\mathbf{0}} \times \mathbf{Q}_p$ we have*

$$\inf_{a \in \mathbf{N}, b \in \mathbf{N} \cup \{0\}} \max\{|a|, |b|\} \cdot |au - b| \cdot |av - b|_p = 0. \tag{4.1}$$

Clearly α satisfies the p-adic Littlewood conjecture (i.e. (1.4)) if and only if $-\alpha$ satisfies the p-adic Littlewood conjecture. For that reason we restrict our attention to positive numbers. Moreover, one can check (see for example [10], a discussion after Theorem 1.2) that if α is a counterexample to the p-adic Littlewood conjecture then $(\alpha^{-1}, 0)$ is a counterexample to the above generalized p-adic Littlewood conjecture. The next proposition goes further and shows that one counterexample α to the p-adic Littlewood conjecture provides a countable collection of counterexamples to the generalized p-adic Littlewood conjecture.

Proposition 4.1. *Let p be a prime number and $\alpha > 0$ an irrational number. Let ε be in $(0, 1/2]$ and assume that*

$$\inf_{q \geq 1} q \cdot ||q\alpha|| \cdot |q|_p > \varepsilon.$$

Then, we have

$$\inf_{a \in \mathbf{N}, b \in \mathbf{N} \cup \{0\}} \max\{a, b\} \cdot \left| a \cdot \frac{||q_n\alpha||}{||q_{n-1}\alpha||} - b \right| \cdot \left| a \cdot \left(\frac{q_n}{q_{n-1}} \right) + b \right|_p > \frac{\varepsilon^2}{4}, \tag{4.2}$$

where $(q_k)_{k \geq 1}$ is the sequence of the denominators of the convergents to α.

Note that, writing $\alpha = [a_0; a_1, a_2, \ldots]$, we have

$$\frac{q_n}{q_{n-1}} = [0; a_n, a_{n-1}, \ldots, a_1] \quad \text{and} \quad \frac{||q_n\alpha||}{||q_{n-1}\alpha||} = [0; a_{n+1}, a_{n+2}, \ldots] \in (0, 1),$$

for every $n \geq 1$.

Proof of Proposition 4.1. We assume that $||q\alpha|| \cdot |q|_p > \varepsilon/q$ for every integer $q \geq 1$. We use the classical estimate from the theory of continued fractions

$$||q_n\alpha|| < q_{n+1}^{-1} < (a_n q_n)^{-1}.$$

It implies that $q_n||q_n\alpha|| < a_n^{-1}$ and hence $a_n < \varepsilon^{-1}$. In other words

$$\alpha \in \boldsymbol{Bad}_N, \quad \text{where } N := [\varepsilon^{-1}] \quad \text{and} \quad q_n < (N+1)q_{n-1}.$$

Now choose some $a \geq 1$, $b \geq 0$, and modify the left hand side of (4.2):

$$\max\{a, b\} \cdot \left| a \frac{||q_n\alpha||}{||q_{n-1}\alpha||} - b \right| \cdot \left| a \left(\frac{q_n}{q_{n-1}} \right) + b \right|_p$$

$$= \frac{\max\{a,b\}}{||q_{n-1}\alpha|| \cdot |q_{n-1}|_p} \cdot \big| a||q_n\alpha|| - b||q_{n-1}\alpha|| \, \big| \cdot |aq_n + bq_{n-1}|_p.$$

Since $||q_{n-1}\alpha|| < (a_{n-1}q_{n-1})^{-1} \le q_{n-1}^{-1}$, the first term in this product is bounded from below by $\max\{a,b\} \cdot q_{n-1}$, which in turn is $\ge (N+1)^{-1} \cdot \max\{aq_n, bq_{n-1}\}$. The second term is estimated as follows:

$$\big| a \cdot ||q_n\alpha|| - b \cdot ||q_{n-1}\alpha|| \, \big| = |(aq_n + bq_{n-1})\alpha - (ap_n + bp_{n-1})| \ge ||(aq_n + bq_{n-1})\alpha||.$$

Therefore, a lower bound of the whole product is

$$\frac{\max\{aq_n, bq_{n-1}\}}{(N+1)} \cdot ||(aq_n + bq_{n-1})\alpha|| \cdot |aq_n + bq_{n-1}|_p$$

$$\ge \frac{\varepsilon}{N+1} \cdot \frac{\max\{aq_n, bq_{n-1}\}}{aq_n + bq_{n-1}} \ge \frac{\varepsilon}{2(N+1)}.$$

Since $N+1 \le \varepsilon^{-1} + 2$ and $\varepsilon \le 1/2$, this proves the proposition. □

In [10] the authors showed that the set of counterexamples to the generalized p-adic Littlewood conjecture is rather small. More precisely, the following theorem (Theorem 5.2 in [10]) was stated there, along with a scheme of proof.

