The equation $\frac{x^n-1}{x-1}=y^q$ has no solution with x square

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Introduction

We study the diophantine equation

$$\frac{x^n-1}{x-1}=y^q, \tag{1}$$

in integers x, $y>1$, $n>2$ and $q\geq 2$. Without loss of generality, we suppose that q is prime.

Saradha & Shorey [S-S] considered equation (1) with $x=z^2$ and proved that it has no solutions in the cases $z>31$ or $z\in\{2,3,4,8,9,16,27\}$*. The purpose of the present paper is to treat completely the remaining cases and to prove the following result.

Theorem 1. *The equation*

$$\frac{x^n-1}{x-1}=y^q,$$

in integers x, $y>1$, $n>2$ and $q\geq 2$, has no solution where $x=z^2$.

Our proof combines many different methods in effective diophantine approximation, including lower bounds for linear forms in two logarithms of algebraic numbers close to 1 and a generalization of a sharp estimate of Le [Le], obtained by hypergeometric means (Skolem's method). Some quite heavy computations are also needed.

By the results of Ljunggren [Lu] for $q=2$ and Saradha-Shorey [S-S] for $q=3$, we already know that (1) has no solution with $x=z^2$ for $q\leq 3$. Thus in the sequel we always assume $q\geq 5$.

Further, Nagell [N] proved that (for $q\geq 5$)

$$3\not| \, n, \quad \text{and} \quad 4\not| \, n.$$

In particular, we have $n\geq 5$. Let n be even, say $n=2m$, with odd $m\geq 3$. Then equation (1) becomes

$$\frac{x^{2m}-1}{x-1}=\left(\frac{x^m-1}{x-1}\right)(x^m+1)=y^q.$$

Since m is odd, we observe that $\frac{x^m-1}{x-1}$ is odd and $\gcd\left(\frac{x^m-1}{x-1}, x^m+1\right)=1$. Thus we have $\frac{x^m-1}{x-1}=y_0^q$ for some integer $y_0>1$. Hence there is no loss of generality in assuming that n is odd, what we do in the sequel.

By putting $x=z^2$ in equation (1), we have

$$\left(\frac{z^n-1}{z-1}\right)\left(\frac{z^n+1}{z+1}\right)=y^q.$$

The factors on the left hand side are relatively prime as n is odd. Thus equation (1) with $x=z^2$ implies that

$$\frac{z^n-1}{z-1}=y_1^q\ , \quad \frac{z^n+1}{z+1}=y_2^q \tag{2}$$

for some integers $y_1>1$ and $y_2>1$ such that $y_1y_2=y$ and $\gcd(y_1,y_2)=1$. ¿From equations (2), we have

$$(z+1)y_2^q-(z-1)y_1^q=2, \tag{3}$$

* Notice that the proof for $z\in\{2,4,8,16\}$ is elementary, whereas for $z\in\{3,9,27\}$ the proof uses a result of Darmon-Merel [D-M] which is a generalization of Wiles' result on Fermat's theorem.

and we observe from (3) that the diophantine equation

$$(z+1)X^q - (z-1)Y^q = 2 \tag{4}$$

has two distinct solutions in positive integers viz., $(X,Y) = (y_2, y_1)$ and $(X,Y) = (1,1)$.

Further, we write

$$\frac{z+1}{z-1} - \left(\frac{Y}{X}\right)^q = \left(\left(\frac{z+1}{z-1}\right)^{\frac{1}{q}} - \frac{Y}{X}\right)\left(\left(\frac{z+1}{z-1}\right)^{\frac{q-1}{q}} + \left(\frac{z+1}{z-1}\right)^{\frac{q-2}{q}}\frac{Y}{X} + \cdots + \left(\frac{Y}{X}\right)^{q-1}\right)$$

to derive from (4) that

$$\left|\left(\frac{z+1}{z-1}\right)^{\frac{1}{q}} - \frac{Y}{X}\right| < \frac{2}{(z-1)X^q}. \tag{5}$$

The plan for the proof of Theorem 1 is as follows:

1) we shall prove that equation (4) has only the solution $(1,1)$ if q is large enough, independently of z. Hence we obtain an upper bound on q,

2) Saradha & Shorey [S-S] proved that (1) with $\gcd\big(x\varphi(x), q\big) > 1$ has no solution; by a method of Le, we show that equation (1) with $\gcd\big(x\varphi(x), q\big) = 1$ implies a lower bound for q in terms of x,

3) combining 1) and 2), we get an upper bound on x, thus only finitely many cases remain,

4) in the remaining cases, we verify that $n \equiv 1 \pmod q$, which means that we have $n = kq + 1$ and thus the diophantine equation

$$xX^q - (x-1)Y^q = 1, \tag{6}$$

has the non-trivial solution $(X,Y) = (x^k, y)$,

5) we prove that equation (6) implies an (absolute) upper bound on q [better than in 1)],

6) using the Thue equation (6), we prove an upper bound on n: now x, q and n are bounded above,

7) by a (difficult) computer verification, we finish the proof.

1. Upper bounds for the equation $(z+1)X^q-(z+1)Y^q=2$: first bounds for q

In order to prove that if q is large enough equation (4) has only the solution $(X,Y)=(1,1)$, we use linear forms in two logarithms and we exploit the fact that the algebraic numbers involved are very closed to 1. This idea was in this context already noticed by Mignotte [Mi2] and Bennett and de Weger [B-W].

We consider the equation (4),

$$(z+1)X^q-(z-1)Y^q=2,$$

with $q\geq 5$ and suppose that it has a non trivial solution, *ie.* $\neq(1,1)$. We notice that

$$\left|\Big(\frac{z+1}{z-1}\Big)^{1/q}-\frac{Y}{X}\right|<\frac{2}{(z-1)X^q}$$

by formula (5), hence Y/X belongs to the principal convergents of $\left(\frac{z+1}{z-1}\right)^{1/q}$. Further,

$$\Lambda=\log\frac{z+1}{z-1}-q\log\frac{Y}{X}$$

satisfies

$$0<\Lambda<\frac{1}{X^q}$$

[see [S-S] formula (44)]. Then we apply the following result [see [Mi1], Theorem 2].

Theorem 1.1. *Consider the linear form*

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D=[\mathbf{Q}(\alpha_1,\alpha_2):\mathbf{Q}]\,/\,[\mathbf{R}(\alpha_1,\alpha_2):\mathbf{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ρ a real number >1. Put $\lambda=\log\rho$, $\chi=h/\lambda$ and suppose that $\chi\geq\chi_0$ for some number $\chi_0\geq 0$ and that

$$\begin{aligned}
&h\geq D\left(\log\Big(\frac{b_1}{a_2}+\frac{b_2}{a_1}\Big)+\log\lambda+f(\lceil K_0\rceil)\right)+0.023,\\
&a_i\geq\max\{1,\rho\,|\log\alpha_i|-\log|\alpha_i|+2D\mathrm{h}(\alpha_i)\},\quad(i=1,2),\\
&a_1a_2\geq\lambda^2
\end{aligned}$$

where

$$f(x)=\log\frac{\big(1+\sqrt{x-1}\big)\sqrt{x}}{x-1}+\frac{\log x}{6x(x-1)}+\frac{3}{2}+\log\frac{3}{4}+\frac{\log\frac{x}{x-1}}{x-1},$$

and

$$K_0=\frac{1}{\lambda}\left(\frac{\sqrt{2+2\chi_0}}{3}+\sqrt{\frac{2(1+\chi_0)}{9}+\frac{2\lambda}{3}\Big(\frac{1}{a_1}+\frac{1}{a_2}\Big)+\frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1a_2}}}\right)^2a_1a_2.$$

