SUMS OF THE ERROR TERM FUNCTION IN THE MEAN SQUARE FOR $\zeta(s)$

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Abstract. Sums of the form $\sum_{n\le x} E^k(n)$ ($k \in \mathbb{N}$ fixed) are investigated, where

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t - T\Bigl(\log\frac{T}{2\pi} + 2\gamma - 1\Bigr)$$

is the error term in the mean square formula for $|\zeta(\frac{1}{2}+it)|$. The emphasis is on the case $k = 1$, which is more difficult than the corresponding sum for the divisor problem. The analysis requires bounds for the irrationality measure of $\mathrm{e}^{2\pi m}$ and for the partial quotients in its continued fraction expansion.

1. Introduction

Recently J. Furuya [3] investigated the sums of $\Delta^k(n)$ ($n \in \mathbb{N}$) when $k = 2$ or $k = 3$, and as usual

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1)$$

denotes the error term in the classical Dirichlet divisor problem ($d(n)$ is the number of divisors of n, and $\gamma = -\Gamma'(1) = 0.577215649\ldots$ is Euler's constant). For an account of the divisor problem, the reader is referred to [7] and [8]. Furuya proved that (c_1, c_2 are suitable constants)

$$\sum_{n\le x} \Delta^2(n) = \int_1^x \Delta^2(u)\,\mathrm{d}u + \frac{1}{6}x\log^2 x + c_1 x\log x + c_2 x + R(x),$$

where $R(x) = O(x^{3/4}\log x)$ and at the same time $R(x) = \Omega_\pm(x^{3/4}\log x)$. Here and below, $f(x) = \Omega_\pm(g(x))$ means that both

$$\limsup_{x\to\infty} f(x)/g(x) > 0, \quad \liminf_{x\to\infty} f(x)/g(x) < 0$$

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hold. This improves much on G.H. Hardy's classical result [5] that $R(x) = O_\varepsilon(x^{1+\epsilon})$, where the subscript means that the O–constant depends only on ε, an arbitrarily small positive constant. The work [3] also brings forth a corresponding asymptotic formula for $\sum_{n\le x}\Delta^3(n)$, where again the error term is precisely determined. It seems natural to try to generalize Furuya's results to sums of $E^k(n)$, where $k\in\mathbb{N}$ is fixed, $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ ($\Re\mathrm{e}\, s > 1$) is the Riemann zeta-function, and

$$E(T) = \int_0^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}t - T\Bigl(\log\frac{T}{2\pi} + 2\gamma - 1\Bigr) \tag{1.1}$$

is the error term in the mean square formula for $|\zeta(\frac12+it)|$. This is motivated by the general analogy between $\Delta(x)$ and $E(T)$, inspired by the pioneering work of F.V. Atkinson [1], who produced an explicit formula for $E(T)$ with a good error term (see also [7] and [8]). Later research established more precise analogies between the two functions, and in particular we refer to the works of M. Jutila [14], [15], and the second author [10], [11].

In considering the sums $\sum_{n\le x} E^k(n)$, it turns out that already for $k=1$ the problem is somewhat different from the corresponding problem for sums of $\Delta(n)$. Namely it reduces to the evaluation of

$$\int_0^T \psi(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \qquad \Bigl(\psi(t) = t - [t] - \tfrac12\Bigr), \tag{1.2}$$

and it would be interesting to investigate also the asymptotic behaviour of the more general integral, where $\psi(t)$ (of period one) is replaced by a general periodic function of arbitrary period. To see how (1.2) arises, note that

$$\begin{aligned}
\sum_{n\le x} E(n) &= \sum_{n\le x}(E(n)-\pi) + \pi x + O(1)\\
&= \pi x + O(1) + \int_0^x (E(u)-\pi)\,\mathrm{d}[u]\\
&= \pi x + O(1) + G(x) - \int_0^x (E(u)-\pi)\,\mathrm{d}\psi(u)\\
&= \pi x + O(\log x) + G(x) + O(|E(x)|) + \int_0^x \psi(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t,
\end{aligned}$$

where

$$G(x) := \int_0^x (E(u)-\pi)\,\mathrm{d}u, \quad G(x) = O(x^{3/4}), \quad G(x) = \Omega_\pm(x^{3/4}). \tag{1.3}$$