Theorem EK. *Let p be a prime number. Then the set of pairs $(u,v) \in \mathbf{R} \times \mathbf{Q}_p$ which do not satisfy*

$$\liminf_{a,b \in \mathbf{Z}} |a| \cdot |au - b| \cdot |av - b|_p = 0. \tag{4.3}$$

is a countable union of sets of box dimension zero.

The outlined proof was based on a theorem due to Einsiedler and Lindenstrauss [11] which at the time of publication of [10] had not appeared yet. In the present paper we reexamine the methods of [10] and provide a more precise result regarding the set of pairs $(u,v) \in [0,1] \times \mathbf{Z}_p$ which do not satisfy (4.1).

Theorem 4.2. *For every prime number p, the set of pairs $(u,v) \in [0,1] \times \mathbf{Z}_p$ which do not satisfy (4.1) is a countable union of sets of box dimension zero. Moreover, for every $\varepsilon > 0$ the set of $(u,v) \in [0,1] \times \mathbf{Z}_p$ which satisfy*

$$\inf_{a \in \mathbf{N}, b \in \mathbf{N} \cup \{0\}} \max\{a,b\} \cdot |au - b| \cdot |av - b|_p \ge \varepsilon. \tag{4.4}$$

has box dimension zero.

The proof of Theorem 4.2 is postponed to Section 5.

Roughly speaking, this theorem together with Proposition 4.1 implies that for every counterexample $\alpha = [0; a_1, a_2, \ldots]$ to the p-adic Littlewood conjecture the set

$$\{[0; a_m, a_{m+1}, \ldots] \ : \ m \ge 1\}$$

has box dimension zero. Let us now show that this in turn implies the statement of Theorem 2.1.

Proof of Theorem 2.1. We will prove the contrapositive of the theorem. So assume that α is a counterexample to (1.4). By the homogeneity of (1.4) we may assume that $\alpha = [a_0; a_1, \ldots]$ is positive. By Proposition 4.1 this leads to a countable collection

$$B = \left\{(\alpha_n, \beta_n) = \left([0; a_{n+1}, a_{n+2}, \ldots], \frac{q_n}{q_{n-1}}\right) : n \geq 1\right\}$$

of pairs in $[0,1] \times \mathbf{Q}_p$ that all satisfy (4.2). By Theorem 4.2, the set

$$B \cap [0,1] \times \mathbf{Z}_p = \{(\alpha_n, \beta_n) : n \geq 1 \text{ and } |q_{n-1}|_p = 1\}$$

has box dimension zero.

Let $N \geq 1$ be such that $a_n \in \{1, \ldots, N\}$ for all $n \geq 1$. We set

$$S' = \{\ell \geq 1 : |q_{\ell-1}|_p = 1\}.$$

Let $\delta > 0$ be arbitrary and let $\pi_\infty : [0,1] \times \mathbf{Z}_p \to [0,1]$ denote the projection to the real coordinate. Then, the definition of box dimension shows that, for all sufficiently large n, the set

$$B' = \{\alpha_\ell : \ell \in S'\} = \pi_\infty(B \cap [0,1] \times \mathbf{Z}_p)$$

can be covered by $s_n \leq e^{\delta n}$ intervals $I_1, \ldots, I_{s_n}$ of size $(1+N)^{-2n}$.

