On Mahler's classification of p-adic numbers

YANN BUGEAUD and **GÜLCAN KEKEÇ**

Abstract

We give transcendence measures for p-adic numbers ξ, having good rational (resp., integer) approximations, that force them to be either p-adic S-numbers or p-adic T-numbers.

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1. Introduction

In 1955, Roth [12] proved that irrational real algebraic numbers cannot be approximable by rational numbers at an order greater than 2.

Theorem 1.1 (Roth [12], 1955). *Let ξ be a real number and ε be a positive real number. Suppose that there exists a sequence $(p_n/q_n)_{n=1}^{\infty}$ of rational numbers such that $2 \le q_1 < q_2 < \cdots$ and*

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < q_n^{-2-\varepsilon} \qquad (n = 1, 2, \ldots).$$

Then ξ is transcendental.

In 1964, under an additional assumption on the growth of the sequence $(q_n)_{n \ge 1}$ in Theorem 1.1, Baker [2] obtained a more precise conclusion than the simple transcendence of ξ. Before stating his result, we shall recall the classifications of transcendental real numbers defined by Mahler [9] in 1932 and by Koksma [7] in 1939. Let d be a positive rational integer and ξ a real number. Then, $w_d(\xi)$ is defined as the supremum of the real numbers w_d for which there exist infinitely many polynomials $P(X)$ with rational integral coefficients and of degree at most d satisfying the inequalities

$$0 < |P(\xi)| \le H(P)^{-w_d},$$

where $H(P)$ denotes the height of the polynomial $P(X)$, that is, $H(P)$ is the maximum of the absolute values of the coefficients of $P(X)$. On the other hand, $w_d^*(\xi)$ is defined as the supremum of the real numbers w_d^* for which there exist infinitely many real algebraic numbers α of degree at most d satisfying the inequalities

$$0 < |\xi - \alpha| \le H(\alpha)^{-w_d^*-1},$$

where $H(\alpha)$ denotes the height of α, that is, $H(\alpha)$ is the height of the minimal polynomial of α over $\mathbb{Z}$. Setting $w(\xi) = \limsup_{d \to \infty} (w_d(\xi)/d)$, Mahler [9] called ξ

- an S-number if $0 < w(\xi) < \infty$,
- a T-number if $w(\xi) = \infty$ and $w_d(\xi) < \infty$ for all positive rational integers d,
- a U-number if $w(\xi) = \infty$ and $w_d(\xi) = \infty$ from some d onward.

Exactly in the same manner, setting $w^*(\xi) = \limsup_{d\to\infty} (w_d^*(\xi)/d)$, and using $w^*(\xi)$ and $w_d^*(\xi)$ instead of $w(\xi)$ and $w_d(\xi)$, Koksma [7] defined the classes of S^*-, T^*-, and U^*-numbers and proved that they coincide with those of S-, T-, and U-numbers, respectively. Thus, the real transcendental numbers are divided into three disjoint classes. (See Bugeaud [3] for details of the classifications of Mahler and Koksma.) Now we can state the result of Baker.

Theorem 1.2 (Baker [2], 1964)**.** *Let ξ be a real number and ε a positive real number. Suppose that there exists a sequence $(p_n/q_n)_{n=1}^{\infty}$ of rational numbers with* $\gcd(p_n, q_n) = 1$ $(n = 1, 2, \ldots)$ *such that $2 \le q_1 < q_2 < \cdots$ and*

$$0 < \left|\xi - \frac{p_n}{q_n}\right| < q_n^{-2-\varepsilon} \qquad (n = 1, 2, \ldots).$$

If

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} < \infty,$$

then there exists a real number c, depending only on ξ and ε, such that

$$w_d^*(\xi) \le \exp\exp\{cd^2\} \qquad (d = 1, 2, \ldots).$$

In particular, ξ is either an S-number or a T-number.

Recently, in 2010, Adamczewski and Bugeaud [1, Théorème 3.1] gave a new proof of Theorem 1.2, based on a new application (Théorème EL in [1]) of a quantitative version of Theorem 1.1 obtained by Evertse [6] and Locher [8]. Moreover, the result of Adamczewski and Bugeaud [1, Théorème 3.1] improves the transcendence measure given by Theorem 1.2. In Theorem 1.4, we give a p-adic analogue of Theorem 1.2 by following the method of the proof of Théorème 3.1 in [1]. Before stating our new result, we recall the classifications of p-adic numbers in analogy with the classifications of Mahler and Koksma of real numbers.