The results on $G(x)$ in (1.3) are to be found in the second author's joint work with J.L. Hafner [4]. In what concerns the summatory function of $\Delta(n)$, we recall that already G.F. Voronoï [19] proved that

$$\sum_{n\le x}\Delta(n) = \tfrac12 x\log x + (\gamma - \tfrac14)x + R_0(x), \tag{1.4}$$

where $R_0(x) = O(x^{3/4})$ and at the same time $R_0(x) = \Omega_\pm(x^{3/4})$.

In working out the proof of (1.4), one actually obtains

$$\sum_{n\le x}\Delta(n) = \tfrac12 x\log x + (\gamma - \tfrac12)x + \Delta(x) + \int_1^x \Delta(t)\,\mathrm{d}t + O(\log x), \tag{1.5}$$

and (1.4) follows from (1.5) when one recalls Voronoï's classical results [19] that

$$\int_1^x \Delta(t)\,\mathrm{d}t = \frac{x}{4} + R_1(x),\quad R_1(x) = O(x^{3/4}),\quad R_1(x) = \Omega_\pm(x^{3/4}). \tag{1.6}$$

The results in (1.3) on $G(x)$ are in fact the analogues of Voronoï's results in (1.6).

Let us point out the basic difference between the (discrete) function $\Delta(n)$ and the (continuous) function $E(n)$ in the context of their respective summatory functions. For the former we start (we may clearly assume that $x\in\mathbb{N}$ in what follows) with

$$\sum_{n\le x}\Delta(n) = \sum_{m\le x} d(m)\Big(\sum_{m\le n\le x} 1\Big) - \sum_{n\le x} n(\log n + 2\gamma - 1). \tag{1.7}$$

The last sum is easily evaluated by the Euler-Maclaurin summation formula. Since we have

$$\sum_{m\le n\le x} 1 = x - m + 1 \qquad (m,\, x\in\mathbb{N}), \tag{1.8}$$

one arrives without any major difficulties at (1.5). But in the analogue of (1.7) for $E(n)$, the first sum on the right-hand side of (1.7) will correspond to

$$\int_0^x |\zeta(\tfrac12 + it)|^2\Big(\sum_{t\le n\le x} 1\Big)\,\mathrm{d}t. \tag{1.9}$$

Observe that in (1.9) t is not an integer (except for $O(x)$ exceptions), and an identity like (1.8) will not hold for the sum over n in (1.9). Instead, the function $[t] = t - \psi(t) - \frac12$ will come into play, producing (1.2). It is rather curious, as will be shown in the next section, that our problem will reduce to bounds for the irrationality measure of $\mathrm{e}^{2\pi m}$ and for the partial quotients in its continued fraction expansion.

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2. The summatory function of $E(n)$

In this section we shall continue the discussion on the summatory function of $E(n)$. In Section 1 we established that

$$\sum_{n\le x} E(n) = \pi x + \int_0^x \psi(t)|\zeta(\tfrac12+it)|^2\,{\rm d}t + G(x) + O(x^{1/3}), \tag{2.1}$$

where $G(x)$ is given by (1.3), and where we used the (weak) bound (see [7] or [8]) $E(x) \ll x^{1/3}$. Since $\psi(t)$ is an oscillating function, one expects that there will be a lot of cancellations in the integral in (2.1), but this turns out to be hard to prove.

