Quadratic approximation in $\mathbb{Q}_p$

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Abstract

Let p be a prime number. Let w_2 and w_2^* denote the exponents of approximation defined by Mahler and Koksma, respectively, in their classifications of p-adic numbers. It is well-known that every p-adic number ξ satisfies $w_2^*(\xi) \le w_2(\xi) \le w_2^*(\xi)+1$, with $w_2^*(\xi) = w_2(\xi) = 2$ for almost all ξ. By means of Schneider's continued fractions, we give explicit examples of p-adic numbers ξ for which the function $w_2 - w_2^*$ takes any prescribed value in the interval $(0, 1]$.

1 Introduction and results

For a real number ξ, the value of Mahler's function w_2 (resp., of Koksma's function w_2^*) at ξ is the supremum of the real numbers w for which

$$0 < |P(\xi)| \le \mathrm{H}(P)^{-w} \qquad (\text{resp.,} \quad 0 < |\xi - \alpha| \le \mathrm{H}(\alpha)^{-w-1})$$

is satisfied for infinitely many integer polynomials $P(X)$ (resp., algebraic numbers α) of degree at most two. Here and below, $\mathrm{H}(P)$ (resp., $\mathrm{H}(\alpha)$) stands for the naïve height of the polynomial $P(X)$ (resp., of the minimal defining polynomial of α over $\mathbb{Z}$).

Throughout this text, p denotes a prime number and $\mathbb{Q}_p$ is the field of p-adic numbers equipped with the p-adic absolute value $|\cdot|_p$ normalized in such a way that $|p|_p = p^{-1}$. For a p-adic number ξ, the value of Mahler's function w_2 (resp., of Koksma's function w_2^*) at ξ is the supremum of the real numbers w for which

$$0 < |P(\xi)|_p \le \mathrm{H}(P)^{-w-1} \qquad (\text{resp.,} \quad (0 < |\xi - \alpha|_p \le \mathrm{H}(\alpha)^{-w-1})$$

is satisfied for infinitely many integer polynomials $P(X)$ (resp., algebraic numbers $\alpha \in \mathbb{Q}_p$) of degree at most two.

For every real number ξ (resp., p-adic number ξ), we have $w_2^*(\xi) \le w_2(\xi) \le w_2^*(\xi) + 1$ and every real number ξ (resp., p-adic number ξ) which

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is not rational or quadratic satisfies $w_2^*(\xi) \geq 2$. References and more information (including the justification of the slight difference between the definitions of w_2 on the real numbers and on the p-adic numbers) can be found in [3, Chapter 3] and in [3, §9.3].

By modifying Baker's variant [1] of Schmidt's construction of T-numbers, Bugeaud [2] established that $w_2 - w_2^*$ takes on the set of real numbers any value in $[0, 1)$. Pejković [11, Theorem 5.2], [12] proved the p-adic analogue of this result. These constructions are quite involved and give no explicit examples of numbers ξ for which $w_2(\xi)$ differs from $w_2^*(\xi)$. Furthermore, they do not cover the case of this difference being 1.

Subsequently, Bugeaud [5, Theorems 4.1 and 4.2] used the theory of continued fractions to give explicit examples of real numbers ξ with any prescribed value (in $[0, 1]$) for $w_2(\xi) - w_2^*(\xi)$. The aim of the present paper is to establish an analogous result in the field $\mathbb{Q}_p$. Our key tool is the following restricted version of Schneider's p-adic continued fractions [14] as described for example in [4, 6] or, more generally, in [15].

Set $\begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and for a sequence $\mathbf{a} = (a_n)_{n \geq 1}$ of positive integers set

$$p_n = p_{n-1} + p^{a_n} p_{n-2}, \quad q_n = q_{n-1} + p^{a_n} q_{n-2}, \qquad (n \geq 1). \tag{1.1}$$

This implies $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ p^{a_n} & 0 \end{pmatrix}$, for $n \geq 1$, and, by induction,

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} 1 & 1 \\ p^{a_i} & 0 \end{pmatrix}, \qquad (n \geq 1), \tag{1.2}$$

where we have put $a_0 = 0$.

