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ON SHRINKING TARGETS FOR $\mathbb{Z}^m$ ACTIONS ON TORI

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Abstract. Let A be an $n \times m$ matrix with real entries. Consider the set $\mathbf{Bad}_A$ of $\mathbf{x} \in [0, 1)^n$ for which there exists a constant $c(\mathbf{x}) > 0$ such that for any $\mathbf{q} \in \mathbb{Z}^m$ the distance between $\mathbf{x}$ and the point $\{A\mathbf{q}\}$ is at least $c(\mathbf{x})|\mathbf{q}|^{-m/n}$. It is shown that the intersection of $\mathbf{Bad}_A$ with any suitably regular fractal set is of maximal Hausdorff dimension. The linear form systems investigated in this paper are natural extensions of irrational rotations of the circle. Even in the latter one-dimensional case, the results obtained are new.

§1. *Introduction.* Consider initially a rotation of the unit circle through an angle α. Identifying the circle with the unit interval $[0, 1)$ and the base point of the iteration with the origin, we are considering the numbers $0, \{\alpha\}, \{2\alpha\}, \ldots$ where $\{\cdot\}$ denotes the fractional part. If α is rational, the rotation is periodic. On the other hand, it is a classic result of Weyl [24] that any irrational rotation of the circle is ergodic. In other words, $\{q\alpha\}_{q\in\mathbb{N}}$ is equidistributed for irrational α. Q1 Q2

Almost every orbit of an ergodic transformation visits any fixed set of positive measure infinitely often. The "shrinking target problem" introduced in [11] formulates the natural question of what happens if the target set—the set of positive measure—is allowed to shrink with time. For example, and more precisely, is there an optimal "shrinking rate" for which almost every orbit visits the shrinking target infinitely often? In the specific case of irrational rotations of the circle, the shrinking target sets correspond to subintervals of $[0, 1)$ whose lengths decay according to some specified function ψ. In other words, the problem translates to considering inequalities of the type

$$\|q\alpha - x\| < \psi(q), \tag{1}$$

where $x \in [0, 1)$ and $\|\cdot\|$ denotes the distance to the nearest integer. The following statement dates back to Khintchine [12] and gives the "optimal" choice of ψ in the non-trivial case where α is irrational and $x \neq s\alpha + t$ for any integers s and t. The inequality

$$\|q\alpha - x\| < \frac{C(\alpha)}{q} \tag{2}$$

is satisfied for infinitely many integers q with $C(\alpha) := \sqrt{1 - 4\lambda(\alpha)^2}/4$; the quantity $\lambda(\alpha) := \liminf_{q\to\infty} q\|q\alpha\|$ is the Markoff constant of α. Note that $\lambda(\alpha)$

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is strictly positive whenever α is badly approximable by rationals. Thus, the above statement strengthens a result of Minkowski [18], namely that (2) has infinitely many solutions with $C(\alpha) = 1/4$. In the trivial case where α is irrational and $x = s\alpha + t$ for some integers s and t, the classic theorem of Hurwitz implies that the inequality

$$\|q\alpha - x\| < \frac{1+\epsilon}{\sqrt{5}q} \quad (\epsilon > 0) \tag{3}$$

is satisfied for infinitely many integers q. Since (3) is weaker than (2), it follows that for any irrational α and any x the inequality (3) has infinitely many solutions. We now describe a metrical statement in which the right-hand side of (3), and indeed (2), can be significantly improved—at a cost!

Kurzweil [16] showed that for any non-increasing function $\psi : \mathbb{N} \to \mathbb{R}_{>0}$ such that $\sum \psi(q) = \infty$ and almost every irrational α, the set of x for which (1) has infinitely many solutions is of full Lebesgue measure. This cannot be improved upon in the sense that there exist irrational α and a function ψ for which $\sum \psi(q) = \infty$ but the "full measure" conclusion fails to hold. Hence, the "almost every" aspect of Kurzweil's result does not extend to all irrationals α without modification; the divergent sum condition is not enough.