We also define another disjoint collection of intervals. To any word $w = w_1 \ldots w_n$ in $\{1, \ldots, N\}^n$, we associate the interval $[w]$ composed of the real numbers in $(0,1)$ whose first n partial quotients are $w_1, \ldots, w_n$. The basic properties of continued fractions show that the length of $[w]$ is at most 2^{-n+2} and at least $(1+N)^{-2n}$. It follows that a given interval I_j from the above list can intersect at most two (neighbouring) intervals of the form $[w]$ for $w \in \{1, \ldots, N\}^n$. This implies that

$$\mathrm{Card}\{a_{\ell+1} \ldots a_{\ell+n} : \ell \in S'\} \leq 2s_n \leq 2e^{\delta n}.$$

To remove the restriction $\ell \in S'$ in the above counting, we note that $\ell \notin S'$ implies $\ell+1 \in S'$ since $q_{\ell-1}$ and q_ℓ are coprime, by the properties of continued fractions. Therefore,

$$p(n, \alpha) = \mathrm{Card}\{a_{\ell+1} \ldots a_{\ell+n} : \ell \geq 0\} \leq 2s_n + 2Ns_{n-1} \leq 2(1+N)e^{\delta n}.$$

As $\delta > 0$ was arbitrary, the theorem follows. □

5. Measure Rigidity and the Proof of Theorem 4.2.

We follow the strategy outlined in [10] (which in turn generalizes the argument from [9]). For this, we set

$$G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{Q}_p),\ \Gamma = \mathrm{SL}_2(\mathbf{Z}[1/p]),\ X = G/\Gamma\,, \tag{5.1}$$

where $\mathbf{Z}[1/p]$ is embedded diagonally via $a \mapsto (a,a)$ in $\mathbf{R} \times \mathbf{Q}_p$. In other words, for $(A,B) \in G$, points $x = (A,B)\Gamma \in X$ are identified with unimodular lattices $(A,B)\mathbf{Z}[1/p]$ in $\mathbf{R}^2 \times \mathbf{Q}_p^2$ that are generated by the column vectors of A and B.

We also set

$$\psi(t,n) = \left(\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}, \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix} \right) \tag{5.2}$$

for $(t,n) \in \mathbf{R} \times \mathbf{Z}$, and define the cone

$$C = \{(t,n) \mid n \geq 0, e^t p^{-n} \geq 1\}. \tag{5.3}$$

Furthermore, for $(u,v) \in \mathbf{R} \times \mathbf{Q}_p$, we define the coset (which we will think of as a point)

$$x_{u,v} = \left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \right) \Gamma.$$

Compact subsets of X can be characterized by the analogue of Mahler's compactness criterion (see [10], Theorem 2.1) so that a subset $K \subset X$ has compact closure if and only if there exists some $\delta > 0$ so that $K \subset K_\delta$, with

$$K_\delta = \left\{ g\Gamma \in K : g\mathbf{Z}[1/p]^2 \cap B_\delta^{\mathbf{R}^2 \times \mathbf{Q}_p^2} = \{0\} \right\},$$

where B_δ denotes the ball of radius δ centered at zero.

In [10], Proposition 2.2, a connection between unboundedness of the cone orbit

$$\psi(C)x_{u,v} = \{\psi(t,n)x_{u,v} : (t,n) \in C\}$$

and (4.3) is given. However, we will need to show the following refinement.

Proposition 5.1. *Let $(u,v) \in (0,1) \times \mathbf{Z}_p$ and $0 < \varepsilon < 1$ be arbitrary. If (u,v) satisfies (4.4), then $\psi(C)x_{u,v} \subset K_\delta$ for $\delta = (\varepsilon/2)^{1/3}$.*

We note that in [10] the converse of the above implication, in a slightly different form, has also been claimed (without a proof). But the other direction is not clear and luckily is also not needed for the proof of our results (or the results of [10]).

Proof of Proposition 5.1. Take $\delta = (\varepsilon/2)^{1/3}$ and suppose that $\psi(C)x_{u,v}$ is not contained in K_δ; that is, there exists a pair (t,n) with $n \geq 0$ and $e^t p^{-n} \geq 1$ such that $\psi(t,n)x_{u,v}\mathbf{Z}[1/p]$ contains a nonzero element in $B_\delta^{\mathbf{R}^2 \times \mathbf{Q}_p^2}$.