To bound the integral (2.1), we first consider subintegrals over intervals of the form $[T,\,2T]$. Then we use the approximate functional equation for $\zeta^2(s)$ (see e.g., [7] and [10]) in the form

$$|\zeta(\tfrac12+it)|^2 = 2\sum_{n\le t/(2\pi)} d(n)n^{-1/2}\cos\left(t\log\frac{t}{2\pi n} - t - \frac{\pi}{4}\right) + \mathcal{R}(t), \tag{2.2}$$

where the error term function $\mathcal{R}(t)$ satisfies the mean square relation

$$\int_1^T \mathcal{R}^2(t)\,{\rm d}t \;=\; AT^{1/2} + O(\log^5 T) \qquad (A>0). \tag{2.3}$$

The asymptotic formula (2.3) was proved by I. Kiuchi [16], who used in his proof precise asymptotic expressions for $\mathcal{R}(t)$ (related to $\Delta(t/(2\pi))$, due to Y. Motohashi [17]. Using (2.3) we obtain, by the Cauchy-Schwarz inequality for integrals,

$$\begin{aligned}\int_T^{2T} \psi(t)|\zeta(\tfrac12+it)|^2\,{\rm d}t \;&=\; O(T^{3/4}) + \\ &+ \sum_{n\le T/\pi} d(n)n^{-1/2}\int_{\max(T,2\pi n)}^{2T} \psi(t)\cos\left(t\log\frac{t}{2\pi n} - t - \frac{\pi}{4}\right){\rm d}t.\end{aligned}$$

For the integral on the right-hand side we use the familiar Fourier series

$$\psi(x) = -\frac{1}{\pi}\sum_{m=1}^\infty \frac{\sin(2\pi m x)}{m} \qquad (x\not\in\mathbb{Z}), \tag{2.4}$$

which is boundedly convergent and thus may be integrated termwise. This leads to

$$\begin{aligned}\int_T^{2T} \psi(t)|\zeta(\tfrac12+it)|^2\,{\rm d}t \;&=\; O(T^{3/4}) - \\ -\frac{1}{\pi}\sum_{m\le \log T}\frac{1}{m}\sum_{n\le T/\pi} d(n)n^{-1/2} &\int\limits_{\max(T,2\pi n)}^{2T} \sin(2\pi m t)\cos\left(t\log\frac{t}{2\pi n} - t - \frac{\pi}{4}\right){\rm d}t.\end{aligned}$$

Namely, when we write the sines and cosines as exponentials, the relevant exponential integrals will be of the form

$$\int_{\max(T,2\pi n)}^{2T} e^{\pm iF(t)}\,dt \qquad \left(1 \le n \le \frac{T}{\pi}\right),$$

where

$$F(t) = t\log\frac{t}{2\pi n} - t - 2\pi mt,\ F'(t) = \log\frac{t}{2\pi n} - 2\pi m,\ F''(t) = \frac{1}{t}.$$

Therefore for $m > \log T$ we have $|F'(t)| \gg m$, hence by the first derivative test for exponential integrals (see e.g., Lemma 2.1 of [7]) the contribution of such m is $O(T^{1/2}\log T)$. Next, for $T/(2\pi) \le n \le T/\pi$, we have

$$|F'(t)| = 2\pi m - \log\frac{t}{2\pi n} \ge 2\pi m - \log\frac{t}{T} \ge 2\pi m - \log 2 \ge \pi m \quad (m \in \mathbb{N}),$$

hence the contribution of such n is again $O(T^{1/2}\log T)$. Likewise the contribution of satisfying $m > (1/(2\pi) + \varepsilon)\log T$ is $O(T^{3/4})$. What remains is

(2.5)
$$\int_T^{2T} \psi(t)|\zeta(\tfrac12 + it)|^2\,dt \;=\; O(T^{3/4})$$
$$-\frac{1}{\pi}\sum_{m\le(1/(2\pi)+\varepsilon)\log T}\frac{1}{m}\sum_{n\le T/(2\pi)}\frac{d(n)}{n^{1/2}}\int_T^{2T}\sin(2\pi mt)\cos\left(t\log\frac{t}{2\pi n} - t - \frac{\pi}{4}\right)dt.$$

The saddle point (the root of $F'(t) = 0$) is $t_0 = 2\pi n e^{2\pi m}$, and it lies in $[T,\,2T]$ for $n \asymp Te^{-2\pi m}$. Using the saddle-point method (direct integration over a suitable contour in the complex plane joining the points T, $2T$ and passing through t_0) we obtain that the expression in (2.5) is asymptotic to sums of the type ($C > 0$ is a generic constant)

$$C\sum_{m\le(1/(2\pi)+\varepsilon)\log T}\frac{e^{\pi m}}{m}\sum_{n\asymp Te^{-2\pi m}} d(n)\exp\left(-2\pi i n e^{2\pi m}\right). \tag{2.6}$$