Over the past few years, there has been much activity in investigating the shrinking target problem associated with irrational rotations of the circle. For example, when $\psi(q) := q^{-v}$ with $v > 1$, Bugeaud [3] and, independently, Schmeling and Trubetskoy [21] obtained the Hausdorff dimension of the set of x for which inequality (1) has infinitely many solutions. Fayad [10], Fan and Wu [9], Kim [13] and Tseng [22, 23] have built upon the work of Kurzweil in various directions. In particular, Kim proved that for any irrational α, the set of x for which

$$\liminf_{q\to\infty} q\|q\alpha - x\| = 0 \tag{4}$$

has full measure. Rather surprisingly, Beresnevich *et al* [1] have shown that this result and, indeed, the dimension result of Bugeaud, Schmeling and Trubetskoy are consequences of the fact that for any irrational α and any x the inequality (3) has infinitely many solutions.

The result of Kim is the underlying motivation for our work. In this paper we investigate the complementary measure-zero set associated with (4), namely

$$\mathbf{Bad}_\alpha := \left\{ x \in [0, 1) : \exists c(x) > 0 \text{ s.t. } \|q\alpha - x\| \geq \frac{c(x)}{q} \ \forall q \in \mathbb{N} \right\}. \tag{5}$$

In fact, we will be concerned with more general actions than rotations of the circle. Broadly speaking, there are two natural ways to generalize circle rotations. One option is to increase the dimension of the torus, i.e. to consider the sequence $\{q\boldsymbol{\alpha}\}$ in $[0, 1)^n$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$. The other option is to increase the dimension of the group acting on the torus, i.e. to consider the sequence $\{\boldsymbol{\alpha} \cdot \mathbf{q}\}$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $\mathbf{q} = (q_1, \ldots, q_m)^T \in \mathbb{Z}^m$.

It is possible to consider both of the aforementioned options at the same time by introducing a $\mathbb{Z}^m$ action on the n-torus by $n \times m$ matrices. Indeed, we may consider the points $\{A\mathbf{q}\} \in [0, 1)^n$ where $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ is fixed and $\mathbf{q}$ runs over $\mathbb{Z}^m$. In this case, the natural analogue of $\mathbf{Bad}_\alpha$ is the set

$$\mathbf{Bad}_A := \left\{\mathbf{x} \in [0, 1)^n : \exists c(\mathbf{x}) > 0 \text{ s.t. } \|A\mathbf{q} - \mathbf{x}\| \geq \frac{c(\mathbf{x})}{|\mathbf{q}|^{m/n}} \ \forall \mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}\right\}.$$

Here and throughout this article, for a vector $\mathbf{x}$ in $\mathbb{R}^n$ we will denote by $|\mathbf{x}|$ the maximum of the absolute values of the coordinates of $\mathbf{x}$, i.e. the infinity norm of $\mathbf{x}$. Also, we write $\|\mathbf{x}\| := \min_{\mathbf{y}\in\mathbb{Z}^n} |\mathbf{x} - \mathbf{y}|$.

The underlying goal of this paper is to show that no matter which of the $\mathbb{Z}^m$ actions defined above we choose, the set $\mathbf{Bad}_A$ is of maximal Hausdorff dimension.

THEOREM 1. *For any $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim \mathbf{Bad}_A = n.$$

In the more familiar setting of irrational rotations of the circle, the theorem reads as follows.

COROLLARY 1. *For any $\alpha \in \mathbb{R}$,*

$$\dim \mathbf{Bad}_\alpha = 1.$$

Note that if α is rational, the set $\mathbf{Bad}_\alpha$ is easily seen to contain all points in the unit interval bounded away from a finite set of points. Thus, for rational α, not only is $\mathbf{Bad}_\alpha$ of full dimension but it is of full Lebesgue measure. In higher dimensions, similar phenomena occur in which the finite set of points is replaced by a finite set of affine subspaces. The reader is referred to [**5**] and §5 below for further details.