Throughout the present paper, p denotes a fixed prime number, and $|\cdot|_p$ denotes the p-adic absolute value on the field $\mathbb{Q}$ of rational numbers, normalized such that $|p|_p = p^{-1}$. We also denote the unique extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ of p-adic numbers, the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$, by the same notation $|\cdot|_p$.

In 1958, Ridout [11] proved the p-adic analogue of Theorem 1.1. For coprime non-zero integers x, y, write $|x, y|$ for the maximum of $|x|$ and $|y|$, that is, for the height of the rational number x/y.

Theorem 1.3 (Ridout [11], 1958)**.** *Let ξ be a p-adic number and ε a positive real number. Suppose that there exists a sequence $(x_n/y_n)_{n=1}^{\infty}$ of rational numbers with* $\gcd(x_n, y_n) = 1$ $(n = 1, 2, \ldots)$

such that $2 \le |x_1, y_1| < |x_2, y_2| < \cdots$ *and*

$$0 < \left| \xi - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-2-\varepsilon} \qquad (n = 1, 2, \ldots).$$

Then ξ *is transcendental.*

Unlike in the real case, we cannot replace $|x_n, y_n|$ by $|y_n|$ in the statement of Theorem 1.3 since, for any irrational p-adic number ξ and any positive real number ε, there exists an integer x such that $|\xi - x/y|_p$ is less than ε.

Mahler [10], in 1934, proposed a classification of transcendental p-adic numbers in analogy with his classification of transcendental real numbers. Let d be a positive rational integer and ξ a transcendental p-adic number. Then $w_d(\xi)$ is defined as the upper limit of the real numbers w_d for which there exist infinitely many polynomials $P(X)$ with rational integral coefficients and of degree at most d satisfying the inequalities

$$0 < |P(\xi)|_p \le H(P)^{-w_d - 1}.$$

Setting $w(\xi) = \limsup_{d \to \infty} (w_d(\xi)/d)$, we then call ξ

- a p-adic S-number if $0 < w(\xi) < \infty$,
- a p-adic T-number if $w(\xi) = \infty$ and $w_d(\xi) < \infty$ for all positive rational integers d,
- a p-adic U-number if $w(\xi) = \infty$ and $w_d(\xi) = \infty$ from some d onward.

The p-adic transcendental numbers are divided into the three disjoint classes S, T, and U. (See Bugeaud [3] for more information about Mahler's classification in $\mathbb{Q}_p$.) On the other hand, in analogy with Koksma's classification of real numbers, let define $w_d^*(\xi)$ as the upper limit of the real numbers w_d^* for which there exist infinitely many p-adic algebraic numbers α of degree at most d satisfying the inequalities

$$0 < |\xi - \alpha|_p \le H(\alpha)^{-w_d^* - 1}.$$

Setting $w^*(\xi) = \limsup_{d \to \infty} (w_d^*(\xi)/d)$, we define the p-adic S^*-numbers, T^*-numbers, and U^*-numbers, respectively, exactly as in the real case. Again, the classes S, T, and U are the same as the classes S^*, T^*, and U^*, respectively. (See Bugeaud [3] and Schlickewei [13].)

A first goal of the present paper is to establish the following p-adic analogue of Baker's Theorem 1.2.

Theorem 1.4. *Let* ξ *be a* p*-adic number and* ε *a positive real number. Suppose that there exists a sequence* $(x_n/y_n)_{n=1}^{\infty}$ *of rational numbers with* $\gcd(x_n, y_n) = 1$ $(n = 1, 2, \ldots)$ *such that* $2 \le |x_1, y_1| < |x_2, y_2| < \cdots$ *and*

$$0 < \left| \xi - \frac{x_n}{y_n} \right|_p < |x_n, y_n|^{-2-\varepsilon} \qquad (n = 1, 2, \ldots). \tag{1.1}$$

If

$$\limsup_{n\to\infty} \frac{\log|x_{n+1}, y_{n+1}|}{\log|x_n, y_n|} < \infty, \tag{1.2}$$

then ξ is transcendental and there exists a real number c, depending only on ξ and ε, such that

$$w_d^*(\xi) \le (2d)^{c\log\log 3d} \qquad (d = 1, 2, \ldots).$$

In particular, ξ is either a p-adic S-number or a p-adic T-number.