Clearly, $\psi(t,n)x_{u,v}$ is generated by

$$\left(\begin{pmatrix} e^{-t} \\ e^t u \end{pmatrix}, \begin{pmatrix} p^n \\ p^{-n} v \end{pmatrix} \right) \text{ and } \left(\begin{pmatrix} 0 \\ e^t \end{pmatrix}, \begin{pmatrix} 0 \\ p^{-n} \end{pmatrix} \right).$$

However, since $\psi(t,n)x_{u,v}$ is a $\mathbf{Z}[1/p]$-module, the vectors

$$\left(\begin{pmatrix} e^{-t} p^{-n} \\ e^t p^{-n} u \end{pmatrix}, \begin{pmatrix} 1 \\ p^{-2n} v \end{pmatrix} \right) \text{ and } \left(\begin{pmatrix} 0 \\ e^t p^{-n} \end{pmatrix}, \begin{pmatrix} 0 \\ p^{-2n} \end{pmatrix} \right)$$

are also generators. Therefore, there exists some nonzero $(a,b) \in \mathbf{Z}[1/p]^2$ such that

$$\left(\begin{pmatrix} e^{-t}p^{-n}a \\ e^{t}p^{-n}(au-b) \end{pmatrix}, \begin{pmatrix} a \\ p^{-2n}(av-b) \end{pmatrix} \right) \in \mathbf{R}^2 \times \mathbf{Q}_p^2$$

is δ-small. In particular, $|a|_p$ is less than δ, which implies that $a \in \mathbf{Z}$. Since $n \geq 0$ and $v \in \mathbf{Z}_p$, the inequality

$$p^{2n}|av-b|_p = |p^{-2n}(av-b)|_p < \delta \tag{5.4}$$

shows that $b \in \mathbf{Z}$ as well. Also, since $|u| < 1$, the inequalities $t \geq 0$, $|e^{-t}p^{-n}a| < \delta$ and

$$|e^{t}p^{-n}(au-b)| < \delta \tag{5.5}$$

imply that

$$e^{-t}p^{-n} \max\{|a|,|b|\} < 2\delta. \tag{5.6}$$

By taking the product of the inequalities (5.4), (5.5) and (5.6), we arrive at

$$\max\{|a|,|b|\} \cdot |au-b| \cdot |av-b|_p < 2\delta^3 = \varepsilon.$$

Also note that $u > 0$, (5.5) and $e^{t}p^{-n} \geq 1$ imply that a and b have the same sign (in the sense that $ab \geq 0$). Without loss of generality we may assume $a, b \geq 0$. Similarly, $b = 0$ implies $a = 0$ and contradicts our choice of (a,b). However, $a \geq 1$ and $b \geq 0$ contradicts (4.4). Consequently, $\psi(C)x_{u,v}$ is contained in K_δ. $\square$

We also need the following partial measure classification result.

Theorem 5.2. *The Haar measure is the only ψ-invariant and ergodic probability measure μ on X for which some $(t,n) \in \mathbf{R} \times \mathbf{Z}$ has positive entropy $h_\mu(\psi(t,n)) > 0$.*

Theorem 5.2 follows from Theorem 1.3 of [11], recalled below. We write ∞ for the archimedean place of $\mathbf{Q}$ and $\mathbf{Q}_\infty = \mathbf{R}$. Moreover, for a finite set S of places we define $\mathbf{Q}_S = \prod_{v \in S} \mathbf{Q}_v$ for the corresponding product of the local fields.

Theorem EL. *Let $\mathbf{G}$ be a $\mathbf{Q}$-almost simple linear algebraic group, let S be a finite set of places containing the archimedean place ∞, let $\Gamma < G = \mathbf{G}(\mathbf{Q}_S)$ be an arithmetic lattice, and let $X = G/\Gamma$. Finally, let A be the direct product of maximal $\mathbf{Q}_v$-diagonalizable algebraic subgroups of $\mathbf{G}(\mathbf{Q}_v)$ for $v \in S$. Let μ be an A-invariant and ergodic probability measure on X. Suppose in addition that μ is not supported on any periodic orbit $g\mathbf{L}(\mathbf{Q}_S)\Gamma$ for any $g \in G$ and proper reductive $\mathbf{Q}$-subgroup $\mathbf{L} < \mathbf{G}$, that* $\operatorname{rank}(A) \geq 2$, *and that $h_\mu(a) > 0$ for some $a \in A$. Then there is a finite index subgroup $L < G$ so that μ is L-invariant and supported on a single L-orbit.*

Proof of Theorem 5.2. We let G, Γ, X be as in (5.1). In the special case $(t,n) = (t,0)$ this follows directly from [14], Theorem 1.1. The method of proof of [14], Theorem 1.1. would in principle also give the general case of Theorem 5.2, but we will instead derive it from the more general Theorem EL.