The problem is thus reduced to the estimation of the sum over n in (2.6). One can use the results of a classic paper by J.R. Wilton [21]. He proved, among other things, the functional equation

$$D(x,\eta) = \eta^{-1}D(\eta^2 x, -\eta^{-1}) + O(x^{1/2}\log x), \quad D(x,\eta) = \sum_{n\le x} d(n)e^{2\pi i\eta n}, \tag{2.7}$$

where $\eta^2 x \gg 1$ and $0 < \eta \le 1$ is real. After taking the conjugate sum in (2.6), (2.7) can be applied with $x = CTe^{-2\pi m}$, $\eta = \{e^{2\pi m}\}$, where $\{y\} = y - [y]$ denotes the fractional part of y. The problem is that it is difficult to find an explicit expression for $\eta = \eta(m)$. On the other hand, Wilton (op. cit.) states that for irrational η one always has

$$D(x,\eta) \;=\; o(x\log x) \qquad (x\to\infty) \tag{2.8}$$

uniformly in $\eta = \eta(m)$. The bound (2.8) can be applied in our case, since the numbers $\eta(m)$ are all transcendental. Namely, since the number e^{π} is transcendental, all the numbers $\mathrm{e}^{2\pi m}$ are also transcendental. The fact that $\mathrm{e}^{\pi} = (-1)^{-i}$ is transcendental follows from the solution of Hilbert's 7th problem (see e.g., C.L. Siegel [18] for a proof) that α^{β} is transcendental for algebraic $\alpha \ne 0, 1$ and irrational β. This was proved by Gel'fond and Schneider independently in 1934. Wilton [21] also proved the following: let

$$\frac{1}{\eta} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots \qquad (0<\eta\le 1)$$

be the expansion of $1/\eta$ as a simple continued fraction. If η is irrational and the partial quotients a_n satisfy $a_n \ll n^{1+K}$ $(K\ge 0)$, then

$$\sum_{n\le x} d(n)\sin(2\pi n\eta) \ll x^{1/2}\log^{2+K}x, \tag{2.9}$$

while if

$$a_n \ll \mathrm{e}^{Kn}\ (K>0) \tag{2.10}$$

holds, then

$$\sum_{n\le x} d(n)\exp(2\pi i n\eta) \ll x^{H}\log x \qquad \left(H = \frac{4K+\log 2}{4K+2\log 2}\right). \tag{2.11}$$

Therefore the information on the size of the a_n's would lead, by (2.9) or (2.11), to good bounds for $\sum_{n\le x} d(n)\sin(2\pi n\eta)$, which is the sum we need to estimate. Additional care should be exerted because in our case $\eta = \mathrm{e}^{2\pi m}$, so that $a_n = a_n(m)$ will depend on m as well. To determine the true order of $a_n = a_n(m)$ seems to be a difficult problem; even the fact that e^{π} is not a Liouville number is not known. In the next section we shall provide an explicit bound for the partial quotients of $\mathrm{e}^{2\pi m}$. Although poorer than (2.10), it is still sufficient to provide a non-trivial bound which in our case improves on the general bound (2.8) in the case when η is irrational.

3. The results on the sum of $E(n)$

We begin by recalling a bound for the irrationality measure of $\mathrm{e}^{\pi m}$. Namely, for any positive integers m, p and q with $p \geq 3$, we have

$$\left|\mathrm{e}^{\pi m} - \frac{p}{q}\right| > \exp\Big\{-2^{72}\,(\log 2m)(\log p)(\log\log p)\Big\}. \tag{3.1}$$

This result was proved by M. Waldschmidt [20, p. 473]. Now for any non-zero integer m let

$$\mathrm{e}^{\pi m} = [a_0(m); a_1(m), \dots]$$

be the expansion of $\mathrm{e}^{\pi m}$ as a continued fraction.