Put

$$v=4\chi+4+1/\chi,\quad m=\max\left\{2^{5/2}(1+\chi)^{3/2},(1+2\chi)^{5/2}/\chi\right\}.$$

Then we have the lower bound

$$\begin{aligned}
\log|\Lambda|\geq&-\frac{1}{\lambda}\left(\frac{v}{6}+\frac{1}{2}\sqrt{\frac{v^2}{9}+\frac{4\lambda v}{3}\Big(\frac{1}{a_1}+\frac{1}{a_2}\Big)+\frac{8\lambda m}{3\sqrt{a_1a_2}}}\right)^2a_1a_2\\
&-\max\Big\{\lambda(1.5+2\chi)+\log\Big(\big((2+2\chi)^{3/2}+(2+2\chi)^2\sqrt{k^*}\big)A+(2+2\chi)\Big),D\log 2\Big\}
\end{aligned}$$

where

$$A=\max\{a_1,a_2\},\quad\text{and}\quad k^*=\frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2+\frac{1}{\lambda}\left(\frac{2}{3\chi}+\frac{2}{3}\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

In the present case we have

$$b_1=q,\ b_2=1,\ \alpha_1=\frac{Y}{X},\ \alpha_2=\frac{z+1}{z-1},\ D=1,$$

and we can take, for a suitable choice of the "radius" ρ,

$$a_1 = (\rho - 1) \log \frac{Y}{X} + 2 \log X, \quad a_2 = (\rho - 1) \log \frac{z+1}{z-1} + 2 \log(z+1).$$

Further, we choose $\chi_0 = 1.5$. Using the lower bound

$$X \geq (z-1)q$$

[which comes from the fact that Y/X is a principal convergent of $\left(\frac{z+1}{z-1}\right)^{1/q}$ and that $1 < \left(\frac{z+1}{z-1}\right)^{1/q} < 1 + \frac{1}{(z-1)q}$], we get the bounds

$$\begin{array}{lll}
q \leq 200, & \text{for} \quad z = 5, & [\text{take } \rho = 19.77] \\
q \leq 179, & \text{for} \quad z = 6, & [\text{take } \rho = 22.52] \\
q \leq 165, & \text{for} \quad z = 7, & [\text{take } \rho = 24.77] \\
q \leq 155, & \text{for} \quad z = 8, & [\text{take } \rho = 23.52] \\
q \leq 147, & \text{for} \quad z = 9, & [\text{take } \rho = 27.02] \\
q \leq 141, & \text{for} \quad z = 10, & [\text{take } \rho = 26.77] \\
q \leq 136, & \text{for} \quad z = 11, & [\text{take } \rho = 27.52] \\
q \leq 132, & \text{for} \quad z = 12, & [\text{take } \rho = 27.77] \\
q \leq 129, & \text{for} \quad z = 13, & [\text{take } \rho = 27.91] \\
q \leq 126, & \text{for} \quad z = 14, & [\text{take } \rho = 28.88] \\
q \leq 124, & \text{for} \quad z = 15, & [\text{take } \rho = 28.92] \\
q \leq 122, & \text{for} \quad z = 16, & [\text{take } \rho = 29.39] \\
q \leq 121, & \text{for} \quad z = 17, & [\text{take } \rho = 30.00] \\
q \leq 119, & \text{for} \quad z = 18, & [\text{take } \rho = 29.91] \\
q \leq 118, & \text{for} \quad z = 19, & [\text{take } \rho = 29.89] \\
q \leq 118, & \text{for} \quad z = 20, & [\text{take } \rho = 30.00] \\
q \leq 117, & \text{for} \quad z = 21, & [\text{take } \rho = 29.63]
\end{array}$$

and it is possible to prove that $q \leq 116$ for $z \geq 22$.

2. Lower bounds on q: sharpening of a lemma by Le

The next lemma is a sharpening of a result of Le [6, assertion I] and Saradha-Shorey [S-S, Lemma 18]. Let p be a prime number. Let $\mathbf{Q}_p$ and $\mathbf{Z}_p$ be the $p-$adic number field and $p-$adic integer ring, respectively. For $\alpha \in \mathbf{Q}_p$, we denote by $\nu(\alpha)$ the p-adic valuation of α normalized by $\nu(p)=1$ and we set $\|\alpha\|_p = p^{-\nu(\alpha)}$. For α, β and γ in $\mathbf{Q}$, we define the hypergeometric function $F(\alpha,\beta,\gamma,X)$ with $X \in \mathbf{Q}_p$ by the series

$$F(\alpha,\beta,\gamma,X) = 1 + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \frac{(\alpha+j)(\beta+j)}{(\gamma+j)} \right) \frac{X^i}{i!}.$$

Let q, s, s_1, s_2 be positive integers with $q>1$ and $s = s_1+s_2$. Let $G(X) = F(-s_2-(1/q), -s_1, -s, X)$, $H(X) = F(-s_1+(1/q), -s_2, -s, X)$ and $E(X) = F(s_2+1, s_1+(q-1)/q, s+2, X)$. Then we have

$$\binom{s}{s_1} G(X) = \sum_{i=0}^{s_1} \binom{s_2+\frac{1}{q}}{i} \binom{s-i}{s_2} (-X)^i, \tag{2.1}$$

$$\binom{s}{s_1} H(X) = \sum_{i=0}^{s_2} \binom{s_1-\frac{1}{q}}{i} \binom{s-i}{s_1} (-X)^i \tag{2.2}$$

and there exists $\theta \in \mathbf{Q}$ such that

$$G(X) - (1-X)^{\frac{1}{q}} H(X) = X^{s+1} \theta E(X) \tag{2.3}$$

for every $X \in \mathbf{Q}_p$ with $\nu(X) \geq 1$ if $q \neq p$ and $\nu(X) \geq 2$ if $q = p$. We show

Lemma 2.1. *Suppose that equation (1) holds. Let $p_1, \ldots, p_\ell, q_1, \ldots, q_m$ be distinct prime divisors of x such that $x = x_1 x_2$ where $x_1 = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$ with $y \equiv 1 \pmod{p_i}$ for $1 \leq i \leq \ell$ and $x_2 = q_1^{\beta_1} \cdots q_m^{\beta_m}$ with $y \not\equiv 1 \pmod{q_j}$ for $1 \leq j \leq m$. Then,*

$$x_1^{n+1-2\beta} \leq \left(\frac{n+3}{4}\right)^2 \left(2+\frac{4}{x}\right)^{n-1} q^{\frac{\alpha q}{q-1}} x_2^{n-1+2\beta},$$

where

$$\alpha = \begin{cases} n+1, & \text{if } q \not| x, \\ 2n, & \text{if } q \mid x, \end{cases} \qquad \text{and} \qquad \beta = \max\{1, \tfrac{n}{q}\}.$$

Under the same hypotheses, if moreover $x \geq 15$ and $n > 3 + 2\sqrt{\frac{16x}{5}+1}$, we also have

$$x_1^n \leq (1+x^{n/q}) q^{\frac{\alpha q}{2(q-1)}} \left(\frac{n+1}{2}\right)^{1/5} e^{1/4}$$
$$\times \frac{1}{\left(\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}\right)^{\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}} \left(\frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}} + \frac{3}{8}}} \left(\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}} x^{\frac{n}{2} - \frac{n-1}{2\sqrt{x+1}} + \frac{2}{3}}.$$

Proof.

We only prove the second estimate since the first one is just that of Lemma 18 of [S-S].