Assume now μ is a ψ-invariant and ergodic probability measure with positive entropy for some $(t,n) \in \mathbf{R} \times \mathbf{Z}$. Strictly speaking $\mathrm{Im}(\psi) = \psi(\mathbf{R} \times \mathbf{Z})$ does not equal the product A of the full diagonal subgroup of $\mathrm{SL}(\mathbf{R})$ and the full diagonal subgroup of $\mathrm{SL}(\mathbf{Q}_p)$. However, $K = A/\mathrm{Im}(\psi)$ is compact which allows us to define the A-invariant and ergodic measure $\mu_A = \int_K a_*\mu \mathrm{d}(a\mathrm{Im}(\psi))$ with positive entropy for $\psi(t,n)$.

Note that a proper nontrivial reductive subgroup $\mathbf{L}$ of SL_2 must be a diagonalisable subgroup. However, if μ_A would be supported on a single orbit $g\mathbf{L}(\mathbf{Q}_S)\Gamma$ this would mean that μ_A is supported on a single periodic orbit for A and would force entropy to be equal to zero. Therefore, all assumptions to Theorem EL are satisfied and it follows that μ_A is invariant under a finite index subgroup of G. However, G does not have any proper finite index subgroup and so μ_A must be the Haar measure on X. The definition of μ_A now expresses the Haar measure as a convex combination of ψ-invariant measures. By ergodicity of the Haar measure under the action of $\psi(\mathbf{R} \times \mathbf{Z})$ this implies that μ equals the Haar measure on X also. $\square$

Finally, for the proof of Theorem 4.2, we need to quote another result highlighting a connection between entropy and box dimension. Recall that given $g \in G$, the *unstable horospherical subgroup for* g is the maximal subgroup of G such that each of its elements h satisfies $g^{-j}hg^j \to 1$ as $j \to \infty$. What follows is a special case of Proposition 4.1 from [10], cf. also Proposition 9.1 from [9]:

Proposition EK. *Let G, Γ, X be as in (5.1), ψ be as in (5.2) and C as in (5.3). Take $(t,n) \in C$ and let $Y \subset X$ be a compact set such that no ψ-invariant and ergodic probability measure supported on Y has positive entropy for $\psi(t,n)$. Then for any compact subset B of the unstable horospherical subgroup for $\psi(t,n)$ and any $x \in X$ the set*

$$\{u \in B : \psi(C)ux \subset Y\}$$

has box dimension zero.

Proof of Theorem 4.2. Fix some $\varepsilon > 0$, let $\delta = \varepsilon^{1/3}$ and $Y = K_\delta \subset X$. Also pick $t > 0$ such that $(t,1) \in C$. Note that $\{x_{u,v} : (u,v) \in \mathbf{R} \times \mathbf{Q}_p\}$ is the unstable horospherical subgroup for $\psi(t,1)$. By Theorem 5.2 and since Y is a proper closed subset of X, there is no ψ-invariant and ergodic probability measure supported on Y, which is precisely the assumption of Proposition EK. By that result we obtain that the set of $(u,v) \in [0,1] \times \mathbf{Z}_p$ with $\psi(C)x_{u,v} \subset Y$ has box dimension zero. However, by Proposition 5.1, this implies that the set of $(u,v) \in (0,1) \times \mathbf{Z}_p$ satisfiying (4.4) has box dimension zero, and Theorem 4.2 follows. $\square$

6. Low complexity case

6.1. Auxiliary results

We begin with two classical lemmata on continued fractions, whose proofs can be found for example in Perron's book [19].

For positive integers $a_1, \ldots, a_n$, denote by $K_n(a_1, \ldots, a_n)$ the denominator of the rational number $[0; a_1, \ldots, a_n]$. It is commonly called a *continuant*.

Lemma 6.1. *For any positive integers $a_1, \ldots, a_n$ and any integer k with $1 \leq k \leq n-1$, we have*

$$K_n(a_1, \ldots, a_n) = K_n(a_n, \ldots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{n-k}(a_{k+1}, \ldots, a_n) &\leq K_n(a_1, \ldots, a_n) \\ &\leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{n-k}(a_{k+1}, \ldots, a_n). \end{aligned}$$

Lemma 6.2. *Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers. Assume that there exists a positive integer n such that $a_i = b_i$ for any $i = 1, \ldots, n$. We then have $|\alpha - \beta| \leq K_n(a_1, \ldots, a_n)^{-2}$.*

A homogeneous linear recurrence sequence with constant coefficients (*recurrence sequence* for short) is a sequence $(u_n)_{n \geq 0}$ of complex numbers such that

$$u_{n+d} = v_{d-1}u_{n+d-1} + v_{d-2}u_{n+d-2} + \ldots + v_0 u_n \quad (n \geq 0),$$

for some complex numbers $v_0, v_1, \ldots, v_{d-1}$ with $v_0 \neq 0$ and with initial values $u_0, \ldots, u_{d-1}$ not all zero. The positive integer d is called the *order* of the recurrence.