LEMMA 1. *There exists an absolute positive constant c such that*

$$\log\log a_n(m) < c(n + \log|m|)\log(n + \log|m|) \tag{3.2}$$

holds for any non-zero integer m and any non-negative integer n.

It is sufficient to establish Lemma 1 for positive values of m, since $\mathrm{e}^{-\pi m} = [0; a_0(m), a_1(m), \dots]$ for $m \geq 1$. Let ξ be a positive real number with continued fraction expansion

$$\xi \;=\; [a_0; a_1, a_2, \dots]$$

and set, for $n \geq 1$,

$$\frac{p_n}{q_n} \;=\; [a_0; a_1, a_2, \dots, a_n].$$

¿From the theory of continued fractions it is known that

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \qquad (n \geq 1). \tag{3.3}$$

It follows from (3.1) that

$$\left|\mathrm{e}^{\pi m} - \frac{p}{q}\right| > \exp\{-2^{75}(\log 2m)(\log q)(\log\log q)\}, \tag{3.4}$$

for all p/q with $q \geq \mathrm{e}^{\pi m}$, since $p \leq 2q\mathrm{e}^{\pi m}$, giving

$$\log p \leq \log 2 + \log q + \pi m \leq \log 2 + 2\log q \leq 3\log q.$$

In particular, (3.4) is true for the convergents $p_h(m)/q_h(m)$ to $\mathrm{e}^{\pi m}$ as soon as $q_h(m) \geq \mathrm{e}^{\pi m}$. Let $n \geq 2$ be an integer. Writing simply p_n/q_n instead of $p_n(m)/q_n(m)$, it then follows from (3.3) and (3.4) that

$$q_n < \exp\Big\{2^{75}(\log 2m)(\log q_{n-1})(\log\log q_{n-1})\Big\}, \tag{3.5}$$

if $q_{n-1} \ge \mathrm{e}^{\pi m}$. We infer from (3.5) that there exists an absolute positive constant c_1 such that

$$\log_2 q_n < c_1 + \log_2 m + \log_2 q_{n-1} + \log_3 q_{n-1}, \tag{3.6}$$

where, for $j \ge 2$, $\log_j x = \log(\log_{j-1} x)$ ($\log_1 x \equiv \log x$) denotes the j-th iteration of the logarithm. On dividing (3.6) by $\log_3 q_n$, we obtain

$$\frac{\log_2 q_n}{\log_3 q_n} < 1 + (c_1 + \log_2 m)\,\frac{1}{\log_3 q_n} + \frac{\log_2 q_{n-1}}{\log_3 q_n}. \tag{3.7}$$

Using repeatedly the fact that $q_n > q_{n-1}$, it follows that

$$\begin{aligned}\frac{\log_2 q_n}{\log_3 q_n} &< n + (c_1 + \log_2 m)\sum_{j=h+1}^{n} \frac{1}{\log_3 q_j} + \frac{\log_2 q_h}{\log_3 q_{h+1}} \\ &< n + n\,\frac{c_1 + \log_2 m}{\log_3 q_h} + \log_2 q_h,\end{aligned} \tag{3.8}$$

where h is the integer such that $q_{h-1} < e^{\pi m}$ and $q_h > e^{\pi m}$. This choice of h and (3.1) imply that

$$q_h < \exp\{c_2 m(\log 2m)^2\}, \tag{3.9}$$

where c_2 (as c_3, c_4 and c_5 below) is an absolute positive constant. Consequently, we infer from (3.8) and (3.9) that

$$\frac{\log_2 q_n}{\log_3 q_n} < n + n\,\frac{c_1 + \log_2 m}{\log_2 m} + c_3 + \log m < c_4(n + \log m).$$

This gives at once the asserted upper bound (3.2), namely

$$\log_2 a_n < \log_2 q_n < c_5(n + \log m)\log(n + \log m).$$

Now we rewrite (3.2) as

$$a_n(m) < \exp\left\{(n + \log m)^{c(n+\log m)}\right\} \tag{3.10}$$

and proceed to estimate, in view of the transformation formula (2.7),

$$D(x, \eta(m)) := \sum_{n\le x} d(n)\exp(2\pi i\eta(m)) \qquad \Bigl(\eta(m) = \mathrm{e}^{-2\pi m},\ 1 \le m \le \frac{\log_2 x}{\log_3 x}\Bigr). \tag{3.11}$$