Taking the determinant of (1.2), we get

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1} \prod_{i=1}^{n} p^{a_i}, \qquad (n \geq 1).$$

It follows easily from (1.1) that p divides neither p_n nor q_n for any $n \geq 0$ so the last equality gives

$$\left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|_p = \frac{1}{|q_n q_{n-1}|_p} |p_n q_{n-1} - p_{n-1} q_n|_p = p^{-\sum_{i=1}^{n} a_i}.$$

Since the p-adic absolute value is non-Archimedean, for $n \geq 0$ and $k \geq 1$, we have

$$\left| \frac{p_{n+k}}{q_{n+k}} - \frac{p_n}{q_n} \right|_p = p^{-\sum_{i=1}^{n+1} a_i}. \tag{1.3}$$

This shows that $(p_n/q_n)_{n\geq 1}$ is a Cauchy sequence and therefore converges to some $\xi_{\mathbf{a}} \in \mathbb{Q}_p$. Then it follows from (1.3) that

$$\left| \xi_{\mathbf{a}} - \frac{p_n}{q_n} \right|_p = p^{-\sum_{i=1}^{n+1} a_i}, \quad (n \geq 1), \tag{1.4}$$

and we can write

$$\xi_{\mathbf{a}} = 1 + \lim_{n\to\infty} \cfrac{p^{a_1}}{1 + \cfrac{p^{a_2}}{1 + \cfrac{p^{a_3}}{\ddots + p^{a_n}}}}.$$

For ease of notation we denote the right hand side of the last equation by $[a_1, a_2, a_3, \ldots]$. In analogy with real continued fraction expansions, we call p_n/q_n the n-th convergent to $\xi_{\mathbf{a}}$ and we observe that

$$\frac{p_n}{q_n} = 1 + \cfrac{p^{a_1}}{1 + \cfrac{p^{a_2}}{1 + \cfrac{p^{a_3}}{\ddots + p^{a_n}}}}$$

and write $p_n/q_n = [a_1, a_2, \ldots, a_n]$. This should not cause any confusion since we do not use real continued fractions in the present paper.

We are now ready to present explicit examples of p-adic numbers for which the values of Mahler's and Koksma's functions differ by any prescribed value from the interval $[0, 1]$. Throughout, $\lfloor x \rfloor$ denotes the greatest integer smaller than, or equal to x.

Theorem 1. *Let $w > (5+\sqrt{17})/2$ be a real number, b a positive integer and $(\varepsilon_j)_{j\geq 0}$ a sequence taking its values in the set $\{0, 1\}$. Define the sequence $(a_{n,w})_{n\geq 1}$ by*

$$a_{n,w} = \begin{cases} b + 3j + 2, & \text{if } n = \lfloor w^j \rfloor \text{ for some } j \in \mathbb{Z}_{\geq 0}, \\ b + 3j + \varepsilon_j, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for some } j \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Set

$$\xi_w = [a_{1,w}, a_{2,w}, a_{3,w}, \ldots] \in \mathbb{Q}_p. \tag{1.5}$$

Then

$$w_2^*(\xi_w) = w - 1 \quad \text{and} \quad w_2(\xi_w) = w. \tag{1.6}$$

Although $(a_{n,w})_{n\geq 1}$ depends on $(\varepsilon_j)_{j\geq 0}$ as well as on w, we do not indicate this dependence explicitly for ease of writing. We introduce the sequence $(\varepsilon_j)_{j\geq 0}$ in order to show that our construction provides us with uncountably many explicitly given p-adic numbers ξ_w for which (1.6) is satisfied.

The next theorem covers the case where the function $w_2 - w_2^*$ takes its values strictly between 0 and 1.

Theorem 2. *Let $w \geq 16$ be a real number, b a positive integer and $(\varepsilon_j)_{j\geq 0}$ a sequence taking its values in the set $\{0,1\}$. Let η be a positive real number with $\eta < \sqrt{w}/4$. Define the sequence $(a_{n,w,\eta})_{n\geq 1}$ by*

$$a_{n,w,\eta} = \begin{cases} b+4j+3, & \text{if } n = \lfloor w^j \rfloor \text{ for some } j \in \mathbb{Z}_{\geq 0}, \\ b+4j+2, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for some } j \in \mathbb{Z}_{\geq 0} \\ & \quad \text{and } (n - \lfloor w^j \rfloor)/\lfloor \eta w^j \rfloor \in \mathbb{Z}, \\ b+4j+\varepsilon_j, & \text{if } \lfloor w^j \rfloor < n < \lfloor w^{j+1} \rfloor \text{ for some } j \in \mathbb{Z}_{\geq 0} \\ & \quad \text{and } (n - \lfloor w^j \rfloor)/\lfloor \eta w^j \rfloor \notin \mathbb{Z}. \end{cases}$$

Set

$$\xi_{w,\eta} = [a_{1,w,\eta}, a_{2,w,\eta}, a_{3,w,\eta}, \ldots] \in \mathbb{Q}_p.$$

Then

$$w_2^*(\xi_{w,\eta}) = \frac{2w-2-\eta}{2+\eta} \quad \text{and} \quad w_2(\xi_{w,\eta}) = \frac{2w-\eta}{2+\eta},$$

hence

$$w_2(\xi_{w,\eta}) - w_2^*(\xi_{w,\eta}) = \frac{2}{2+\eta}.$$

Since the equality $w_2(\xi) = w_2^*(\xi)$ holds for almost all (with respect to the Haar measure) p-adic numbers ξ (see e.g. [3, Theorem 9.5]), the previous two theorems immediately yield the next corollary.