Fix i with $1 \leq i \leq \ell$. Let $p = p_i$ and $\gamma = \alpha_i$. We observe that

$$\gamma \geq \begin{cases} 1, & \text{if } p \neq q, \\ 2, & \text{if } p = q. \end{cases} \tag{2.4}$$

We re-write equation $\frac{x^n-1}{x-1} = y^q$ as

$$(1-x)y^q = 1-x^n$$

and we observe from $y \equiv 1 \pmod{p}$ and (2.4) that $(\frac{1-x}{1-x^n})^{\frac{1}{q}} y$ is an element of $\mathbf{Z}_p$ satisfying $X^q - 1 = 0$. If $q \neq p$, we apply Hensel's Lemma to conclude that

$$(1-x)^{\frac{1}{q}} y = (1-x^n)^{\frac{1}{q}}. \tag{2.5}$$

The relation (2.5) is also valid if $q=p$ since $\mathbf{Z}_p$ has no p-th roots of unity other than 1. By (2.5) and (2.4), we derive that

$$(1-x)^{\frac{1}{q}}y = 1 + p^{\gamma n - \delta}\theta' \tag{2.6}$$

where $\theta' \in \mathbf{Z}_p$ and $\delta = 0$ or 1 according as $q \neq p$ or $q = p$, respectively. We take $s = n-1$, $\frac{n-1}{2} \geq s_1 \geq \frac{n-3}{2}$ with $s = s_1 + s_2$. We observe that $s_2 \geq s_1 \geq 1$ since $n \geq 5$. By (2.3) with $X = x$, (2.6), $y \equiv 1 \pmod p$ and (2.4), we have

$$\left\| y\binom{s}{s_1}G(x) - \binom{s}{s_1}H(x)\right\|_p \leq \max\left\{\left\|x^n\binom{s}{s_1}\theta E(x)\right\|_p, \left\|p^{\gamma n-\delta}\binom{s}{s_1}H(x)\right\|_p\right\}. \tag{2.7}$$

We observe from (2.3), (2.1) and (2.2) that $X^n\binom{s}{s_1}\theta E(X)$ is a power series in X with rational coefficients whose denominators are powers of q. In fact, power of q in the denominator of the coefficient of X^{n+i} in this power series is at most $(n+i)q/(q-1)$ for $i \geq 0$. Hence, by (2.4), we have

$$\left\|x^n\binom{s}{s_1}\theta E(x)\right\|_p \leq \begin{cases} p^{-\gamma n}, & \text{if } p \neq q, \\ p^{-\gamma n + nq/(q-1)}, & \text{if } p = q, \end{cases}$$

and

$$\left\|p^{\gamma n-\delta}\binom{s}{s_1}H(x)\right\|_p \leq p^{-\gamma n+\delta}.$$

Therefore we see from (2.7) that

$$\left\| y\binom{s}{s_1}G(x) - \binom{s}{s_1}H(x)\right\|_p \leq \begin{cases} p^{-\gamma n}, & \text{if } p \neq q, \\ p^{-\gamma n + nq/(q-1)}, & \text{if } p = q. \end{cases} \tag{2.8}$$

We define

$$N = q^\mu\left(y\binom{s}{s_1}G(x) - \binom{s}{s_1}H(x)\right) \tag{2.9}$$

where

$$\mu = \begin{cases} s_2 + \left[\frac{s_2}{q-1}\right], & \text{if } q \not| x \\ 0, & \text{if } q \mid x. \end{cases} \tag{2.10}$$

We observe from (2.1) and (2.2) that N is an integer. We note that there are two choices of (s_1, s_2) viz., $(\frac{n-3}{2}, \frac{n+1}{2})$ and $(\frac{n-1}{2}, \frac{n-1}{2})$ and there exists at least one choice of s_1, s_2 for which $N \neq 0$. We fix (s_1, s_2) such that $N \neq 0$. Further, by considering the contributions of all $p = p_i$ with $1 \leq i \leq \ell$ in N, we derive from (2.9) and (2.8) that

$$|N| \geq \begin{cases} (p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell})^n = x_1^n, & \text{if } p_i \neq q \text{ for } 1 \leq i \leq \ell, \\ \frac{(p_1^{\alpha_1}\cdots p_\ell^{\alpha_\ell})^n}{q^{nq/(q-1)}} = \frac{x_1^n}{q^{nq/(q-1)}}, & \text{if } p_i = q \text{ for some } i \text{ with } 1 \leq i \leq \ell. \end{cases}$$

¿From (2.1),

$$\left|\binom{s}{s_1}G(x)\right| = \left|\sum_{i=0}^{s_1}\binom{s_2+\frac{1}{q}}{i}\binom{s-i}{s_2}(-x)^i\right|.$$

For $0 \leq i \leq s_1$, put

$$u_i = \binom{s_2+\frac{1}{q}}{i}\binom{s-i}{s_2}x^i.$$

Therefore

$$\frac{u_{i+1}}{u_i} = \frac{s_2+\frac{1}{q}-i}{i+1} \times \frac{s_1-i}{s-i} \times x.$$

It is easy to check that the function $i \mapsto u_{i+1}/u_i$ is decreasing. Notice that $u_1/u_0 = (s_2+\frac{1}{q})\frac{s_1 x}{s} > 1$ and that $\frac{u_{s_1}}{u_{s_1-1}} = \frac{s_2+\frac{1}{q}-s_1+1}{s_1} \times \frac{1}{s-s_1+1} \times x < 1$, thus the minimal index i_0 such that $u_{i_0} = \max u_i$ satisfies $0 < i_0 \leq s_1$ and we have $u_0 < u_1 < \ldots < u_{i_0} \geq u_{i_0+1} > \ldots > u_{s_1}$. A moment of reflexion shows that $\binom{s}{s_1}|G(x)| \leq |u_{i_0}|$. Now we try to estimate i_0. Put $j = s_1 - i$ and $K(j) = (s_2 + \frac{1}{q} - i)(s_1 - i)x - (i+1)(s-i)$, then

$$K(j) = (x+1)j^2 + \left(x\left(s_2 - s_1 + \tfrac{1}{q}\right) + s_2 - s_1 - 1\right) j - (s_1+1)s_2.$$

We distinguish two cases

$$s_1 = \begin{cases} s_2 : & \text{case (i)}, \\ s_2 - 2 : & \text{case (ii)}. \end{cases}$$

Notice that

$$K\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) = 2\lambda s_1\sqrt{x+1} + \lambda^2(x+1) + \left(x\left(s_2 - s_1 + \tfrac{1}{q}\right) + s_2 - s_1 - 1\right)\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) - s_1 - (s_2 - s_1)(s_1+1).$$

In case (i), $s_1 = s_2$ and

$$K\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) = 2\lambda s_1\sqrt{x+1} + \lambda^2(x+1) + \left(\tfrac{x}{q} - 1\right)\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) - s_1.$$

If $\lambda \geq 0$ then

$$\begin{aligned} K\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) &\geq 2\lambda s_1\sqrt{x+1} + \lambda^2(x+1) - \frac{s_1}{\sqrt{x+1}} - \lambda - s_1 \\ &\geq 4\lambda s_1 + 4\lambda^2 - \frac{s_1}{2} - \lambda - s_1, \end{aligned}$$

since the middle expression is an increasing function of x. This shows that

$$K\left(\frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right) \geq 0.$$

Moreover

$$K\left(\frac{s_1}{\sqrt{x+1}} - \frac{1}{4}\right) = -\frac{s_1}{2}\sqrt{x+1} + \frac{x+1}{16} + \frac{xs_1}{q\sqrt{x+1}} - \frac{x}{4q} - \frac{s_1}{\sqrt{x+1}} + \frac{1}{4} - s_1 \leq 0$$

for $s_1 \geq \sqrt{x+1}$, since the middle expression is a decreasing function of s_1.