Then $1/\eta(m) = e^{2\pi m}$, and the above discussion applies with m replaced by $2m$. The above range for m suffices, since by trivial estimation the contribution of $m > \log_2 T/\log_3 T$ to (2.6) is

$$\ll \sum_{m>\log_2 T/\log_3 T} \frac{e^{\pi m}}{m} \cdot \frac{T}{e^{2\pi m}} \log T \ll T \log T \exp\Big\{-C\frac{\log_2 T}{\log_3 T}\Big\}.$$

¿From Wilton's bounds [21, eqs. (6.3_{III}) and (6.3_{IV})] we have

$$D(x,\eta(m)) \ll x^{1/2}\log^2 x + \min\Big(a_N(2m)x^{1/2}\log x,\ 2^{-\frac{1}{2}N}x\log x\Big), \tag{3.12}$$

for large N satisfying $N \ll \log x$. We use (3.10) to deduce that, for

$$N < \Big(\frac{1}{c} + o(1)\Big)\frac{\log_2 x}{\log_3 x},$$

we have

$$\begin{aligned} a_N(2m)x^{1/2}\log x &\ll x^{1/2}\log x \exp\left\{\Big(N + \frac{\log_2 x}{\log_3 x}\Big)^{c(N+\log_2 x/\log_3 x)}\right\} \\ &\ll x\log x\exp\left(-C\frac{\log_2 x}{\log_3 x}\right) \qquad (C = C(c) > 0), \end{aligned}$$

while for $N \ge (1/c + o(1))\log_2 x/\log_3 x$ obviously

$$2^{-\frac{1}{2}N}x\log x \ll x\log x\exp\left(-C\frac{\log_2 x}{\log_3 x}\right).$$

This means that we obtain the following non-trivial result, which improves on (2.8), although only by a quantity that is less than $(\log x)^\varepsilon$ for any given $\varepsilon > 0$.

THEOREM 1. *If $D(x,\eta(m))$ is given by* (3.11), *then for $1 \le m \le \frac{\log_2 x}{\log_3 x}$ and some constant $C > 0$ we have*

$$D(x,\eta(m)) \ll x\log x\exp\left(-C\frac{\log_2 x}{\log_3 x}\right). \tag{3.13}$$

To apply Theorem 1 we transform the sum over n in (2.6) by (2.7) (in the range $1 \le m \le \log_2 T/\log_3 T$) with $x = CTe^{-2\pi m}$, $\eta = e^{2\pi m}$. We obtain $D(x,\eta(m))$ with $\eta(m) = e^{-2\pi m}$, $x = CTe^{2\pi m}$, plus an error term of $O(T^{1/2}\log T)$. We use (3.13) for this $D(x,\eta(m))$, estimate the contribution of the range $m \ge \log_2 T/\log_3 T$ trivially, and arrive at

THEOREM 2. *We have*

$$\sum_{n\le x} E(n) \;=\; \pi x + H(x), \tag{3.14}$$

where, for some $C>0$, unconditionally

$$H(x) \;\ll\; x\log x \exp\left(-C\frac{\log_2 x}{\log_3 x}\right). \tag{3.15}$$

If (2.9) holds with $\eta = \mathrm{e}^{2\pi m}$ uniformly for $1\le m\ll \log x$, then we have

$$H(x) \;=\; O(x^{3/4}). \tag{3.16}$$

Remark 1. The intrinsic difficulty in this problem is to bound the sum $D(x,\eta(m))$. In fact, unconditionally our bound for $H(x)$ in (3.15) is poorer than the main term πx in (3.14). However, if the bound in (2.9) is true in our case, then (3.16) can be complemented with $H(x) = \Omega_\pm(x^{3/4})$. Namely instead of using (2.2)–(2.3), we may use an (unsymmetric) approximate functional equation for $|\zeta(\frac{1}{2}+it)|^2$ with a sharp error term (see e.g., [7, Chapter 4]). In fact it may be conjectured that instead of (3.15) with (3.16) one has

$$\sum_{n\le x} E(n) \;=\; \pi x + G(x) + O_\varepsilon(x^{1/2+\varepsilon}),$$

where $G(x)$ is given by (1.3).