Corollary 1. *For every ε in $[0,1]$ there are uncountably many p-adic numbers ξ_ε for which $w_2(\xi_\varepsilon) - w_2^*(\xi_\varepsilon) = \varepsilon$.*

Corollary 1 was already established in [11, 12] (except for the case $\varepsilon = 1$), by means of a much more complicated proof. The present approach is simpler and, unlike the other one, it is constructive in the sense that we are able to give explicit examples of p-adic numbers with the required properties.

We stress that there is an important difference between our theorems and [5, Theorems 4.1 and 4.2]. Namely, the continued fractions constructed in [5] have bounded partial quotients, while the sequences $(a_{n,w})_{n\geq 1}$ and $(a_{n,w,\eta})_{n\geq 1}$ defined in Theorems 1 and 2 are unbounded. As we explain in Remark 2 below, we are forced to work with unbounded sequences, since otherwise we would be unable to estimate sufficiently precisely the heights of the quadratic numbers involved in our proofs and would not get the exact values of the exponents of approximation.

Our theorems and the previous papers [4, 6] show that Schneider's continued fractions appear to be a powerful tool for constructing explicit examples of p-adic numbers with various, prescribed Diophantine properties.

The present paper is organized as follows. In Section 2 we gather results on quadratic numbers in $\mathbb{Q}_p$ with precise estimates for the heights of special families of p-adic quadratic numbers. In Section 3, we improve the classical Liouville's inequality in the special case where one of the quadratic numbers involved is p-adically close to its Galois conjugate. Finally, the proofs of our theorems are given in Section 4.

Throughout this text, the constants implied by $\ll$, $\gg$ are absolute.

2 Quadratic numbers in $\mathbb{Q}_p$ and estimates for their height

For a p-adic number $\xi = [a_1, a_2, a_3, \ldots]$ defined as above and an integer $i \geq 1$, put $\xi_i = [a_i, a_{i+1}, \ldots]$. Using (1.2), we get

$$\xi = \frac{p_{i-1}\xi_{i+1} + p^{a_i} p_{i-2}}{q_{i-1}\xi_{i+1} + p^{a_i} q_{i-2}}, \quad (i \geq 1). \tag{2.1}$$

Let $r \geq 0, s \geq 1$ be integers and $a_1, \ldots, a_{r+s}$ be positive integers. Set $\xi = [a_1, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}]$, where the bar indicates that the block $a_{r+1}, \ldots, a_{r+s}$ is repeated infinitely many times. Then, ξ is a quadratic p-adic number (see [7] or [15]). To see this, apply (2.1) first for ξ and $i = r$ and then for ξ and $i = r + s$ to get

$$\xi = \frac{p_{r-1}\xi_{r+1} + p^{a_r} p_{r-2}}{q_{r-1}\xi_{r+1} + p^{a_r} q_{r-2}} = \frac{p_{r+s-1}\xi_{r+s+1} + p^{a_{r+s}} p_{r+s-2}}{q_{r+s-1}\xi_{r+s+1} + p^{a_{r+s}} q_{r+s-2}}.$$

Then, by eliminating $\xi_{r+1} = \xi_{r+s+1}$, we obtain

$$\begin{aligned}
&(p^{a_{r+s}} q_{r-1} q_{r+s-2} - p^{a_r} q_{r-2} q_{r+s-1})\xi^2 \\
&+ (p^{a_r} p_{r-2} q_{r+s-1} + p^{a_r} p_{r+s-1} q_{r-2} - p^{a_{r+s}} p_{r-1} q_{r+s-2} - p^{a_{r+s}} p_{r+s-2} q_{r-1})\xi \\
&\qquad + (p^{a_{r+s}} p_{r-1} p_{r+s-2} - p^{a_r} p_{r-2} p_{r+s-1}) = 0. \qquad (2.2)
\end{aligned}$$

Denote by ξ' the Galois conjugate of ξ. It is well defined since ξ is not a rational number [7, 15]. Thus $\xi' \in \mathbb{Q}_p$ satisfies the same equation (2.2) as ξ.

The square of the p-adic distance between the p-adic roots of the polynomial $aX^2 + bX + c$ is $|\frac{b^2-4ac}{a^2}|_p$. We apply this fact to (2.2) in order to find $|\xi - \xi'|_p$.