Lemma 6.3. *Let $(u_n)_{n \geq 1}$ be a recurrence sequence of order d of rational integers. Then, for every prime number p and every positive integer k, the period of the sequence $(u_n)_{n \geq 1}$ modulo p^k is at most equal to $(p^d - 1)p^{k-1}$.*

Proof. See Everest et al. [12], page 47. □

Lemma 6.4. *Let $\alpha = [a_0; a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}, b_0, \ldots, b_{s-1}, \ldots]$ be a quadratic irrational number and denote by $(p_n/q_n)_{n \geq 0}$ the sequence of its convergents. Then, there exists an integer t such that*

$$q_{n+2s} - t q_{n+s} + (-1)^s q_n = 0$$

for $n \geq r$. In particular, the sequence $(q_n)_{n \geq 0}$ satisfies a linear recurrence with constant integral coefficients.

Proof. This result is included in the proof of Theorem 1 in [13]. □

6.2. Proofs

Preliminaries.

Without any loss of generality, we consider real numbers in $(0, 1)$. We associate to every real irrational number $\alpha := [0; a_1, a_2, \ldots]$ the infinite word $\mathbf{a} := a_1 a_2 \ldots$ formed by the sequence of the partial quotients of its fractional part. Set

$$p_{-1} = q_0 = 1, \quad p_0 = q_{-1} = 0,$$

and

$$\frac{p_n}{q_n} = [0; a_1, \ldots, a_n], \quad \text{for } n \geq 1.$$

By the theory of continued fractions, we have

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_1].$$

This is one of the key tools of our proofs.

Proof of Theorems 2.3 and 3.3.

Assume that the infinite word $a_{m+1}a_{m+2}\ldots$ is recurrent. Then, there exists an increasing sequence of positive integers $(n_j)_{j\geq 1}$ such that

$$a_{m+1}a_{m+2}\ldots a_{m+n_j} \text{ is a suffix of } a_{m+1}a_{m+2}\ldots a_{m+n_{j+1}}, \text{ for } j \geq 1.$$

Say differently, there are finite words $V_1, V_2, \ldots$ such that

$$a_{m+1}a_{m+2}\ldots a_{m+n_{j+1}} = V_j a_{m+1}a_{m+2}\ldots a_{m+n_j}, \text{ for } j \geq 1.$$

Actually, these properties are equivalent.

Let $\ell \geq 2$ be an integer. Let $k \geq \ell^2 + 1$ be an integer. By Dirichlet's *Schubfachprinzip*, there exist integers i, j with $1 \leq i < j \leq k$ such that

$$q_{m+n_i} \equiv q_{m+n_j} \pmod{\ell}, \quad q_{m+n_i-1} \equiv q_{m+n_j-1} \pmod{\ell}$$

and j is minimal with this property.

Setting

$$Q := |q_{m+n_i}q_{m+n_j-1} - q_{m+n_i-1}q_{m+n_j}|,$$

we observe that

$$\ell \text{ divides } Q, \tag{6.1}$$

and we derive from Lemma 6.2 that

$$\begin{aligned} 0 < Q &= q_{m+n_i}q_{m+n_j}\left|\frac{q_{m+n_j-1}}{q_{m+n_j}} - \frac{q_{m+n_i-1}}{q_{m+n_i}}\right| \\ &\leq q_{m+n_i}q_{m+n_j}K(a_{m+n_i}, \ldots, a_{m+1})^{-2}, \end{aligned}$$

since the n_i first partial quotients of q_{m+n_j-1}/q_{m+n_j} and q_{m+n_i-1}/q_{m+n_i} are the same, namely $a_{m+n_i}, \ldots, a_{m+1}$. Furthermore, we have

$$||Q\alpha|| \leq ||q_{m+n_i}(q_{m+n_j-1}\alpha)|| + ||q_{m+n_i-1}(q_{m+n_j}\alpha)|| \leq 2q_{m+n_i}q_{m+n_j}^{-1}.$$