Inspired by the works of Kleinbock and Weiss [**14**] and Kristensen *et al* [**15**], we shall deduce Theorem 1 as a simple consequence of a general statement concerning the intersection of $\mathbf{Bad}_A$ with compact subsets of $\mathbb{R}^n$. The latter includes exotic fractal sets such as the Sierpinski gasket and the van Koch curve.

§2. *The setup and main result.* Let (X, d) be a metric space and let (Ω, d) be a compact subspace of X which supports a non-atomic finite measure μ. Throughout, $B(c, r)$ will denote a closed ball in X with centre c and radius r. The measure μ is said to be *δ-Ahlfors regular* if there exist strictly positive constants δ and r_0 such that for $c \in \Omega$ and $r < r_0$,

$$ar^\delta \leq \mu(B(c, r)) \leq br^\delta$$

where $0 < a \leq 1 \leq b$ are constants independent of the ball. It is easily verified that if μ is δ-Ahlfors regular, then the Hausdorff dimension of Ω is δ, i.e.

$$\dim \Omega = \delta. \tag{6}$$

For further details, including the definition of Hausdorff dimension, the reader is referred to [17].

In the above, take $X = \mathbb{R}^n$ and let $\mathcal{L}$ denote a generic $(n-1)$-dimensional hyperplane. For $\epsilon > 0$, let $\mathcal{L}^{(\epsilon)}$ denote the ϵ-neighbourhood of $\mathcal{L}$. The measure μ is said to be *absolutely α-decaying* if there exist strictly positive constants C, α and r_0 such that for any hyperplane $\mathcal{L}$, any $\epsilon > 0$, any $x \in \Omega$ and any $r < r_0$,

$$\mu(B(x, r) \cap \mathcal{L}^{(\epsilon)}) \leq C\left(\frac{\epsilon}{r}\right)^{\alpha} \mu(B(x, r)).$$

It is worth mentioning that if μ is δ-Ahlfors regular and absolutely α-decaying, then μ is an absolutely friendly measure as defined in [20].

Armed with the notions of Ahlfors regular and absolutely decaying, we are in a position to state our main result.

THEOREM 2. *Let $K \subseteq [0, 1]^n$ be a compact set which supports an absolutely α-decaying, δ-Ahlfors regular measure μ such that $\delta > n - 1$. Then, for any $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$,*

$$\dim(\mathbf{Bad}_A \cap K) = \delta.$$

In view of (6), the theorem can be interpreted as stating that within K the set $\mathbf{Bad}_A$ is of maximal dimension. It is easily seen that Theorem 1 is a consequence of Theorem 2: simply take $K = [0, 1]^n$ and μ to be n-dimensional Lebesgue measure. Trivially, n-dimensional Lebesgue measure is n-Ahlfors regular and absolutely 1-decaying. More exotically, the natural measures associated with self-similar sets in $\mathbb{R}^n$ satisfying the open set condition are absolutely α-decaying and δ-Ahlfors regular; see [14, 20]. Thus, Theorem 2 is applicable to these sets, which in general are of fractal nature.

Although Theorem 2 constitutes our main result, we state an "auxiliary" result in this section for the simple reason that it is new and of independent interest. In short, it strengthens and generalizes a theorem of Pollington [19] and de Mathan [7, 8] that answers a question of Erdős. A sequence

$$\{\mathbf{y}_i\} := \{\mathbf{y}_i := (y_{1,i}, \ldots, y_{n,i})^T \in \mathbb{Z}^n \backslash \{\mathbf{0}\}\}$$

is said to be *lacunary* if there exists a constant $\lambda > 1$ such that

$$|\mathbf{y}_{i+1}| \geq \lambda |\mathbf{y}_i| \quad \text{for all } i \in \mathbb{N}.$$