Assume that $a_r \neq a_{r+s}$. It is easily checked that

$$|p^{a_{r+s}} q_{r-1} q_{r+s-2} - p^{a_r} q_{r-2} q_{r+s-1}|_p = p^{-\min\{a_r, a_{r+s}\}},$$

and

$$\begin{aligned}&\big|(p^{a_r}p_{r-2}q_{r+s-1}+p^{a_r}p_{r+s-1}q_{r-2}-p^{a_{r+s}}p_{r-1}q_{r+s-2}-p^{a_{r+s}}p_{r+s-2}q_{r-1})^2\\&-4(p^{a_{r+s}}q_{r-1}q_{r+s-2}-p^{a_r}q_{r-2}q_{r+s-1})(p^{a_{r+s}}p_{r-1}p_{r+s-2}-p^{a_r}p_{r-2}p_{r+s-1})\big|_p\\&=\Bigg|\bigg(p^{a_r}q_{r-2}q_{r+s-1}\Big(\frac{p_{r+s-1}}{q_{r+s-1}}-\frac{p_{r-2}}{q_{r-2}}\Big)+p^{a_{r+s}}q_{r-1}q_{r+s-2}\Big(\frac{p_{r+s-2}}{q_{r+s-2}}-\frac{p_{r-1}}{q_{r-1}}\Big)\bigg)^2\\&\quad-4p^{a_r+a_{r+s}}q_{r-2}q_{r-1}q_{r+s-2}q_{r+s-1}\Big(\frac{p_{r+s-1}}{q_{r+s-1}}-\frac{p_{r-1}}{q_{r-1}}\Big)\Big(\frac{p_{r+s-2}}{q_{r+s-2}}-\frac{p_{r-2}}{q_{r-2}}\Big)\Bigg|_p\\&=p^{-2\sum_{i=1}^r a_i},\end{aligned}$$

by (1.3). Consequently, we have established that the p-adic distance between the quadratic number $\xi=[a_1,\dots,a_r,\overline{a_{r+1},\dots,a_{r+s}}]$ and its Galois conjugate ξ' satisfies

$$|\xi-\xi'|_p=p^{-\sum_{i=1}^r a_i}p^{\min\{a_r,a_{r+s}\}},\quad \text{if } a_r\neq a_{r+s}. \tag{2.3}$$

Remark 1. Keeping the above notation, it can be shown that $1-\xi'_{r+1}=[\overline{a_{r+s},\dots,a_{r+1}}]$ and $|\xi'_{r+1}|_p<1$, where ξ'_{r+1} denotes the Galois conjugate of ξ_{r+1}.

The following two lemmas provide us with lower and upper bounds for the height of the quadratic numbers occurring in the proofs of Theorems 1 and 2.

Lemma 1. *Let $a_1,\dots,a_{n+1}$ be positive integers such that $a_n>a_{n+1}$. Let $\xi=[a_1,\dots,a_n,\overline{a_{n+1}}]\in\mathbb{Q}_p$ and denote by $(p_k/q_k)_{k\geq 1}$ the sequence of its convergents. Then, the height of ξ satisfies*

$$p_n^{-2}p^{2\sum_{i=1}^n a_i}\ll \mathrm{H}(\xi)\ll p^{a_n}p_n^2. \tag{2.4}$$

Proof. Let $P_\xi(X)=AX^2+BX+C=A(X-\xi)(X-\xi')$ denote the minimal polynomial of ξ over $\mathbb{Z}$. Since $q_n^2P_\xi(p_n/q_n)$ is a nonzero integer, we have

$$\begin{aligned}\left|P_\xi\left(\frac{p_n}{q_n}\right)\right|_p&=\left|q_n^2P_\xi\left(\frac{p_n}{q_n}\right)\right|_p=|Ap_n^2+Bp_nq_n+Cq_n^2|_p\\&\geq\frac{1}{|Ap_n^2+Bp_nq_n+Cq_n^2|}\geq\frac{1}{3\,\mathrm{H}(P_\xi)p_n^2}.\end{aligned}\tag{2.5}$$

It follows from (1.4) and (2.3) that

$$\begin{aligned}\left|P_\xi\left(\frac{p_n}{q_n}\right)\right|_p&=|A|_p\left|\xi-\frac{p_n}{q_n}\right|_p\left|\xi'-\frac{p_n}{q_n}\right|_p\\&\leq 1\cdot p^{-\sum_{i=1}^{n+1}a_i}\cdot p^{-\sum_{i=1}^n a_i}p^{a_{n+1}}\leq p^{-2\sum_{i=1}^n a_i}.\end{aligned}\tag{2.6}$$

Combining (2.5) and (2.6) gives

$$p_n^{-2}p^{2\sum_{i=1}^n a_i}\leq 3\,\mathrm{H}(\xi),$$

while the right hand side of (2.4) is a consequence of (2.2). □

Lemma 2. *Let $a_1, \ldots, a_{n+m}$ be positive integers such that*

$$a_n > a_{n+m} > a_{n+1} = \ldots = a_{n+m-1}.$$

Let

$$\zeta = [a_1, \ldots, a_n, \overline{a_{n+1}, \ldots, a_{n+m}}] \in \mathbb{Q}_p$$

and denote by $(p_k/q_k)_{k\geq 1}$ the sequence of its convergents. Then, the height of ζ satisfies

$$p_n^{-2} p^{-a_n - a_{n+m}} p^{\frac{3}{2}\sum_{i=1}^{n} a_i + \frac{1}{2}\sum_{i=1}^{n+m-1} a_i} \ll \mathrm{H}(\zeta) \ll p_n p_{n+m}. \tag{2.7}$$