Remark 2. Another discrete sum involving the divisor function was investigated by Coppola–Salerno [2] and A. Ivić [12]. The former have shown that, for $T^\varepsilon \le h\le \frac{1}{2}\sqrt{T}$, $L=\log T$,

$$\sum_{T\le n\le 2T} (\Delta(n+U)-\Delta(n))^2 = \frac{8}{\pi^2}TU\log^3\Bigl(\frac{\sqrt{T}}{U}\Bigr) + O(TUL^{5/2}\sqrt{L}). \tag{3.17}$$

This is an asymptotic formula with a weak error term, improved in [12] by the second author to

$$\begin{aligned}\sum_{T\le n\le 2T} (\Delta(n+U)-\Delta(n))^2 &= TU\sum_{j=0}^{3} c_j\log^j\Bigl(\frac{\sqrt{T}}{U}\Bigr)\\ &\quad + O_\varepsilon(T^{1/2+\varepsilon}U^2) + O_\varepsilon(T^{1+\varepsilon}U^{1/2}),\end{aligned} \tag{3.18}$$

where $1\ll U\le \frac{1}{2}\sqrt{T}$ with $c_3 = 8\pi^{-2}$. However, the analogue of (3.17) or (3.18) for the sum

$$\sum_{T\le n\le 2T} (E(n+U)-E(n))^2 \tag{3.19}$$

does not carry over, for the same reason for which we had difficulties in evaluating sums of $E(n)$; because $E(T)$ is a continuous function, while $\Delta(x)$ is not, having jumps at natural numbers of order at most $O_\varepsilon(x^\varepsilon)$. The true order of magnitude of the sum in (3.19) seems elusive.

4. Sums of $E^k(n)$ for $k > 1$

We begin by noting that in general, for $k \in \mathbb{N}$ fixed, one has

$$\sum_{n\le x} E^k(n) = \int_0^x E^k(u)\,\mathrm{d}[u] = \int_0^x E^k(u)\,\mathrm{d}u - \int_0^x E^k(u)\,\mathrm{d}\psi(u). \tag{4.1}$$

Integration by parts gives

$$\begin{aligned}\int_0^x E^k(u)\,\mathrm{d}\psi(u) &= E^k(x)\psi(x) - k\int_0^x \psi(u)E^{k-1}(u)E'(u)\,\mathrm{d}u\\ &\ll 1 + |E(x)|^k + \int_{10}^x |E(u)|^{k-1}\Bigl(|\zeta(\tfrac12+iu)|^2 + \log(\frac{u}{2\pi}) + 2\gamma\Bigr)\,\mathrm{d}u,\end{aligned} \tag{4.2}$$

where we used the defining property (1.1). Here we distinguish two cases.

If $k = 2, 4$, then by the Cauchy-Schwarz inequality for integrals the last integral above is

$$\begin{aligned}&\ll \left\{\int_{10}^x E^{2k-2}(u)\,\mathrm{d}u \cdot \int_{10}^x \Bigl(|\zeta(\tfrac12+iu)|^4 + \log^2 u\Bigr)\,\mathrm{d}u\right\}^{1/2}\\ &\ll x^{k/3} + \Bigl(x^{1+\frac12(k-1)}\cdot x\log^4 x\Bigr)^{1/2} \ll x^{k/3} + x^{(k+3)/4}\log^2 x\\ &\ll x^{(k+3)/4}\log^2 x,\end{aligned}$$

where we used (see the works of D.R. Heath-Brown [6], Ivić–Sargos [13] and W. Zhai [22]),

$$\int_0^x |E(u)|^k\,\mathrm{d}u \ll x^{1+k/4} \qquad (k\in\mathbb{N},\ 1\le k\le 9). \tag{4.3}$$