We point out that the bound on $w_d^*(\xi)$ in Theorem 1.4 does not depend on p. It has the same shape as the bound obtained in the real case in Theorem 3.1 of [1]. The same remarks hold for our next result.

Theorem 1.5. *Let ξ be a p-adic number and let $\xi = \sum_{k\ge -k_0} a_k p^k = \sum_{j\ge 1} a_{n_j} p^{n_j}$ denote its Hensel expansion, where $k_0 \ge 0$, $a_{-k_0} \ne 0$ if $k_0 > 0$, $(n_j)_{j\ge1}$ is strictly increasing, and a_{n_j} is in $\{1, \ldots, p-1\}$ for $j \ge 1$. Let ε be a positive real number. Suppose that there exists an increasing sequence $\mathcal{J} = (j_k)_{k\ge1}$ of positive integers such that $n_{j+1} \ge (1+\varepsilon)n_j$ for j in $\mathcal{J}$ and*

$$\limsup_{k\to\infty} \frac{j_{k+1}}{j_k} < \infty.$$

Then, ξ is transcendental and there exists a real number c, depending only on ξ and ε, such that

$$w_d^*(\xi) \le (2d)^{c\log\log 3d} \qquad (d = 1, 2, \ldots).$$

In particular, ξ is either a p-adic S-number or a p-adic T-number.

We display a straightforward application of Theorem 1.5.

Corollary 1.1. *For any real number $c > 1$, the p-adic number $\sum_{i=0}^{+\infty} p^{\lfloor c^i \rfloor}$ is either an S-number or a T-number.*

The proof of Theorems 1.4 and 1.5 follow a method introduced in [1] and depend on a quantitative version of Ridout's theorem given in Theorem 2.1 in the next section.

We take this opportunity to give, in addition, an application of Theorem 2.1 to the number of digit changes in the Hensel expansion of irrational algebraic p-adic numbers, thereby improving [4, Theorem 2].

Let ξ be a p-adic number and denote by

$$\xi = \sum_{k=-k_0}^{+\infty} a_k p^k, \quad (a_k \in \{0, 1, \ldots, p-1\}, k_0 \ge 0, a_{-k_0} \ne 0 \text{ if } k_0 > 0),$$

its Hensel expansion. For a positive integer n, set

$$\mathrm{nbdc}(n, \xi, p) = \mathrm{Card}\{1 \le k \le n : a_k \ne a_{k+1}\},$$

and

$$\mathcal{S}(n,\xi,p)=\sum_{k=1}^{n} a_k.$$

Theorem 1.6. *Let p be a prime number. Let ξ be an algebraic irrational number in $\mathbb{Q}_p$. For any positive real number δ with $\delta < 1/2$ and any sufficiently large integer n, we have*

$$\mathrm{nbdc}(n,\xi,p) \geq (\log n)^{1+\delta},$$

and there are at least $(\log n)^{1+\delta}$ non-zero digits among the first n digits of the Hensel expansion of ξ, and, moreover,

$$(\log n)^{1+\delta} \leq \mathcal{S}(n,\xi,p) \leq n(p-1)-(\log n)^{1+\delta}.$$

The proof of Theorem 1.6 follows the same lines as that of [5, Theorem 3.1]. We omit the details. The good approximations to ξ are obtained by truncating its Hensel expansion and repeating the last digit. They are rational numbers, whose denominator divides $p-1$, and we apply Theorem 2.1 to $(p-1)\xi$.

Corollary 1.2. *For every real numbers $c > 1$ and $\eta > 2/3$, the p-adic number*

$$\sum_{j\geq 1} p^{\lfloor c^{j^\eta}\rfloor}$$

is transcendental.

2. Auxiliary result

The following theorem is a consequence of [5, Proposition A.1] and is the key point in the proof of Theorems 1.4 and 1.5. It can be regarded as a p-adic analogue of Théorème EL in [1].