Proof. The resultant of the minimal polynomials of ζ and of the p-adic number $\xi = [a_1, \ldots, a_n, \overline{a_{n+1}}]$ is a nonzero integer which can be bounded from above using its representation as a determinant (see e.g. [11, p. 20]). Therefore, we get

$$|\zeta - \xi|_p |\zeta' - \xi|_p |\zeta - \xi'|_p |\zeta' - \xi'|_p \geq \frac{1}{81\,\mathrm{H}(\zeta)^2\,\mathrm{H}(\xi)^2}, \tag{2.8}$$

where ζ' and ξ' denote the Galois conjugates of ζ and ξ, respectively.

Since $a_{n+1} < a_{n+m}$, we deduce from (1.4) that

$$\begin{aligned} |\zeta - \xi|_p &= \max\left\{ \left|\zeta - \frac{p_{n+m-1}}{q_{n+m-1}}\right|_p, \left|\xi - \frac{p_{n+m-1}}{q_{n+m-1}}\right|_p \right\} \\ &= \max\{p^{-\sum_{i=1}^{n+m} a_i}, p^{-\sum_{i=1}^{n+m-1} a_i - a_{n+1}}\} \\ &= p^{-\sum_{i=1}^{n+m-1} a_i - a_{n+1}}, \end{aligned}$$

while (2.3) gives

$$\begin{aligned} |\zeta - \zeta'|_p &= p^{-\sum_{i=1}^{n} a_i + \min\{a_n, a_{n+m}\}} = p^{-\sum_{i=1}^{n} a_i + a_{n+m}}, \\ |\xi - \xi'|_p &= p^{-\sum_{i=1}^{n} a_i + \min\{a_n, a_{n+1}\}} = p^{-\sum_{i=1}^{n} a_i + a_{n+1}}, \end{aligned}$$

since $a_n > a_{n+m}$ and $a_n > a_{n+1}$. Consequently,

$$\begin{aligned} |\zeta' - \xi|_p &= \max\{|\zeta - \zeta'|_p, |\zeta - \xi|_p\} = p^{-\sum_{i=1}^{n} a_i + a_{n+m}}, \\ |\zeta - \xi'|_p &= \max\{|\zeta - \xi|_p, |\xi - \xi'|_p\} = p^{-\sum_{i=1}^{n} a_i + a_{n+1}}, \\ |\zeta' - \xi'|_p &= \max\{|\zeta - \zeta'|_p, |\zeta - \xi'|_p\} = p^{-\sum_{i=1}^{n} a_i + a_{n+m}}. \end{aligned}$$

Combining these results and the right hand side of (2.4) with (2.8), we get

$$p^{2a_n} p_n^4\, \mathrm{H}(\zeta)^2 \gg p^{-2a_{n+m} + 3\sum_{i=1}^{n} a_i + \sum_{i=1}^{n+m-1} a_i},$$

which is exactly the lower bound in (2.7), while the upper bound is a consequence of (2.2). □

Next, we investigate the bounds on the numerator p_n and on the denominator q_n of the n-th convergent p_n/q_n to the number $\xi_w = [a_{1,w}, a_{2,w}, a_{3,w}, \dots]$ defined in Theorem 1. The discussion concerning the numbers $\xi_{w,\eta}$ defined in Theorem 2 is analogous and we give only the necessary results at the end of this section.

Let $\beta_j = (1 + \sqrt{1 + 4p^{b+3j+\varepsilon_j}})/2$ be the largest of the two eigenvalues of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ p^{b+3j+\varepsilon_j} & 0 \end{smallmatrix}\right)$.

For a positive integer n, let k be the integer defined by the inequalities $\lfloor w^k \rfloor \leq n < \lfloor w^{k+1} \rfloor$. An induction based on (1.1) gives the lower bound

$$p_n \geq \prod_{j=0}^{k-1} \beta_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} \cdot \beta_k^{n - \lfloor w^k \rfloor}. \tag{2.9}$$