In case (ii), $s_2 = s_1 + 2$ and

$$\begin{aligned} K\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) &= 2\lambda s_1\sqrt{x+1} + \lambda^2(x+1) + \left(x\left(2 + \tfrac{1}{q}\right) + 1\right)\left(\frac{s_1}{\sqrt{x+1}} + \lambda\right) - 3s_1 - 2 \\ &= \left(2 + \frac{1}{q} + 2\lambda\right) s_1\sqrt{x+1} + \lambda^2(x+1) + \lambda\left(x\left(2 + \tfrac{1}{q}\right) + 1\right) - \left(1 + \frac{1}{q}\right)\frac{s_1}{\sqrt{x+1}} - 3s_1 - 2. \end{aligned}$$

If $\lambda = -1/4$ and $s_1 \geq \sqrt{x+1}$ then

$$K\left(\frac{s_1}{\sqrt{x+1}} - \frac{1}{4}\right) \geq -\frac{s_1}{2}\sqrt{x+1} + \frac{x+1}{16} + \left(x\left(2 + \tfrac{1}{q}\right) + 1\right)\frac{3s_1}{4\sqrt{x+1}} - 3s_1 - 2.$$

The partial derivative relative to s_1 of this last expression is

$$-\frac{\sqrt{x+1}}{2} + \left(x\left(2 + \tfrac{1}{q}\right) + 1\right)\frac{3}{4\sqrt{x+1}} - 3,$$

and is non-negative for $x \geq 15$. Thus for $\lambda = -1/4$, $s_1 \geq \sqrt{x+1}$ and $x \geq 15$ we have

$$K\left(\frac{s_1}{\sqrt{x+1}} - \frac{1}{4}\right) \geq -\frac{x+1}{2} + \frac{x+1}{16} + \frac{3x}{2} + \frac{3x}{4q} + \frac{3}{4} - 3\sqrt{x+1} - 2 \geq 0.$$

Whereas,

$$\begin{aligned} K\left(\frac{s_1}{\sqrt{x+1}} - \frac{7}{6}\right) &= \left(2 + \frac{1}{q} - \frac{7}{3}\right) s_1\sqrt{x+1} + \frac{49}{36}(x+1) - \frac{7}{6}\left(x\left(2 + \tfrac{1}{q}\right) + 1\right) - \left(1 + \frac{1}{q}\right)\frac{s_1}{\sqrt{x+1}} - 3s_1 - 2 \\ &\leq \frac{49}{36}(x+1) - \frac{7}{6}\left(x\left(2 + \tfrac{1}{q}\right) + 1\right) - \left(1 + \frac{1}{q}\right)\frac{s_1}{\sqrt{x+1}} - 3s_1 - 2 \\ &\leq -\frac{35}{36}x - \left(1 + \frac{1}{q}\right)\frac{s_1}{\sqrt{x+1}} - 3s_1 < 0. \end{aligned}$$

To summarize, when $x \geq 15$ and $s_1 \geq \sqrt{x+1}$ we have

$$s_1 - \frac{s_1}{\sqrt{x+1}} - \frac{3}{8} \leq i_0 \leq s_1 - \frac{s_1}{\sqrt{x+1}} + \frac{7}{6}.$$

Notice that this implies

$$i_0 \geq s_1 - \frac{s_1}{4} - \frac{3}{8} \geq \frac{s_2+1}{2},$$

since $n \geq 19$. We also have

$$|u_{i_0}| = \binom{s_2 + \frac{1}{q}}{i_0}\binom{s-i_0}{s_2} x^{i_0} \leq \binom{s_2+1/5}{i_0}\binom{s-i_0}{s_2} x^{i_0}.$$

We use the inequality

$$\binom{m}{k} \leq \frac{m^m}{k^k(m-k)^{m-k}}$$

which is true for any positive integer k and any real $m > k$. Proof: For z real positive, by comparison with an integral,

$$\sum_{j=1}^{k} \log\left(1+\tfrac{z}{j}\right) \leq \log(1+z) + \int_1^k \log\left(1+\tfrac{z}{t}\right)\, dt = (z+k)\log(z+k) - k\log k - z\log(z+1),$$

putting $x = z+k$ in this inequality gives the result.

Notice also that the function $k \mapsto k^k(m-k)^{m-k}$ of the real variable k is decreasing in the interval $0 < k < m/2$ and non-decreasing in the interval $m/2 \leq k < m$.

In our case, combining the above results we get

$$\begin{aligned}
|u_{i_0}| &\leq \frac{(s_2+1/5)^{s_2+1/5}}{\left(s_1 - \frac{s_1}{\sqrt{x+1}} - \frac{3}{8}\right)^{s_1 - \frac{s_1}{\sqrt{x+1}} - \frac{3}{8}} \left(s_2 - s_1 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right)^{s_2 - s_1 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}}} \\
&\quad \times \frac{\left(s - s_1 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right)^{s - s_1 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}}}{s_2^{s_2}\left(\frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{s_1}{\sqrt{x+1}} + \frac{3}{8}}} x^{s_1 - \frac{s_1}{\sqrt{x+1}} + \frac{7}{6}} \\
&\leq s_2^{1/5}\left(1 + \frac{1}{5s_2}\right)^{s_2+1/5} \frac{1}{\left(s_1 - \frac{s_1}{\sqrt{x+1}} - \frac{3}{8}\right)^{s_1 - \frac{s_1}{\sqrt{x+1}} - \frac{3}{8}} \left(\frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{2s_1}{\sqrt{x+1}} + \frac{3}{8}}} \\
&\quad \times \left(s_2 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}\right)^{s_2 + \frac{s_1}{\sqrt{x+1}} + \frac{3}{8}} x^{s_1 - \frac{s_1}{\sqrt{x+1}} + \frac{7}{6}}
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\binom{s}{s_1} G(x)\right| &\leq \left(\frac{n+1}{2}\right)^{1/5} e^{1/4} \frac{1}{\left(\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}\right)^{\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}} \left(\frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}} + \frac{3}{8}}} \\
&\quad \times \left(\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}} x^{\frac{n}{2} - \frac{n-1}{2\sqrt{x+1}} + \frac{2}{3}}
\end{aligned}$$

Similarly we estimate from (2.2) that

$$\begin{aligned}
\left|\binom{s}{s_1} H(x)\right| &\leq \left(\frac{n+1}{2}\right)^{1/5} e^{1/4} \frac{1}{\left(\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}\right)^{\frac{n-3}{2} - \frac{n-3}{2\sqrt{x+1}} - \frac{3}{8}} \left(\frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}} + \frac{3}{8}}} \\
&\quad \times \left(\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}\right)^{\frac{n+1}{2} + \frac{n-3}{2\sqrt{x+1}} + \frac{3}{8}} x^{\frac{n}{2} - \frac{n-1}{2\sqrt{x+1}} + \frac{2}{3}}
\end{aligned}$$

Consequently

$$|N| \le q^{\mu}(1+y)\left(\frac{n+1}{2}\right)^{1/5} e^{1/4}\frac{1}{\left(\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}\right)^{\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}}\left(\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}}+\frac{3}{8}}}$$
$$\times\left(\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}} x^{\frac{n}{2}-\frac{n-1}{2\sqrt{x+1}}+\frac{2}{3}}.$$

which, together with $y^q \le 2x^{n-1} \le x^n$ by (1), implies that

$$|N| \le q^{\mu}(1+x^{n/q})\left(\frac{n+1}{2}\right)^{1/5} e^{1/4}\frac{1}{\left(\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}\right)^{\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}}\left(\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}}+\frac{3}{8}}}$$
$$\times\left(\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}} x^{\frac{n}{2}-\frac{n-1}{2\sqrt{x+1}}+\frac{2}{3}}.$$

Finally we combine the lower and upper bound for N and use (2.10) with $s_2 \le \frac{n+1}{2}$ to obtain the estimate of the lemma. □

Now we apply the previous lemma to equation (1) when the condition $\gcd(x\varphi(x), q) = 1$ is satisfied.