Using that

$$q_{m+n_i} \leq 2q_m K(a_{m+n_i}, \ldots, a_{m+1}),$$

by Lemma 6.1, we finally get

$$Q \cdot ||Q\alpha|| \leq 8q_m^2. \tag{6.2}$$

It then follows from (6.1) and (6.2) that

$$Q \cdot ||Q\alpha|| \cdot |Q|_\ell \leq 8q_m^2 \ell^{-1}, \tag{6.3}$$

where $|Q|_\ell$ is equal to ℓ^{-a} if ℓ^a divides Q but ℓ^{a+1} does not. Since ℓ can be an arbitrary prime power, this proves Theorem 2.3.

Our proof shows that there are arbitrarily large integers q such that

$$q\ell \cdot ||q(\ell\alpha)|| \leq 8q_m^2,$$

which implies that

$$c(\ell\alpha) \leq \frac{8q_m^2}{\ell},$$

and establishes Theorem 3.3. □

Proof of Corollary 2.4.

Let $\mathbf{a}$ be an infinite Sturmian word. We first claim that every prefix of finite length of $\mathbf{a}$ occurs infinitely often in $\mathbf{a}$. Indeed, otherwise, there would exist a positive integer n, a finite word W and an infinite word $\mathbf{a}'$ such that $\mathbf{a} = W\mathbf{a}'$ and $p(n, \mathbf{a}') \leq n$, which would imply that $\mathbf{a}'$ is ultimately periodic, a contradiction with the assumption that $\mathbf{a}$ is Sturmian.

Let $\mathbf{a}$ be an infinite word on a finite alphabet $\mathcal{A}$ such that there are positive integers k and n_0 with

$$p(n, \mathbf{a}) = n + k, \quad \text{for } n \geq n_0.$$

Then, by a result of Cassaigne [6], there exist finite words W, W_0, W_1 on $\mathcal{A}$ and a Sturmian word $\mathbf{s}$ on $\{0, 1\}$ such that

$$\mathbf{a} = W\phi(\mathbf{s}),$$

where $\phi(\mathbf{s})$ denotes the infinite word obtained by replacing in $\mathbf{s}$ every 0 by W_0 and every 1 by W_1. We conclude by applying Theorem 2.3 with m being the length of W. □

Proof of Corollary 2.5.

It is sufficient to note that, if $a_1 \ldots a_n$ and $a_1 \ldots a_{n'}$ are palindromes with $n' > 2n$, then $a_{n'-n+1} \ldots a_{n'} = a_n \ldots a_1 = a_1 \ldots a_n$. The corollary then follows from Theorem 2.3 applied with $m = 0$. □

Proof of (1.3) when α is a quadratic irrationality and $\mathcal{D}$ is bounded.

Since $\mathcal{D}$ is bounded, every product $e_n = \prod_{1 \leq k \leq n} d_k$ is divisible by a finite collection of prime numbers. Let $p_1, \ldots, p_h$ be these primes and denote by S the set of integers which are divisible only by primes from $\{p_1, \ldots, p_h\}$. Let α be a quadratic real number.

By Lemma 6.4, the sequence $(q_n)_{n\geq 0}$ of denominators of convergents to α is eventually a recurrence sequence of positive integers. By Lemma 6.3, there exists a positive integer C_1 such that, for $i = 1, \ldots, h$ and $v \geq 1$, the sequence $(q_n)_{n\geq 0}$ is eventually periodic modulo p_i^v, with period length at most equal to $C_1 p_i^v$.

Consequently, there exists a positive integer C_2 such that, for every positive integer ℓ in S, the sequence $(q_n)_{n\geq 1}$ modulo ℓ is eventually periodic of period at most $C_2\ell$.

We need to slightly modify the proof of Theorem 2.3. Take $\ell = e_n \in S$. Denote by m the length of the preperiod of $(q_n)_{n\geq 1}$ and by d the length of the period of $(q_n)_{n\geq 1}$ modulo ℓ. Observe that

$$q_m \equiv q_{m+d} \pmod{\ell}, \quad q_{m+1} \equiv q_{m+d+1} \pmod{\ell}.$$

We then set

$$Q := |q_m q_{m+d+1} - q_{m+1} q_{m+d}|$$

and proceed exactly as in the proof of Theorem 2.3 to get that

$$Q \cdot ||Q\alpha|| \leq 2q_{m+1}^2.$$

Noticing that $|Q|_{\mathcal{D}} \leq \ell^{-1}$ and $Q \leq C_3^\ell$, for some integer C_3 depending only on $p_1, \ldots, p_h$, this establishes (1.3). □

Proof of Theorem 2.7.