To see this, observe that $p_1 = 1 + p^{a_{1,w}} \geq 1$ and $p_2 = 1 + p^{a_{1,w}} + p^{a_{2,w}} \geq \beta_0$. Let $n \geq 2$ be such that (2.9) holds for p_{n-1} and p_n. In order to prove (2.9) for p_{n+1}, we distinguish two cases, namely $n = \lfloor w^k \rfloor$ and $\lfloor w^k \rfloor < n < \lfloor w^{k+1} \rfloor$. In the first case, we get

$$\begin{aligned} p_{n+1} = p_n + p^{a_{n+1,w}} p_{n-1} &\geq \prod_{j=0}^{k-1} \beta_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} (1 + p^{b+3k+\varepsilon_k}/\beta_{k-1}) \\ &\geq \prod_{j=0}^{k-1} \beta_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} \cdot \beta_k, \end{aligned}$$

while, in the second, we obtain

$$\begin{aligned} p_{n+1} = p_n + p^{a_{n+1,w}} p_{n-1} &\geq \prod_{j=0}^{k-1} \beta_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} \cdot \beta_k^{n - \lfloor w^k \rfloor} (1 + p^{b+3k+\varepsilon_k}/\beta_k) \\ &\geq \prod_{j=0}^{k-1} \beta_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} \cdot \beta_k^{n+1 - \lfloor w^k \rfloor}. \end{aligned}$$

This proves (2.9). Throughout, we denote by $\log_p$ the base p logarithm. For $n \geq 1$, set

$$S_n = \sum_{j=0}^{k-1} (\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor) \log_p \beta_j + (n - \lfloor w^k \rfloor) \log_p \beta_k,$$

and observe that (2.9) can be rewritten as $p_n \geq p^{S_n}$.

In order to find an upper bound on p_n, we take a (submultiplicative) matrix norm $\|\cdot\|$ on $\mathbb{R}^{2\times 2}$ such that for every 2×2 matrix A the inequality

$$\rho(A) \leq \|A\| \leq \rho(A) + 1 \tag{2.10}$$

is valid (see [8, Lemma 5.6.10]). We denote by $\rho(A)$ the spectral radius of the matrix A.

It then follows from (1.2) that there exists a positive number M, independent of n, such that

$$\begin{aligned}
&\left\|\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix}\right\| = \left\|\prod_{i=0}^{n}\begin{pmatrix} 1 & 1 \\ p^{a_{i,w}} & 0\end{pmatrix}\right\| \\
&\le \left\|\begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}\right\| \cdot \prod_{j=1}^{k-1}\left\|\begin{pmatrix} 1 & 1 \\ p^{b+3j+\varepsilon_j} & 0\end{pmatrix}^{\lfloor w^{j+1}\rfloor - \lfloor w^j\rfloor - 1}\right\| \\
&\qquad \cdot \left\|\begin{pmatrix} 1 & 1 \\ p^{b+3k+\varepsilon_k} & 0\end{pmatrix}^{n-\lfloor w^k\rfloor}\right\| \cdot \prod_{j=0}^{k}\left\|\begin{pmatrix} 1 & 1 \\ p^{b+3j+2} & 0\end{pmatrix}\right\| \\
&< 2\prod_{j=0}^{k-1}\left(\beta_j^{\lfloor w^{j+1}\rfloor - \lfloor w^j\rfloor - 1} + 1\right)\cdot\left(\beta_k^{n-\lfloor w^k\rfloor} + 1\right)\prod_{j=1}^{k+1}(\beta_j + 1) \\
&< 2^{2k+3}\prod_{j=0}^{k-1}\beta_j^{\lfloor w^{j+1}\rfloor - \lfloor w^j\rfloor}\cdot\beta_k^{n-\lfloor w^k\rfloor+1}\cdot\beta_{k+1} \\
&\le n^M p^{S_n},
\end{aligned} \tag{2.11}$$

where we used the submultiplicativity, applied (2.10) and the well known fact that $\rho(A^k) = \rho(A)^k$ holds for any positive integer k. Since any two norms on $\mathbb{R}^{2\times 2}$ are equivalent, the bound (2.11) also holds (by increasing M by an absolute constant if necessary) for the sup norm $\|\cdot\|_\infty$. Therefore,

$$p^{S_n} \le p_n \le n^M p^{S_n}, \quad (n \ge 1). \tag{2.12}$$

For $(a_{n,w})_{n\ge 1}$ as in the definition of ξ_w in (1.5), define the quadratic number

$$\xi_{w,j} = [a_{1,w}, \ldots, a_{\lfloor w^j\rfloor,w}, \overline{a_{\lfloor w^j\rfloor+1,w}}] \tag{2.13}$$

Then Lemma 1 and (2.12) give

$$\lfloor w^j\rfloor^{-2M} p^{2\sum_{i=1}^{\lfloor w^j\rfloor} a_{i,w} - 2S_{\lfloor w^j\rfloor}} \ll \mathrm{H}(\xi_{w,j}) \ll p^{a_{\lfloor w^j\rfloor,w}}\lfloor w^j\rfloor^{2M} p^{2S_{\lfloor w^j\rfloor}}.$$