Given a sequence $\{\mathbf{y}_i\}$ in $\mathbb{Z}^n$, let

$$\mathbf{Bad}_{\{\mathbf{y}_i\}} := \{\mathbf{x} \in [0, 1]^n : \exists c(\mathbf{x}) > 0 \text{ s.t. } \|\mathbf{y}_i \cdot \mathbf{x}\| \geq c(\mathbf{x})\ \forall i \in \mathbb{N}\}.$$

THEOREM 3. *Let $\{\mathbf{y}_i\}$ be a lacunary sequence in $\mathbb{Z}^n$. Furthermore, let $K \subseteq [0, 1]^n$ be a compact set which supports an absolutely α-decaying, δ-Ahlfors regular measure μ such that $\delta > n - 1$. Then*

$$\dim(\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K) = \delta.$$

Upon taking $n = 1$, $K = [0, 1]$ and μ to be one-dimensional Lebesgue measure, Theorem 3 corresponds to the theorem of Pollington and de Mathan referred to above.

§3. *Preliminaries for Theorem 3*. The proof of Theorem 3 makes use of the general framework developed in [15] for establishing dimension statements for a large class of badly approximable sets. In this section, we provide a simplification of the framework that is geared towards the particular application we have in mind. In turn, this will avoid excessive referencing to the conditions imposed in [15] and thereby improve the clarity of our exposition.

As in §2, let (X, d) be a metric space and (Ω, d) a compact subspace of X which supports a non-atomic finite measure μ. Let $\mathcal{R} := \{R_\alpha \in X : \alpha \in J\}$ be a family of subsets R_α of X indexed by an infinite countable set J. The sets R_α will be referred to as *resonant sets*. Next, let $\beta : J \to \mathbb{R}_{>0} : \alpha \mapsto \beta_\alpha$ be a positive function on J such that the number of $\alpha \in J$ with β_α bounded above is finite. Thus, β_α tends to infinity as α runs through J. We are now ready to define the badly approximable set

$$\mathbf{Bad}(\mathcal{R}, \beta) := \left\{ x \in \Omega : \exists c(x) > 0 \text{ s.t. } d(x, R_\alpha) \geq \frac{c(x)}{\beta_\alpha} \ \forall \alpha \in J \right\},$$

where $d(x, R_\alpha) := \inf_{a \in R_\alpha} d(x, a)$. Loosely speaking, $\mathbf{Bad}(\mathcal{R}, \beta)$ consists of points in Ω that "stay clear" of the family $\mathcal{R}$ of resonant sets by a factor governed by β.

The goal is to determine conditions under which $\dim \mathbf{Bad}(\mathcal{R}, \beta) = \dim \Omega$, i.e. the set of badly approximable points in Ω is of maximal dimension. With this in mind, we begin with some useful notation. For any fixed integer $k > 1$ and any integer $n \geq 1$, let $B_n := \{x \in \Omega : d(c, x) \leq 1/k^n\}$ denote a generic closed ball in Ω of radius $1/k^n$ with centre c in Ω. For any $\theta \in \mathbb{R}_{>0}$, let $\theta B_n := \{x \in \Omega : d(c, x) \leq \theta/k^n\}$ denote the ball B_n scaled by θ. Finally, let $J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\}$. The following statement is a simple consequence of combining [15, Theorem 1 and Lemma 7] and realizes the above goal.

THEOREM KTV. *Let (X, d) be a metric space and let (Ω, d) be a compact subspace of X which supports a δ-Ahlfors regular measure μ. Let k be sufficiently large. Then for any $\theta \in \mathbb{R}_{>0}$, any $n \geq 1$ and any ball B_n, there exists a collection $\mathcal{C}(\theta B_n)$ of disjoint balls $2\theta B_{n+1}$ contained within θB_n such that $\#\mathcal{C}(\theta B_n) \geq \kappa_1 k^\delta$. In addition, suppose that for some $\theta \in \mathbb{R}_{>0}$ we also have*

$$\#\left\{ 2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : \min_{\alpha \in J(n+1)} d(c, R_\alpha) \leq 2\theta k^{-(n+1)} \right\} \leq \kappa_2 k^\delta , \qquad (7)$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, assume

$$\dim \Big(\bigcup_{\alpha \in J} R_\alpha \Big) < \delta. \qquad (8)$$

Then

$$\dim \mathbf{Bad}(\mathcal{R}, \beta) = \delta.$$

Note that this theorem, together with (6), implies that $\dim \mathbf{Bad}(\mathcal{R}, \beta) = \dim \Omega$.