Larger even values of k could be, of course, also considered, but for such values we do not have precise formulas for the integrals of E^k, as we have when $k = 2$ or $k = 4$. Namely we have (see [8])

$$\int_0^x E^2(u)\,\mathrm{d}u = C_2x^{3/2} + O(x\log^4 x) \qquad (C_2 > 0), \tag{4.4}$$

and (see W. Zhai [22, Part III])

$$\int_0^x E^4(u)\,\mathrm{d}u = C_4x^2 + O_\varepsilon(x^{2-3/28+\varepsilon}) \qquad (C_4 > 0). \tag{4.5}$$

If $k = 3, 5, 7, 9$ then $k-1$ is even, and we have $|E(u)|^{k-1} = E^{k-1}(u)$. Since (4.3) holds and $E(x) \ll x^{1/3}$, we see that the last integral in (4.2) is

$$\int_{10}^x E^{k-1}(u)\Bigl(E'(u) + O(\log u)\Bigr)\,\mathrm{d}u \ll |E(x)|^k + x^{(k+3)/4}\log x \ll x^{k/3}\log x.$$

Note that we also have (see W. Zhai [23], who improved the exponent $7/4-1/12$ of Ivić–Sargos [13] in the error term)

(4.6)
$$\int_0^x E^3(u)\,\mathrm{d}u = C_3x^{7/4} + O_\varepsilon(x^{7/4-83/393+\varepsilon}) \quad \left(\frac{83}{393} = 0.211195\ldots, C_3 > 0\right),$$

and, in general (see W. Zhai [22]) for $5 \le k \le 9$,

$$\int_0^x E^k(u)\,\mathrm{d}u = C_kx^{1+k/4} + O(x^{1+k/4-\delta(k)}) \qquad (C_k > 0), \tag{4.7}$$

where $\delta(k)$ is a positive constant which may be explicitly evaluated. ¿From (4.1)–(4.7) it follows that we have proved

THEOREM 3. *We have*

$$\sum_{n\le x} E^2(n) = C_2x^{3/2} + O(x^{5/4}\log^2 x),$$
$$\sum_{n\le x} E^3(n) = C_3x^{7/4} + O_\varepsilon(x^{7/4-83/393+\varepsilon}),$$
$$\sum_{n\le x} E^4(n) = C_4x^2 + O_\varepsilon(x^{2-3/28+\varepsilon}),$$

and for $k = 5, 7, 9$,

$$\sum_{n\le x} E^k(n) = C_kx^{1+k/4} + O(x^{1+k/4-\delta(k)}) + O(x^{k/3}\log x).$$

Naturally, other values of k could be also considered, but as already remarked the results would not be very good in view of the existing results on $\int_0^x E^k(u)\,\mathrm{d}u$.

Note. We are grateful to the referee for pointing to us that the true order of the sum in (3.19) can be determined. Namely

$$\sum_{T\le n\le 2T} (E(n+U) - E(n))^2 = \int_T^{2T} (E(t+U) - E(t))^2\,\mathrm{d}[t]$$
$$= \int_T^{2T} (E(t+U) - E(t))^2\,\mathrm{d}t - \int_T^{2T} (E(t+U) - E(t))^2\,\mathrm{d}\psi(t)$$
$$= I_1(T,U) - I_2(T,U),$$

say. For $I_1(T,U)$ we have an asymptotic formula for $T^\varepsilon \le U = U(T) \le T^{1/2-\varepsilon}$ by [12], the main term being a multiple of $TU\log^3(\sqrt{T}/U)$. Integration by parts shows that

$$I_2(T,U) = 2\int_T^{2T} (E(t+U) - E(t))(E'(t+U) - E'(t))\psi(t)\,\mathrm{d}u + O(T^{2/3}),$$

and the Cauchy-Schwarz inequality shows that $I_2(T,U) \ll TU^{1/2}\log^{7/2} T$, hence $I_2(T,U)$ is of a lower order of magnitude than $I_1(T,U)$.

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