To estimate the difference $\sum_{i=1}^{\lfloor w^j\rfloor} a_{i,w} - 2S_{\lfloor w^j\rfloor}$ we apply

$$\begin{aligned}
0 < 2\log_p\frac{1+\sqrt{1+4p^x}}{2} - x &= \log_p\frac{(1+\sqrt{1+4p^x})^2}{4p^x} \\
&= \frac{1}{\log p}\log\left(1 + \frac{1+\sqrt{1+4p^x}}{2p^x}\right) \\
&< \frac{1}{\log p}\cdot\frac{1+\sqrt{1+4p^x}}{2p^x} < \frac{2}{\log p}\cdot\frac{1}{p^{x/2}}, \quad x > 0,
\end{aligned}$$

which for $x = b + 3i + \varepsilon_i$ gives

$$2\log_p \beta_i - (b + 3i + \varepsilon_i) < \frac{2}{\log p} \cdot p^{-\frac{b+3i+\varepsilon_i}{2}}.$$

Thus,

$$\begin{aligned}\Big|\sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w} - 2S_{\lfloor w^j \rfloor}\Big| &< \sum_{i=0}^{j-1} \big(2\log_p \beta_i - (b + 3i + \varepsilon_i)\big)(\lfloor w^{i+1} \rfloor - \lfloor w^i \rfloor) \\ &\qquad + 2j + b + 3j + 2 \\ &< 5(b + j) + \frac{2}{\log p \cdot p^{b/2}} \sum_{i=0}^{j-1} p^{-\frac{3i+\varepsilon_i}{2}} \lfloor w^{i+1} \rfloor \\ &< (w/2)^j \quad \text{for } j \text{ large enough.}\end{aligned} \tag{2.14}$$

This implies

$$p^{2S_{\lfloor w^j \rfloor} - 3(w/2)^j} \ll \mathrm{H}(\xi_{w,j}) \ll p^{2S_{\lfloor w^j \rfloor} + (w/2)^j} \tag{2.15}$$

Furthermore, the real numbers $E_1(j)$ defined by

$$\begin{aligned}\sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w} &= \sum_{i=0}^{j-1} (b + 3i + \varepsilon_i)(\lfloor w^{i+1} \rfloor - \lfloor w^i \rfloor) + \sum_{i=0}^{j-1} (2 - \varepsilon_i) + b + 3j + 2 \\ &= \big(b + 3j + E_1(j)\big) w^j,\end{aligned} \tag{2.16}$$

satisfy $|E_1(j)| < 10$ when j is sufficiently large.

Remark 2. Let us define the p-adic number $\xi_w^c = [a_{1,w}, a_{2,w}, \ldots]$ in a similar way as what was done in [5], that is, by putting $a_{\lfloor w^j \rfloor, w} = c$ for $j \in \mathbb{Z}_{\geq 0}$ and $a_{i,w} = b$ otherwise, where b and c are distinct positive integers. Unlike for real continued fractions, we are unable to decide whether $w_2(\xi_w^c) - w_2^*(\xi_w^c) = 1$. The reason for this can be explained as follows. Define $\xi_{w,j}^c$ as in (2.13) and denote by $(\xi_{w,j}^c)'$ its Galois conjugate. Since $(1 + \sqrt{1 + 4p^b})/2 > p^{b/2}$, bounds similar to those in (2.12) show that (2.4) is not tight enough to get that $|\xi_{w,j}^c - (\xi_{w,j}^c)'|_p^{-1}$ is close to $\mathrm{H}(\xi_{w,j}^c)$ when j tends to infinity (cf. (2.3)), a condition required in order to apply Lemma 5 below. Thus, we would need to show that the coefficients in (2.2) in this case ($r = n = \lfloor w^j \rfloor$, $s = 1$) are all actually much smaller than p_n^2, i.e., of the same size as $p^{\sum_{i=1}^n a_{i,w}}$ (which still leaves the left hand side of (2.4) problematic), or that this polynomial is not the minimal polynomial of $\xi_{w,j}^c$. This seems to be out of our reach.

For the p-adic numbers $\xi_{w,\eta}$ defined in Theorem 2, we obtain similar results, which we briefly summarize below. Choose and fix w and η as in the statement of Theorem 2. Denote by $(p_k/q_k)_{k\geq 1}$ the convergents to $\xi_{w,\eta}$

and set $\gamma_j = (1 + \sqrt{1 + 4p^{b+4j+\varepsilon_j}})/2$ for $j \geq 0$. For a positive integer n, let k be the integer defined by the inequalities $\lfloor w^k \rfloor \leq n < \lfloor w^{k+1} \rfloor$. As above, we get

$$p_n \geq \prod_{j=0}^{k-1} \gamma_j^{\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor} \cdot \gamma_k^{n - \lfloor w^k \rfloor}.$$