§4. *Proof of Theorem 3.* We are given a lacunary sequence $\{\mathbf{y}_i\}$. For each index $i \in \mathbb{N}$ and any integer p, consider the hyperplane $\mathcal{L}_{p,i} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{y}_i \cdot \mathbf{x} = p\}$. It is easily verified that $\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K$ is equivalent to the set of $\mathbf{x}$ in K for which there exists a constant $c(\mathbf{x}) > 0$ such that $\mathbf{x}$ avoids the $(c(\mathbf{x})/|\mathbf{y}_i|_2)$-neighbourhood of $\mathcal{L}_{p,i}$ for every choice of i and p; that is,

$$\mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K = \Big\{\mathbf{x} \in K : \exists c(\mathbf{x}) > 0 \text{ s.t. } \min_{\mathbf{y}\in\mathcal{L}_{p,i}} |\mathbf{x} - \mathbf{y}|_2 \geq \frac{c(\mathbf{x})}{|\mathbf{y}_i|_2}\ \forall (p, i) \in \mathbb{Z} \times \mathbb{N}\Big\}.$$

Here $|\cdot|_2$ denotes the standard Euclidean norm in $\mathbb{R}^n$. With reference to §3, set

$$X := \mathbb{R}^n, \qquad \Omega := K, \qquad d := |\cdot|_2, \qquad J := \{(p, i) \in \mathbb{Z} \times \mathbb{N}\},$$
$$\alpha := (p, i) \in J, \qquad R_\alpha := \mathcal{L}_{p,i} \quad \text{and} \quad \beta_\alpha := |\mathbf{y}_i|_2.$$

It follows that

$$\mathbf{Bad}(\mathcal{R}, \beta) = \mathbf{Bad}_{\{\mathbf{y}_i\}} \cap K.$$

The upshot of this is that the proof of Theorem 3 is reduced to showing that the conditions of Theorem KTV are satisfied.

For $k > 1$ and $m \geq 1$, let B_m be a generic closed ball with radius k^{-m} and centre in K. For sufficiently large k and any $\theta \in \mathbb{R}_{>0}$, Theorem KTV guarantees the existence of a collection $\mathcal{C}(\theta B_m)$ of disjoint balls $2\theta B_{m+1}$ contained within θB_m such that

$$\#\mathcal{C}(\theta B_m) \geq \kappa_1 k^\delta.$$

The positive constant κ_1 is independent of k and n. We now endeavour to show that the additional condition (7) on the collection $\mathcal{C}(\theta B_m)$ is satisfied. To this end, set $\theta := (2k)^{-1}$ and proceed as follows. Fix $m \geq 1$ and assume there exists an index i such that

$$k^m \leq |\mathbf{y}_i|_2 < k^{m+1}. \tag{9}$$

If this were not the case, the left-hand side of (7) would be zero and the additional condition would be trivially satisfied. Associated with the index i is the family of hyperplanes $\{\mathcal{L}_{p,i} : p \in \mathbb{Z}\}$. The distance between any two such hyperplanes is at least $|\mathbf{y}_i|_2^{-1} > k^{-(m+1)}$. The diameter of the ball θB_m is $k^{-(m+1)}$. Thus, for any element of the sequence $\{\mathbf{y}_i\}$ satisfying (9), there is at most one hyperplane passing through θB_m. Assume that the hyperplane $\mathcal{L}_{p,i}$ passes through θB_m, and consider the counting function

$$\omega(m, p, i) := \#\{2\theta B_{m+1} \subset \mathcal{C}(\theta B_m) : 2\theta B_{m+1} \cap \mathcal{L}_{p,i} \neq \emptyset\}.$$