Lemma 2.2. *Equation (1) with* $\gcd(x\varphi(x), q) = 1$ *and with*

$$(x \ge 49 \text{ and } n \ge 25x) \quad \text{or} \quad x \ge 961,$$

implies that

$$x \le \frac{49}{19} q \le 2.579\, q.$$

Proof. Suppose that equation (1) with $x \ge 49$, $n \ge 25x$, and $\gcd(x\varphi(x), q) = 1$ is satisfied. By subtracting 1 from both the sides of equation (1), we have

$$x\frac{x^{n-1}-1}{x-1} = (y-1)\left(\frac{y^q-1}{y-1}\right). \tag{2.11}$$

By $\gcd(x\varphi(x), q) = 1$ and the fact that every prime divisor of $(y^q-1)/(y-1)$ is either q or $\equiv 1 \pmod q$, we observe that $y \equiv 1 \pmod x$. Then we observe from equation (1) that $y^q < 2x^{n-1} \le 2(y-1)^{n-1} \le y^n$ which implies

$$n \ge q+2 \tag{2.12}$$

since n and q are odd. Thus, in the notation of Lemma 2.1, we have $x_1 = x$, $x_2 = 1$, $\alpha = n+1$ and $\beta = \frac{n}{q}$. Now we infer from Lemma 2.1 that

(2.13)

$$x^n \le (1+x^{\frac{n}{q}})q^{\frac{(n+1)q}{2(q-1)}}\left(\frac{n+1}{2}\right)^{1/5} e^{1/4}$$
$$\times\frac{1}{\left(\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}\right)^{\frac{n-3}{2}-\frac{n-3}{2\sqrt{x+1}}-\frac{3}{8}}\left(\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n-3}{\sqrt{x+1}}+\frac{3}{8}}}\left(\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}\right)^{\frac{n+1}{2}+\frac{n-3}{2\sqrt{x+1}}+\frac{3}{8}} x^{\frac{n}{2}-\frac{n-1}{2\sqrt{x+1}}+\frac{2}{3}}.$$

Then we conclude by a computer verification for $49 \le x \le 960$. For $x \ge 961$ the result is contained in Lemma 19 of [S-S]. □

We shall apply the following immediate consequence of Lemma 2.2 in the proof of Theorem 1.

Corollary 2.1. *Equation (1) with* $x = z^2$ *and with*

$$(z \ge 7 \text{ and } \quad n \ge 25x) \text{ or } x \ge 961,$$

and $\gcd\bigl(x\varphi(x), q\bigr) = 1$ *implies that*

$$z \leq 1.606\ q^{1/2}.$$

Indeed, we get more precise information. With the hypotheses of the above Corollary we get

$$\begin{aligned}
z = 7 &\implies q \notin \{5, 7, 11, 13, 17, 19\}, \\
z = 8 &\implies q \notin \{5, 7, 11, 13, 17, 19, 23, 29, 31\}, \\
z = 9 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 47\}, \\
z = 10 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 61\}, \\
z = 11 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 83\}, \\
z = 12 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 103\}, \\
z = 13 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 127\}, \\
z = 14 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 151\}, \\
z = 15 &\implies q \notin \{3, 5, 7, 11, 13, 17, 19, \ldots, 179\}
\end{aligned}$$

and $q \notin \{3, 5, 7, 11, 13, 17, 19, \ldots, 199\}$ for $x = z^2$ with $z \geq 16$.

3. An upper bound on x

As stated in Section 1, it has been proved in [S-S] that (1) with $\gcd(x\varphi(x), q) > 1$ has no solution. Thus we may suppose that $\gcd(x\varphi(x), q) = 1$. Now we compare the estimates of sections 1 and 2. This shows that there is no non-trivial solution for $x = z^2$ with $z \geq 12$. Now, in the finite number of remaining cases, we get (with the help of a computer) some big lower bound on n. More precisely, we verify that in all the remaining cases we have $n \geq 20000$.

To test whether numbers are pure powers we proceed in the following way.

We have to check a lot of numbers $w = \frac{z^n-1}{z-1}$ to be pure powers or not. Thus we have huge numbers w and we want to know whether or not there exists q such that $w = y^q$. In our special case, we have an upper bound on q but this is not very important (anyway we always have $q \leq \log_2 w$ and, looking for powers of small primes in w, one can much improve this upper bound). In a first try we used transcendental functions. But this method has two serious drawbacks: there is some danger with the rounding of real numbers and we need an enormous precision, this is quite time consuming. Notice that in our case we expect (or hope?) that w is not a pure power and we just want to be sure our guess is correct. Thus we use the following "probabilistic" algorithm: to test that $w \neq y^q$, for fixed q, we compute $r := w^{\frac{p-1}{q}} \bmod p$ for at most 30 primes of the type $p = 2kq + 1$. If it happens that $r = 0$ we compute $v = \mathrm{valuation}(w, p)$ and verify that $q \nmid v$; if both $r \neq 0, 1$ then w cannot be a q–power.

By Lemma 2.2, for $n \geq 20000$ we get

$$\begin{aligned}
z = 7 &\implies q \notin \{5, 7, 11, 13, 17, 19\},\\
z = 8 &\implies q \notin \{5, 7, 11, 13, 17, 19, 23, 29, 31\},\\
z = 9 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 47\},\\
z = 10 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 61\},\\
z = 11 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 83\},\\
z = 12 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 103\},\\
z = 13 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 127\},\\
z = 14 &\implies q \notin \{5, 7, 11, 13, 17, 19, \ldots, 151\},\\
z = 15 &\implies q \notin \{3, 5, 7, 11, 13, 17, 19, \ldots, 179\}
\end{aligned}$$

and $q \notin \{3, 5, 7, 11, 13, 17, 19, \ldots, 199\}$ for $x = z^2$ with $z \geq 16$. (Notice that the estimate of Lemma 2.2 becomes better for large n, this is the reason why we verified that n must be large.)

By the study of Section 1, after some computation one sees that:

$$\begin{aligned}
z = 5, &\quad q \leq 124 \ \text{ if } \ \log y \geq 500,\\
z = 6, &\quad q \leq 106 \ \text{ if } \ \log y \geq 1400,\\
z = 7, &\quad q \leq 101 \ \text{ if } \ \log y \geq 500,\\
z = 8, &\quad q \leq 95 \ \text{ if } \ \log y \geq 500,\\
z = 9, &\quad q \leq 90 \ \text{ if } \ \log y \geq 500,\\
z = 10, &\quad q \leq 87 \ \text{ if } \ \log y \geq 500,\\
z = 11, &\quad q \leq 86 \ \text{ if } \ \log y \geq 500,\\
z = 12, &\quad q \leq 83 \ \text{ if } \ \log y \geq 500.
\end{aligned}$$

(Again, the estimates of Section 1 become better for large n. And, $n > 20000$ implies $\log y > 500$ for $z > 9$, in the range $5 \leq z \leq 9$ we extended the verification up to $n = 110000$ which implies $\log y > 1400$.)