Theorem 2.1 (Bugeaud and Evertse [5], 2008)**.** *Let α be a p-adic algebraic number of degree d and height H, and let ε be a positive real number. Then the inequality*

$$\left|\alpha-\frac{x}{y}\right|_p \leq |x,y|^{-2-\varepsilon}$$

has at most

$$2^{26}(1+2/\varepsilon)^3\log(2d+4)\log\left((1+2/\varepsilon)\log(2d+4)\right)$$

solutions $(x,y)\in\mathbb{Z}^2$ with $\gcd(x,y)=1$ and

$$|x,y| \geq \max\left(\left(2\sqrt{d+1}H\right)^{1/((d+1)(d+2))}, 4^{2/\varepsilon}\right).$$

Likewise, the inequality

$$|\alpha - x|_p \leq |x|^{-1-\varepsilon}$$

has at most

$$2^{26}(1+1/\varepsilon)^3\log(2d+4)\log\left((1+1/\varepsilon)\log(2d+4)\right)$$

solutions $x \in \mathbb{Z}$ with

$$|x| \geq \max\left(\left(2\sqrt{d+1}H\right)^{1/((d+1)(d+2))}, 4^{1/\varepsilon}\right).$$

Theorem 2.1 follows from [5, Proposition A.1] in a same way as [5, Corollary 5.2] eventually follows from [5, Proposition A.1]. We omit the details.

3. Proof of Theorem 1.4

Let all the hypotheses of Theorem 1.4 be satisfied. It follows from Theorem 1.3 that ξ is a p-adic transcendental number. By (1.2), there exists a real number $c > 1$ such that

$$|x_n, y_n| < |x_{n+1}, y_{n+1}| \leq |x_n, y_n|^c \qquad (n = 1, 2, \ldots). \tag{3.1}$$

Let d be a positive rational integer and α be a p-adic algebraic number of degree d. We choose α with sufficiently large height $H(\alpha)$ so that

$$|x_1, y_1| \leq \left(2\sqrt{d+1}H(\alpha)\right)^{1/((d+1)(d+2))}$$

is satisfied. Let $j \geq 2$ be the unique integer satisfying

$$|x_{j-1}, y_{j-1}| \leq \left(2\sqrt{d+1}H(\alpha)\right)^{1/((d+1)(d+2))} < |x_j, y_j|. \tag{3.2}$$

We suppose that we choose α with sufficiently large height $H(\alpha)$ so that the inequality

$$|x_j, y_j| > \max\left(4^{2/\varepsilon}, 4^c(d+1)^c\right) \tag{3.3}$$

is satisfied. Let χ be the real number defined by

$$|\xi - \alpha|_p = H(\alpha)^{-\chi}.$$

We suppose that $\chi > 1$ and we will bound χ from above.

Let K be the largest of the positive integers h satisfying $|x_{j+h}, y_{j+h}|^{2+\varepsilon} < H(\alpha)^\chi$. Then

$$|\xi - \alpha|_p = H(\alpha)^{-\chi} < |x_{j+h}, y_{j+h}|^{-2-\varepsilon} \qquad (h = 1, 2, \ldots, K). \tag{3.4}$$

By (1.1) and (3.4),

$$\left|\alpha - \frac{x_{j+h}}{y_{j+h}}\right|_p \leq \max\left(\left|\xi - \frac{x_{j+h}}{y_{j+h}}\right|_p, |\xi - \alpha|_p\right) < |x_{j+h}, y_{j+h}|^{-2-\varepsilon}$$

for $h = 1, 2, \ldots, K$. As a result, the inequality

$$\left|\alpha - \frac{x}{y}\right|_p < |x, y|^{-2-\varepsilon}$$

has at least K rational solutions x/y with $\gcd(x, y) = 1$ and $|x, y| > |x_j, y_j|$. Hence, by (3.2), (3.3), and Theorem 2.1,

$$K \leq 2^{26}(1 + 2/\varepsilon)^3 \log(2d + 4) \log\left((1 + 2/\varepsilon) \log(2d + 4)\right). \tag{3.5}$$

On the other hand, by $\chi > 1$ and the choice of K, the inequalities (3.1), (3.2), and (3.3) imply that

$$|x_j, y_j|^{(2+\varepsilon)c^{K+1}} \geq |x_{j+K+1}, y_{j+K+1}|^{2+\varepsilon} \geq H(\alpha)^{\chi} \geq 2^{-\chi}(d+1)^{-\chi/2} |x_j, y_j|^{\chi/c} \geq |x_j, y_j|^{\chi/(2c)},$$

and so

$$\chi \leq 2(2+\varepsilon)c^{K+2}. \tag{3.6}$$

It follows from (3.5) and (3.6) that there exists a real number c', depending only on ξ and ε, such that

$$w_d^*(\xi) \leq (2d)^{c' \log\log 3d} \qquad (d = 1, 2, \ldots).$$

Then ξ is either a p-adic S-number or a p-adic T-number. This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.5