The balls $2\theta B_{m+1}$ are disjoint and each is of diameter $4\theta k^{-(m+1)}$. Thus, upon setting $\epsilon := 8\theta k^{-(m+1)}$, we have

$$\omega(m, p, i) \leq \#\big\{2\theta B_{m+1} \subset \mathcal{C}(\theta B_m) : 2\theta B_{m+1} \subset \mathcal{L}_{p,i}^{(\epsilon)}\big\}$$
$$\leq \frac{\mu(\theta B_m \cap \mathcal{L}_{p,i}^{(\epsilon)})}{\mu(2\theta B_{m+1})}.$$

On making use of the fact that μ is absolutely α-decaying and δ-Ahlfors regular, it is readily verified that

$$\omega(m, p, i) \le \kappa k^{\delta-\alpha}.$$

The absolute constant κ is dependent only on α and δ. Next, let $\upsilon(m, \{\mathbf{y}_i\})$ denote the number of elements of the sequence $\{\mathbf{y}_i\}$ that satisfy (9). Since $\{\mathbf{y}_i\}$ is lacunary, we find that for k sufficiently large,

$$\upsilon(m, \{\mathbf{y}_i\}) \le 1 + \log(\sqrt{n}k)/\log\lambda < \frac{\kappa_1}{2\,\kappa} k^{\alpha}.$$

Here, $\lambda > 1$ is the lacunarity constant and we have used the fact that $|\mathbf{y}| \le |\mathbf{y}|_2 \le \sqrt{n}\,|\mathbf{y}|$ for $\mathbf{y} \in \mathbb{Z}^n$. On combining the above upper bound estimates, we obtain that

$$\begin{aligned} \text{left-hand side of (7)} &< \upsilon(m, \{\mathbf{y}_i\}) \times \omega(m, p, i) \\ &\le \frac{\kappa_1}{2\kappa} k^{\alpha} \times \kappa k^{\delta-\alpha} = \frac{1}{2}\kappa_1 k^{\delta}. \end{aligned}$$

Thus, with $\theta := (2k)^{-1}$, the collection $\mathcal{C}(\theta B_m)$ satisfies (7). Finally, note that the collection $\{\mathcal{L}_{p,i} : (p, i) \in \mathbb{Z} \times \mathbb{N}\}$ of hyperplanes (resonant sets) is countable and hence

$$\dim\Big(\bigcup \mathcal{L}_{p,i}\Big) = n - 1.$$

We are given that $\delta > n - 1$, so (8) is trivially satisfied. Thus, the conditions of Theorem KTV are satisfied and Theorem 3 follows.

§5. *Preliminaries for Theorem 2.* The proof of Theorem 2 makes use of the existence of "special" sequences which, for the most part, are constructed in [**5**]. Throughout, $\mathrm{Mat}^*_{n\times m}(\mathbb{R})$ will denote the collection of matrices $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ such that the associated group $G := A^T\mathbb{Z}^n + \mathbb{Z}^m$ has rank $n + m$. In [**5**, §3], it is shown that for each matrix $A \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ there exists a sequence $\{\mathbf{y}_i\}$ of integer vectors $\mathbf{y}_i = (y_{1,i}, \ldots, y_{n,i})^T \in \mathbb{Z}^n$ satisfying the following properties.

(i) $1 = |\mathbf{y}_1| < |\mathbf{y}_2| < |\mathbf{y}_3| < \cdots$.

(ii) $\|A^T\mathbf{y}_1\| > \|A^T\mathbf{y}_2\| > \|A^T\mathbf{y}_3\| > \cdots$.