If we combine the two previous tables we see that the present situation is the following (recall that q is a prime number):

$$\begin{aligned}
z = 5, &\quad q \leq 113,\\
z = 6, &\quad q \leq 103,\\
z = 7, &\quad 19 < q \leq 101,\\
z = 8, &\quad 31 < q \leq 89,\\
z = 9, &\quad 47 < q \leq 89,\\
z = 10, &\quad 61 < q \leq 83,
\end{aligned}$$

and that solutions cannot exist for $z \geq 11$.

4. A computer interlude

¿From now on we shall not consider equation (1) with $x = z^2$ anymore but consider only the original equation

$$\frac{x^n - 1}{x - 1} = y^q \tag{1}$$

or its "companion"

$$\frac{x^n + 1}{x + 1} = y^q \tag{1'}$$

with $q \geq 5$. We come back to the notation x for the first variable. We consider the cases (x, q) which remain at the end of the previous section.

Using again congruences with primes p of the form $kq + 1$ for each of the remaining equations we could easily verify that, in all cases, we must have

$$n \equiv 1 \pmod q \quad \text{and} \quad n \geq 10^{11}.$$

This verification took less than one minute for each case.

Such a method was also used in [S], see the proof of Theorem 3 of this paper.

5. Study of the equation $(a+1)X^q - aY^q = 1$, new upper bounds on q

In the previous section we verified that for our finite set of equations (1) and (1') we have $n = kq + 1$ for some rational integer k. Equations (1) and (1′) give respectively

$$x \cdot (x^k)^q - (x-1) \cdot y^q = 1, \quad \text{and} \quad x \cdot (x^k)^q - (x+1)y^q = -1.$$

These two equations are both of the type

$$(a+1)X^q - aY^q = 1 \tag{5.1}$$

with $a = x - 1$ in the first case and $a = x$ in the second one (we consider these two cases for reasons which will become clear in the next section.) Equation (5.1) has been studied recently by Benett and de Weger [B-W] (see also [Mi2]) and their main result implies that this equation has only trivial solutions for $q \leq 13$ (the proof of this result involved the complete resolution of a finite number of Thue equations). Thus we may now restrict ourselves to the cases where $q \geq 17$.

Moreover, we can again apply the Theorem 1.1 of Section 1. The results are better than those of Section 1 for the equation $(a+1)X^q - (a-1)Y^q$ because $\log \frac{a+1}{a}$ is smaller than $\log \frac{a+1}{a-1}$. To be precise, applying Theorem 1.1 to the equation (5.1), we can prove that, under the hypothesis

$$\max\{\log X, \log Y\} > 10^{10}$$

(which is true in our cases since $n > 10^{11}$), we have

$$\begin{aligned}
a = 4 &\implies q \leq 72,\\
a = 5 &\implies q \leq 70,\\
a = 6 &\implies q \leq 68,\\
a = 7 &\implies q \leq 66,\\
a = 8 &\implies q \leq 64,\\
a = 9 &\implies q \leq 62.
\end{aligned}$$

In the case $z = 10$ we proved in Section 3 that $q > 61$ and the case $a = 9$ here implies $q \leq 62$, hence Theorem 1 holds for $z = 10$. In the Introduction we noticed that the cases $z = 8$ and $z = 9$ were solved in [S-S], thus we only have to treat the cases $z \in \{5, 6, 7\}$. We choose to consider equation (1) for $z = 5$ and 6 and equation (1′) for $z = 7$. Thus we have to study equation (5.1) in the following cases (recall that q is prime):

$$\begin{aligned}
a = 4, &\quad 17 \leq q \leq 71,\\
a = 5, &\quad 17 \leq q \leq 67,\\
a = 7, &\quad 17 \leq q \leq 61.
\end{aligned}$$

We are not able to solve completely these equations, but we shall give bounds on Y and this will lead to bounds on n for equation (1) or (1′). And then we shall conclude by a computer verification. See the next two sections.

6. Study of the equations $(a+1)X^q - aY^q = 1$, upper bounds on n

The aim of this Section is to provide a sharp upper bound for n if (x, y, n, q) satisfies (1) with $x \in \{5^2, 6^2, 7^2\}$ and $q \in \{17, \dots, 71\}$. Indeed, we are led to estimate the size of integer solutions of Thue equations. Our approach follows the classical lines (see *e.g.* [BG2]), however we try to use all the informations given by the fields involved. First, we state some auxiliary results.

Proposition 6.1. *Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers, not 0 or 1. Let $d \geq [\mathbf{Q}(\alpha_1, \dots, \alpha_n) : \mathbf{Q}]$ and define the modified height* h' *by*

$$\mathrm{h}'(\alpha) = \max\left\{\mathrm{h}(\alpha), \frac{|\log \alpha|}{d}, \frac{1}{d}\right\},$$

for every non-zero α in $\mathbf{Q}(\alpha_1, \dots, \alpha_n)$. Let $b_1, \dots, b_n$ be rational integers not all 0 and with absolute value less than $B\,(B \geq 3)$. Setting

$$\Lambda = \alpha_1^{b_1} \dots \alpha_n^{b_n} - 1,$$

we have

$$\log|\Lambda| > -C(n, d)\, \mathrm{h}'(\alpha_1) \dots \mathrm{h}'(\alpha_n)\, \log(nB),$$

with

$$C(n, d) = 1810\,(n+2)!\,\big(32\,d\,(n+1)\big)^{n+2} \log\big(2(n+1)\,d\big).$$

Proof : Denote by log the principal determination of the logarithm. If $|\Lambda| < 1/3$, then there exists an integer b_0, with $|b_0| \leq n\,B$, such that

$$\Omega := |b_0 \log(-1) + b_1 \log \alpha_1 + \dots + b_n \log \alpha_n|$$

satisfies $|\Lambda| \geq \Omega/2$. Noticing that $\mathrm{h}'(-1) = \pi/d$, we deduce our lower bound from the Theorem of Baker & Wüstholz [Ba-W]. □

In the following two lemmas, we denote by $\mathbf{K}$ a number field of degree $d \geq 2$. Let $R_{\mathbf{K}}$ (*resp.* $D_{\mathbf{K}}$, r) be its regulator (*resp.* class number, unit rank). There is a positive constant $\delta_{\mathbf{K}}$, depending only on $\mathbf{K}$, such that for every non-zero algebraic number $\alpha \in \mathbf{K}$ which is not a root of unity we have $\mathrm{h}(\alpha) \geq \delta_{\mathbf{K}}/d$. Voutier [V] proved that we may take $\delta_{\mathbf{K}} = 2/(\log 3d)^3$. The purpose of Lemma 6.1 is to improve this estimate when $|D_{\mathbf{K}}|$ is large. In the sequel, we denote by $\delta'_{\mathbf{K}}$ a positive constant such that, for every non-zero algebraic integer α generating $\mathbf{K}$ and which is not a root of unity, we have $\mathrm{h}(\alpha) \geq \delta'_{\mathbf{K}}/d$.

Lemma 6.1. *For every algebraic integer η which generates $\mathbf{K}$ we have*

$$\delta'_{\mathbf{K}} \geq \frac{\log|D_{\mathbf{K}}| - d \log d}{2(d-1)}.$$

Proof : As in Mignotte [Mi3], we use Hadamard inequality to obtain

$$|D_{\mathbf{K}}| \leq \mathrm{Discr}(1, \eta, \dots, \eta^{d-1})^2 \leq d^d\, \mathbf{M}(\eta)^{2(d-1)},$$

and we conclude by $d \log \mathbf{M}(\alpha) = \mathrm{h}(\alpha)$. □

Lemma 6.2 below asserts the existence of an independent system of units with small height and is well known. Notice that in their paper on Thue equations, Bugeaud & Győry [BG] use fundamental systems of units. However, this is not needed, and this remark (already noted in [B]) allows us to remove a factor $2^{-r+1}\,r!$ without further complications.