Let all the hypotheses of Theorem 1.5 be satisfied. It follows from Theorem 2.1 that ξ is a p-adic transcendental number. For $J \geq 1$, set $\xi_J := \sum_{j=1}^{J} a_{n_j} p^{n_j}$. By assumption, there exists a real number $c > 1$ such that

$$\xi_j < \xi_{j+1} \leq \xi_j^c \qquad (j = 1, 2, \ldots). \tag{4.1}$$

Without loss of generality, we assume that $k_0 = 0$. Consequently, we have

$$1 \leq \xi_j < p^{1+n_j} \qquad (j = 1, 2, \ldots) \tag{4.2}$$

and there exists j_0 such that

$$|\xi - \xi_j|_p = p^{-n_{j+1}} < \xi_j^{-n_{j+1}/(1+n_j)} < \xi_j^{-1-\varepsilon/2} \qquad (j \geq j_0). \tag{4.3}$$

Let d be a positive rational integer and α be a p-adic algebraic number of degree d. We choose α with sufficiently large height $H(\alpha)$ so that

$$\xi_1 \leq \left(2\sqrt{d+1}H(\alpha)\right)^{1/((d+1)(d+2))}$$

is satisfied. Let $j \geq 2$ be the unique integer satisfying

$$\xi_{j-1} \leq \left(2\sqrt{d+1}H(\alpha)\right)^{1/((d+1)(d+2))} < \xi_j. \tag{4.4}$$

We suppose that we choose α with sufficiently large height $H(\alpha)$ so that the inequality

$$\xi_j > \max\left(4^{2/\varepsilon}, 4^c(d+1)^c\right) \tag{4.5}$$

is satisfied. Let χ be the real number defined by

$$|\xi - \alpha|_p = H(\alpha)^{-\chi}.$$

We suppose that $\chi > 1$ and we will bound χ from above.

Let K be the largest of the positive integers h satisfying $\xi_{j+h}^{1+\varepsilon/2} < H(\alpha)^{\chi}$. Then

$$|\xi - \alpha|_p = H(\alpha)^{-\chi} < \xi_{j+h}^{-1-\varepsilon/2} \qquad (h = 1, 2, \ldots, K). \tag{4.6}$$

By (4.3) and (4.6),

$$|\alpha - \xi_{j+h}|_p \leq \max\left(|\xi - \xi_{j+h}|_p, |\xi - \alpha|_p\right) < \xi_{j+h}^{-1-\varepsilon/2}$$

for $h = 1, 2, \ldots, K$. As a result, the inequality

$$|\alpha - x|_p < x^{-1-\varepsilon/2}$$

has at least K solutions in positive integers x with $x > \xi_j$. Hence, by (4.4), (4.5), and Theorem 2.1,

$$K \leq 2^{26}(1 + 2/\varepsilon)^3 \log(2d + 4) \log\left((1 + 2/\varepsilon)\log(2d + 4)\right). \tag{4.7}$$

On the other hand, by $\chi > 1$ and the choice of K, the inequalities (4.1), (4.4), and (4.5) imply that

$$\xi_j^{(1+\varepsilon/2)c^{K+1}} \geq \xi_{j+K+1}^{1+\varepsilon/2} \geq H(\alpha)^{\chi} \geq 2^{-\chi}(d+1)^{-\chi/2}\xi_j^{\chi/c} \geq \xi_j^{\chi/(2c)},$$

and so

$$\chi \leq 2(1 + \varepsilon/2)c^{K+2}. \tag{4.8}$$

It follows from (4.7) and (4.8) that there exists a real number c', depending only on ξ and ε, such that

$$w_d^*(\xi) \leq (2d)^{c' \log\log 3d} \qquad (d = 1, 2, \ldots).$$

Then ξ is either a p-adic S-number or a p-adic T-number. This completes the proof of Theorem 1.5.

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YANN BUGEAUD, Université de Strasbourg, CNRS, IRMA UMR 7501, 7, rue René Descartes, 67000 Strasbourg, France
e-mail: bugeaud@math.unistra.fr

GÜLCAN KEKEÇ, Istanbul University, Faculty of Science, Department of Mathematics, 34134, Vezneciler-Istanbul/TURKEY
e-mail: gulkekec@istanbul.edu.tr