(iii) For all non-zero $\mathbf{y} \in \mathbb{Z}^n$ with $|\mathbf{y}| < |\mathbf{y}_{i+1}|$ we have that $\|A^T\mathbf{y}\| \ge \|A^T\mathbf{y}_i\|$.

Such a sequence $\{\mathbf{y}_i\}$ is referred to as a *sequence of best approximations* to A. In the one-dimensional case ($n = m = 1$), when A is an irrational number α, the sequence of best approximations is precisely the sequence of denominators associated with the convergents of the continued fraction representing α.

Let $\{\mathbf{y}_i\}$ be a sequence of best approximations to a matrix $A \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$. A further property enjoyed by $\{\mathbf{y}_i\}$ is that

$$\|A^T\mathbf{y}_i\| \le |\mathbf{y}_{i+1}|^{-m/n} \quad \text{for all } i \in \mathbb{N}. \tag{10}$$

This property is easily deduced via Dirichlet's box principle; see [**5**, §3] for the details.

The following result, which is taken from [**5**, §5], enables us to extract a lacunary subsequence from a given sequence of best approximations. This will allow us to utilize Theorem 3 in the course of establishing Theorem 2.

LEMMA BL. *Let $A \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ and let $\{\mathbf{y}_i\}$ be a sequence of best approximations to A. Then there exists an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(1) = 1$ and, for all $i \geq 2$,*

$$|\mathbf{y}_{\phi(i)}| \geq \sqrt{9n}\, |\mathbf{y}_{\phi(i-1)}| \quad \textit{and} \quad |\mathbf{y}_{\phi(i-1)+1}| \geq \frac{|\mathbf{y}_{\phi(i)}|}{9n}. \tag{11}$$

It is clear that the sequence $\{\mathbf{y}_{\phi(i)}\}$ is lacunary and that it also satisfies (10), i.e.

$$\|A^T \mathbf{y}_{\phi(i)}\| \leq |\mathbf{y}_{\phi(i)+1}|^{-m/n} \quad \text{for all } i \in \mathbb{N}. \tag{12}$$

The next inequality follows directly from the definition of the norms involved. For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$, we have that

$$\|\mathbf{x} \cdot \mathbf{y}\| < k|\mathbf{x}|\|\mathbf{y}\|. \tag{13}$$

We end this section with a short discussion that allows us to assume $A \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ when proving Theorem 2. With this in mind, suppose $A \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ and that the rank of the associated group $G := A^T\mathbb{Z}^n + \mathbb{Z}^m$ is strictly less than $n + m$. Then it is easily verified that $\{A\mathbf{q} : \mathbf{q} \in \mathbb{Z}^m\}$ is restricted to at most a countable family of positively separated, parallel hyperplanes in $\mathbb{R}^n$. Let F denote the set of these hyperplanes. Then

$$K\backslash F = \mathbf{Bad}_A \cap K.$$

We are given that $\delta > n - 1$, which, together with (6), implies that $\dim K$ is strictly greater than $\dim F$. Thus $\dim(K\backslash F) = \dim K$, and the statement of Theorem 2 follows for any $A \notin \mathrm{Mat}^*_{n\times m}(\mathbb{R})$.

§6. *Proof of Theorem 2.* Without loss of generality, assume that $A \in \mathrm{Mat}^*_{n\times m}(\mathbb{R})$ and let $\{\mathbf{y}_i\}$ be a sequence of best approximations to A. In view of Lemma BL, there exists a lacunary subsequence $\{\mathbf{y}_{\phi(i)}\}$ of the sequence of best approximations. For any $c > 0$, let

$$\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) := \{\mathbf{x} \in K : \|\mathbf{y}_{\phi(i)} \cdot \mathbf{x}\| \geq c \ \forall i \in \mathbb{N}\}.$$

It is readily verified that $\mathbf{Bad}_{\{\mathbf{y}_{\phi(i)}\}} \cap K = \bigcup_{c>0} \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and that

$$\dim \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \to \dim(\mathbf{Bad}_{\{\mathbf{y}_{\phi(i)}\}} \cap K) \quad \text{as } c \to 0.$$