Lemma 6.2. *Assume that d is prime. There exists in $\mathbf{K}$ an independent system $\{\varepsilon_1, \dots, \varepsilon_r\}$ of units with the following properties :*

$$\prod_{i=1}^{r} \mathrm{h}(\varepsilon_i) \leq d^{-r}\, r!\, R_{\mathbf{K}},$$

$$\mathrm{h}(\varepsilon_i) \leq d^{-1}\, r!\, \delta'^{-r+1}_{\mathbf{K}}\, R_{\mathbf{K}}, \quad i = 1, \dots, r,$$

the absolute values of the entries

of the inverse matrix of $(\log|\varepsilon_i|_{v_j})_{i,j=1,\dots,s-1}$ do not exceed $r!\,\delta'^{-1}_{\mathbf{K}}$.

Moreover, for every non-zero algebraic integer $\alpha \in \mathbf{K}$, there exists a unit ε in the multiplicative subgroup of $\mathbf{K}^*$ generated by the ε_i such that

$$\mathrm{h}(\varepsilon\alpha) \leq |\mathrm{N}(\alpha)|^{1/2d} \exp\{d^{-1}\, r\, r!\, \delta'^{-r+1}_{\mathbf{K}}\, R_K\}$$

Proof : See Lemme 1 and Lemme 2 in [B]. Notice that the first three statements can easily be deduced from the proof of Lemma 1 of [BG1] and that the last one follows from the second one and the last inequality in the proof of Lemma 2 in [B]. Further, we observe that since d is prime each unit ε_i generates $\mathbf{K}$. That is the reason why we have replaced $\delta_{\mathbf{K}}$ by $\delta'_{\mathbf{K}}$ in the statement of the lemma. □

In order to bound the regulator of the fields we will meet, which are all Kummer fields, we need two further lemmas.

Lemma 6.3. *Let q be an odd prime number and let a be an integer coprime with q and free of q-th powers. Let Δ be the discriminant of the field generated by the real root of $X^q - a$ and denote by $\mathrm{sfp}(a)$ the square-free part of a. Then, if $a^{q-1} \not\equiv 1 \bmod q^2$, we have $|\Delta| = q^q\,(\mathrm{sfp}(a))^{q-1}$, and, else, we have $|\Delta| = q^{q-2}\,(\mathrm{sfp}(a))^{q-1}$.*

Proof : This is in the work of Wegner [W], see at the end of the third Section. □

Lemma 6.4. *Let t denote the number of distinct complex embeddings of $\mathbf{K}$ and set $\Delta = (2/\pi)^t\,|D_{\mathbf{K}}|^{1/2}$. Then we have*

$$R_{\mathbf{K}} \le \Delta\,(\log\Delta)^{d-1-t}\,(d-1-\log\Delta)^t\,/(d-1)!$$

Proof : This is Theorem 6.5 in Lenstra [L]. □

As explained in Section 5, we apply the preceeding Proposition and Lemmas to obtain an explicit upper estimate for the size of the solutions of the Thue equations

$$(a+1)\,X^q - a\,Y^q = 1, \tag{5.1}$$

where $q \ge 3$ is a prime number and $a \ge 2$ is is a rational integer. Hence, we work with the equation

$$X^q \pm a'\,Y^q = \pm m, \tag{6.1}$$

where $a' = a\,(a+1)^{q-1}$ and $m = (a+1)^{q-1}$. Further, we assume that the polynomial $X^q - a'Y^q$ is irreducible.

We denote by ζ a primitive q-th root of unity and by A a q-th root of a' satisfying

$$|X - A\,Y| = \min_{0\le j\le q-1} |X - \zeta^j\,A\,Y|. \tag{6.2}$$

Let $\mathbf{K}$ be the number field generated by A. We denote by $D_{\mathbf{K}}$ (*resp.* $R_{\mathbf{K}}$, r) the discriminant of $\mathbf{K}$ (*resp.* its regulator, unit rank), and we observe that $q = 2r+1$. We may assume that

$$|Y| \ge \exp\{10^{20}\,q^2\,r!\,\delta'^{-r+1}_{\mathbf{K}}\} + |m|^{10} + |a'|^{10}. \tag{6.3}$$

We deduce from (6.2) and (6.3) that for $j \ne 0$ we have $|Y|\,|A\,\zeta^j - A| \le |X - \zeta^j\,AY| + |X - AY| \le 2\,|X - \zeta^j\,AY|$, and

$$\begin{aligned} |X - AY| &= \frac{|m|}{\prod_{1\le l\le q-1} |X - \zeta^j\,AY|} \\ &\le \left|\frac{2}{AY}\right|^{q-1} \cdot \frac{|m|}{\prod_{1\le l\le q-1} |\zeta^j - 1|}. \end{aligned} \tag{6.4}$$

Let $\varepsilon_1, \dots, \varepsilon_r$ be an independent system of units in $\mathbf{K}$ satisfying the properties stated in Lemma 6.2. Then there exist rational integers $b_1, \dots, b_r$ and an algebraic integer $\beta \in \mathbf{K}$ such that

$$X - AY = \beta\,\varepsilon_1^{b_1} \dots \varepsilon_r^{b_r} \tag{6.5}$$

with

$$\mathrm{h}(\beta) \le q^{-1}\,r\,r!\,\delta'^{-r+1}_{\mathbf{K}}\,R_{\mathbf{K}} + (\log|m|)/2. \tag{6.6}$$

Set $\varepsilon = \varepsilon_1^{b_1} \dots \varepsilon_r^{b_r}$ and $B = \max\{|b_1|, \dots, |b_r|, 3\}$. Denote by the superscripts $^{(1)}$ and $^{(2)}$ the embeddings from $\mathbf{K}$ into $\mathbf{C}$ corresponding respectively to $\sigma(A) = \zeta\,A$ and $\sigma(A) = \zeta^2\,A$. Working in the field $\mathbf{Q}(A, \zeta)$, of degree not greater than $q(q-1)$, we write the so-called Siegel identity and, immediately, we obtain

$$\left|\frac{\zeta^2 - \zeta}{\zeta - 1} \cdot \frac{\beta\,\varepsilon}{\beta^{(2)}\,\varepsilon^{(2)}}\right| = \left|\frac{(\zeta^2 - 1)\,\beta^{(1)}}{(\zeta - 1)\,\beta^{(2)}} \cdot \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} - 1\right| =: \Lambda. \tag{6.7}$$

On the one hand, using (6.3), (6.4) and $q \geq 3$, we have

$$\log \Lambda \leq \log|\beta\,\varepsilon| + \log|X - \zeta^2 A\,Y| \leq -(q - 5/2)\log|Y| \leq -0.5\,\log|Y|, \tag{6.8}$$

and on the other hand, we infer from (6.5), (6.7) and Proposition 1 that

$$\log \Lambda \geq -C\big(r+1, q(q-1)\big) \prod_{j=1}^{r} \mathrm{h}'\left(\frac{\varepsilon_j^{(1)}}{\varepsilon_j^{(2)}}\right) \mathrm{h}'\left(\frac{(\zeta^2-1)\,\beta^{(1)}}{(\zeta-1)\,\beta^{(2)}}\right) \log((r+1)B). \tag{6.9}$$

However, we easily see that $\mathrm{h}'(\varepsilon_j^{(1)}/\varepsilon_j^{(2)}) \leq 2\,\mathrm{h}'(\varepsilon_j)$, for $1 \leq j \leq r$, and that, by $q \geq 3$ and (6.6), we also have

$$\mathrm{h}'\left(\frac{(\zeta^2-1)\,\beta^{(1)}}{(\zeta-1)\,\beta^{(2)}}\right) \leq 2\,\mathrm{h}(\beta) + \log 4 \leq 2\,r\,r!\,{\delta'}_{\mathbf{K}}^{-r+1}\,R_{\mathbf{K}} + \log|4m|. \tag{6.10}$$