We keep the notation of the proof of Theorem 2.3. By assumption, we can select a suitable sequence $(n_j)_{j\geq 1}$ with the property that $n_j < C_1^j$ for some integer $C_1 \geq 2$ and every $j \geq 1$. Then, there are positive constants C_2, C_3, depending only on C_1, such that

$$Q \leq C_2^{n_i + n_j},$$

thus,

$$\log\log Q \leq C_3 \ell^2,$$

since i and j are at most equal to $\ell^2 + 1$. Combined with (6.3), this proves the theorem. □

References

[1] J.-P. Allouche and J. Shallit, Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, Cambridge, 2003.

[2] D. Badziahin and S. Velani, *Multiplicatively badly approximable numbers and the mixed Littlewood conjecture*, Adv. Math. 228 (2011), 2766–2796.

[3] Y. Bugeaud, Approximation by algebraic numbers, Cambridge Tracts in Mathematics 160, Cambridge, 2004.

[4] Y. Bugeaud, *Around the Littlewood conjecture in Diophantine approximation*, Publ. Math. Besançon Algèbre Théorie Nr. (2014), 5–18.

[5] Y. Bugeaud, M. Drmota, and B. de Mathan, *On a mixed Littlewood conjecture in Diophantine approximation*, Acta Arith. 128 (2007), 107–124.

[6] J. Cassaigne, *Sequences with grouped factors.* In: DLT'97, Developments in Language Theory III, Thessaloniki, Aristotle University of Thessaloniki, 1998, pp. 211–222.

[7] J. W. S. Cassels and H. P. F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London, Ser. A, 248 (1955), 73–96.

[8] M. Einsiedler, L. Fishman, and U. Shapira, *Diophantine approximation on fractals*, Geom. Funct. Anal. 21 (2011), 14–35.

[9] M. Einsiedler, A. Katok, and E. Lindenstrauss, *Invariant measures and the set of exceptions to the Littlewood conjecture*, Ann. of Math. 164 (2006), 513–560.

[10] M. Einsiedler and D. Kleinbock, *Measure rigidity and p-adic Littlewood-type problems*, Compositio Math. 143 (2007), 689–702.

[11] M. Einsiedler and E. Lindenstrauss, *On measures invariant under tori on quotients of semi-simple groups*, Ann. of Math. To appear.

[12] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, Recurrence sequences. Mathematical Surveys and Monographs, 104. American Mathematical Society, Providence, RI, 2003.

[13] H. W. Lenstra and J. O. Shallit, *Continued fractions and linear recurrences*, Math. Comp. 61 (1993), 351–354.

[14] E. Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2) 163 (2006), 165–219.

[15] E. Lindenstrauss, *Equidistribution in homogeneous spaces and number theory.* In: Proceedings of the International Congress of Mathematicians. Volume I, 531–557, Hindustan Book Agency, New Delhi, 2010.

[16] B. de Mathan et O. Teulié, *Problèmes diophantiens simultanés*, Monatsh. Math. 143 (2004), 229–245.

[17] M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. 60 (1938), 815–866.

[18] M. Morse and G. A. Hedlund, *Symbolic dynamics II*, Amer. J. Math. 62 (1940), 1–42.

[19] O. Perron, Die Lehre von den Ketterbrüchen. Teubner, Leipzig, 1929.

[20] A. D. Pollington and S. Velani, *On a problem in simultaneous Diophantine approximation: Littlewood's conjecture*, Acta Math. 185 (2000), 287–306.

Dmitry Badziahin
University of Durham
Department of Mathematical Sciences
South Rd
Durham DH1 3LE
UK

dzmitry.badziahin@durham.ac.uk

Yann Bugeaud
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 Strasbourg Cedex
France

bugeaud@math.unistra.fr

Manfred Einsiedler
ETH Zürich, Departement Mathematik
Rämistrasse 101
CH-8092 Zürich
Switzerland

manfred.einsiedler@math.ethz.ch

Dmitry Kleinbock
Brandeis University
Department of Mathematics
Waltham, MA 02454
USA

kleinboc@brandeis.edu