For c sufficiently small, suppose for the moment that

$$\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \subseteq \mathbf{Bad}_A \cap K. \tag{14}$$

From Theorem 3, it follows that

$$\dim(\mathbf{Bad}_A \cap K) \geq \dim \mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \to \delta \quad \text{as } c \to 0.$$

The upshot of this is that $\dim(\mathbf{Bad}_A \cap K) \geq \delta$. For the complementary upper bound statement, trivially we have

$$\dim(\mathbf{Bad}_A \cap K) \leq \dim K \overset{(6)}{=} \delta.$$

This completes the proof of Theorem 2 modulo the inclusion (14).

To establish (14), fix a point $\mathbf{x}$ in $\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and let $\mathbf{q}$ be any non-zero integer vector. For c sufficiently small, there exists an index $i \in \mathbb{N}$ such that

$$|\mathbf{y}_{\phi(i)}| \leq 9n\left(\frac{2m}{c}\right)^{m/n} |\mathbf{q}|^{m/n} < |\mathbf{y}_{\phi(i+1)}|. \tag{15}$$

The existence of such an index is guaranteed by the first of the inequalities in (11) as long as c is sufficiently small. By the definition of $\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c)$ and the trivial equality

$$\mathbf{y}_{\phi(i)} \cdot \mathbf{x} = \mathbf{q} \cdot A^T \mathbf{y}_{\phi(i)} - \mathbf{y}_{\phi(i)} \cdot (A\mathbf{q} - \mathbf{x}),$$

we immediately have that

$$0 < c \leq \|\mathbf{y}_{\phi(i)} \cdot \mathbf{x}\| = \|\mathbf{q} \cdot A^T \mathbf{y}_{\phi(i)} - \mathbf{y}_{\phi(i)} \cdot (A\mathbf{q} - \mathbf{x})\|. \tag{16}$$

On applying the triangle inequality and making use of (13), it follows that

$$c \leq m|\mathbf{q}|\|A^T \mathbf{y}_{\phi(i)}\| + n|\mathbf{y}_{\phi(i)}|\|A\mathbf{q} - \mathbf{x}\|. \tag{17}$$

However,

$$\begin{aligned} m|\mathbf{q}|\|A^T \mathbf{y}_{\phi(i)}\| &\overset{(12)}{\leq} m|\mathbf{q}||\mathbf{y}_{\phi(i)+1}|^{-n/m} \\ &\overset{(15)}{\leq} \frac{m}{(9n)^{n/m}(2m/c)}\left(\frac{|\mathbf{y}_{\phi(i+1)}|}{|\mathbf{y}_{\phi(i)+1}|}\right)^{n/m} \overset{(11)}{\leq} \frac{c}{2} \end{aligned}$$

and

$$n|\mathbf{y}_{\phi(i)}|\|A\mathbf{q} - \mathbf{x}\| \overset{(15)}{\leq} 9n^2\left(\frac{2m}{c}\right)^{m/n} |\mathbf{q}|^{m/n}\|A\mathbf{q} - \mathbf{x}\|,$$

which, together with (17), yields that

$$\|A\mathbf{q} - \mathbf{x}\| > \frac{c^{m/n+1}}{9n^2(2m)^{m/n}}|\mathbf{q}|^{-m/n}.$$

In other words, for any c sufficiently small,

$$\mathbf{B}_{\{\mathbf{y}_{\phi(i)}\}}(c) \subseteq \left\{\mathbf{x} \in K : \exists c(\mathbf{x}) > 0 \text{ s.t. } \|A\mathbf{q} - \mathbf{x}\| \geq \frac{c(\mathbf{x})}{|\mathbf{q}|^{m/n}} \ \forall \mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}\right\}.$$

The right-hand side is $\mathbf{Bad}_A \cap K$, and this establishes (14), which in turn completes the proof of Theorem 2.

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