Further, for $1 \leq j \leq r$, using that $|\log\varepsilon_j|^2 \leq \mathrm{h}(\varepsilon_j)^2 + \pi^2$, we verify that

$$\mathrm{h}'(\varepsilon_j) \leq \max\{1, 3.3\,{\delta'}_{\mathbf{K}}^{-1}\}. \tag{6.11}$$

Arguing as in [12], we bound B in terms of $|Y|$ by Lemma 3, (6.3), (6.1) and (6.5) , and we obtain

$$\begin{aligned} B &\leq 2\,q\,r!\,{\delta'}_{\mathbf{K}}^{-1}\,(\mathrm{h}(\beta) + \mathrm{h}(X - AY)) \\ &\leq 6\,q\,r!\,{\delta'}_{\mathbf{K}}^{-1}\,\log|Y|, \end{aligned}$$

whence by (6.3)

$$\log((r+1)B) \leq 2\,\log\big((\log 2|Y|)/\log 3\big).$$

Using this together with $q \geq 3$, (6.8), (6.9), (6.10) and (6.11), we get

$$\begin{aligned} \log|2Y| \leq 2\,C\big(r+1, q(q-1)\big)\,2^r \prod_{j=1}^{r} \mathrm{h}(\varepsilon_j)\,\max\{1, 3.3\,{\delta'}_{\mathbf{K}}^{-1}\}^r \times \\ (2r)\,(r!)\,{\delta'}_{\mathbf{K}}^{-r+1}\,R_{\mathbf{K}}^2\,\log\big((\log 2|Y|)/\log 3\big) \end{aligned} \tag{6.12}$$

and we can deduce an upper bound for $|Y|$ and for $|X|$.

However, for the problem considered in this paper, we know (see Section 5) that either $X = (a+1)^{(n-1)/q}$ or $Y = a^{(n-1)/q}$, depending on with which equation (5.1) we work. In both cases, we get $(n-1)/q \leq (\log 2|Y|)/\log 2$, whence, by (6.12), Lemma 6.2, Lemma 6.4 and the Proposition, we obtain the estimate

$$n \leq 4\,r\,\Theta\,\log\Theta,$$

where

$$\begin{aligned} \Theta = 2472\,r\,(2r+1)^3\,\log(2r+2)\,(r!)^2\,(r+3)!\,\big(128\,r\,(r+2)\big)^{r+3} \times \\ {\delta'}_{\mathbf{K}}^{-r+1}\,\max\{1, 3.3\,{\delta'}_{\mathbf{K}}^{-1}\}^r\,R_{\mathbf{K}}^2. \end{aligned} \tag{6.13}$$

According to Section 5, we have to consider the following Thue equations

$$5X^q - 4Y^q = 1 \quad \text{with } X = 5^{(n-1)/q},$$

$$6X^q - 5Y^q = 1 \quad \text{with } X = 6^{(n-1)/q}$$

and

$$8X^q - 7Y^q = 1 \quad \text{with } Y = 7^{(n-1)/q}, \tag{6.14}$$

where q is a prime number with $17 \leq q \leq 71$. Observe that we deal with (6.14) rather than with $7X^q - 6Y^q = 1$ since the discriminant of the field generated by the q-th root of $6/7$ is larger than the one of $\mathbf{Q}(\sqrt[q]{8/7})$.

Further, except for equation (6.14) with $q = 29$, we are always in the first case of Lemma 6.3. Hence, denoting by f the square-free part of $a(a+1)$, we get $|D_{\mathbf{K}}| = f^{2r}\,(2r+1)^{2r+1}$, whence, by Lemma 6.1, ${\delta'}_{\mathbf{K}} \geq (\log f)/2$ and, by (6.13), we have

$$\begin{aligned} \Theta \leq 155\,r\,(2r+1)^3\,\log(2r+2)\,(r!)^2\,(r+3)!\,\big(256\,r\,(r+2)\big)^{r+3}\,(\log f)^{-r+1} \times \\ \max\{1, 6.6\,\log^{-1} f\}^r\,\Delta^2\,(\log\Delta)^{2r}\,(2r + \log\Delta)^{2r}\,\big((2r)!\big)^{-2}, \end{aligned} \tag{6.15}$$

with $\Delta = (2/\pi)^r\,f^r\,(2r+1)^{r+1/2}$. Some easy calculation shows that estimate (6.15) remains true for equation (6.14) with $q = 29$, since we know by Lemma 6.3 that the discriminant of the corresponding field is equal to $\pm 14^{28}\,29^{26}$.

7. The last battle

By the study ended in Section 5, to finish the proof of Theorem 1, we consider the finite set of equations

$$
\begin{aligned}
5^n - 1 &= 4y^q, \quad 17 \le q \le 71,\\
6^n - 1 &= 5y^q, \quad 17 \le q \le 67,\\
7^n + 1 &= 8y^q, \quad 17 \le q \le 61.
\end{aligned}
\tag{7.1}
$$

¿From the study of Section 6, we can obtain the following bounds on n for these equations

q	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$x=5$	200	200	210	250	270	330	370	390	430	490	555	575	640	680
$x=6$	200	200	220	265	285	345	385	405	450	515	580	600	670	
$x=7$	200	200	215	255	275	335	380	395	435	500	565	580		

where for the column q and the line x the corresponding value, say $N = N_{x,q}$ means that $n \le 10^N$ for the associated equation of type (7.1).

The computing problem which remains is not so simple. The bounds are rather big for the **exponent** n. Of course, we have to use congruences mod primes $p \equiv 1 \pmod q$. But we have to use a lot of them and we had important difficulties to automatize the procedure.

More precisely, to prove that, except the trivial solution $n = 1$, we must have $n > 10^N$ we have to collect information for many p's. At each time we have a table T of possible values of n modulo some number M. We may initialize the procedure with $T = \{1\}$ and $M = 2$ for example. Taking a new prime p, we compute the order of x mod p, say v. If we use this value of p, we have to test for each element of T whether or not the corresponding equation (7.1) has a solution mod p; then we get a new table T' of "survivants" n with the modulus $M' = \operatorname{lcm}(M, v)$. But in the present situation, we have to deal with a table T which contains (maybe a lot of) integers of several hundreds of digits and, if we are not careful enough, the size of the table T may explode. So we have to reject p when it leads (or may lead) to too big a table. On the opposite, if we reject too many primes to keep T not too big then the modulus M does not increase enough fastly and we can not reach the necessary condition $M > 10^N$. Indeed, it took us several days before we had a satisfactory strategy.

Then, after a long week-end of computation on several fast stations (using PARI) we could prove the following result which completes the proof of Theorem 1.

Proposition 7.1. *The set of diophantine equations*

$$
\begin{aligned}
5^n - 1 &= 4y^q, \quad 3 \le q \le 71,\\
6^n - 1 &= 5y^q, \quad 3 \le q \le 67,\\
7^n + 1 &= 8y^q, \quad 3 \le q \le 61.
\end{aligned}
$$

has only the trivial solution $n = 1$.

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Added in proof: When the results of this paper were presented at Luminy in May 1998, M. Bennett announced a general result on the equation $ax^n - (a+1)y^n = 1$ for $n \geq 3$ and a positive, namely that this diophantine equation has only the trivial positive solutions $x = y = 1$, this result easily implies the main result of this paper (see Sections 5 and 6 of the present paper).