Setting

$$\widetilde{S}_n = \sum_{j=0}^{k-1} (\lfloor w^{j+1} \rfloor - \lfloor w^j \rfloor) \log_p \gamma_j + (n - \lfloor w^k \rfloor) \log_p \gamma_k,$$

then, arguing as before, there exists a real number $\widetilde{M}$, independent of n, such that

$$p^{\widetilde{S}_n} \ll p_n \ll n^{\widetilde{M}} p^{\widetilde{S}_n}. \tag{2.17}$$

For the sequence $(a_{n,w,\eta})_{n \geq 1}$ defined in Theorem 2, set

$$\xi_{w,\eta,j} = [a_{1,w,\eta}, \ldots, a_{\lfloor w^j \rfloor, w,\eta}, \overline{a_{\lfloor w^j \rfloor + 1, w,\eta}, \ldots, a_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor, w,\eta}}]. \tag{2.18}$$

Then, Lemma 2 and (2.17) give

$$\begin{aligned} \lfloor w^j \rfloor^{-2\widetilde{M}} p^{-2\widetilde{S}_{\lfloor w^j \rfloor} - a_{\lfloor w^j \rfloor, w,\eta} - a_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor, w,\eta} + \frac{3}{2} \sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w,\eta} + \frac{1}{2} \sum_{i=1}^{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor - 1} a_{i,w,\eta}} & \\ \ll \mathrm{H}(\xi_{w,\eta,j}) & \\ \ll \lfloor w^j \rfloor^{\widetilde{M}} p^{\widetilde{S}_{\lfloor w^j \rfloor}} (\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor)^{\widetilde{M}} p^{\widetilde{S}_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor}}. & \end{aligned} \tag{2.19}$$

We also have

$$\begin{aligned} \left| \sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w,\eta} - 2\widetilde{S}_{\lfloor w^j \rfloor} \right| &< (w/2)^j, \\ \left| \sum_{i=1}^{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor} a_{i,w,\eta} - 2\widetilde{S}_{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor} \right| &< (1+\eta)(w/2)^j, \end{aligned} \tag{2.20}$$

for j large enough and the real numbers $\widetilde{E}_1(j), \widetilde{E}_2(j)$ defined by

$$\begin{aligned} \sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w,\eta} &= \big(b + 4j + \widetilde{E}_1(j)\big) w^j, \\ \sum_{i=1}^{\lfloor w^j \rfloor + \lfloor \eta w^j \rfloor} a_{i,w,\eta} &= \big(b + 4j + \widetilde{E}_2(j)\big)(1+\eta) w^j, \end{aligned} \tag{2.21}$$

satisfy $|\widetilde{E}_1(j)|, |\widetilde{E}_2(j)| < 10 + \frac{1}{\eta}$, for j large enough.

3 Liouville's inequality and application

Lemmas 4 and 5 below are the p-adic analogues of Lemmas 7.1 and 7.3 from [5]. Their proofs are, up to the non-Archimedean nature of the p-adic absolute value, almost identical to the proofs of these lemmas. We include them for the sake of completeness.

We need the following facts which we collect in one lemma.

Lemma 3. *a) (cf. [10, p. 341]) Let $P(X) = a_nX^n + \cdots + a_1X + a_0 = a_n(X - \alpha_1)\cdots(X - \alpha_n) \in \mathbb{Q}_p[X]$. Then for any non-empty set $I \subseteq \{1, \ldots, n\}$, we have*

$$\prod_{i\in I} |\alpha_i|_p \leq \frac{\max_{j\in\{0,1,\ldots,n\}} |a_j|_p}{|a_n|_p}.$$

b) For any $x, y \in \mathbb{Q}_p$, $|x - y|_p \leq \max\{1, |x|_p\} \max\{1, |y|_p\}$.
c) (cf. [11, Lemma 3.2]) For any quadratic p-adic number α with Galois conjugate α',

$$\frac{1}{\mathrm{H}(\alpha)\sqrt{5}} \leq |\alpha - \alpha'|_p \leq \mathrm{H}(\alpha).$$

The next lemma provides us with a strengthening of the Liouville inequality.

Lemma 4. *Let α and β be p-adic quadratic numbers. Denote by $P_\alpha(X) := a(X-\alpha)(X-\alpha')$ and $P_\beta(X) := b(X-\beta)(X-\beta')$ their minimal polynomials over $\mathbb{Z}$. Assume that $\alpha, \alpha', \beta, \beta'$ are distinct. Then we have*

$$|\alpha - \beta|_p \geq \frac{1}{81} \max\{|\alpha - \alpha'|_p^{-1}, 1\} \,\mathrm{H}(\alpha)^{-2}\,\mathrm{H}(\beta)^{-2}. \tag{3.1}$$

Proof. The resultant $\mathrm{Res}(P_\alpha, P_\beta)$ is a nonzero integer which can be bounded from above using its representation as a determinant (see e.g. [11, p. 20]). Therefore,

$$\begin{aligned}\frac{1}{3^4\,\mathrm{H}(\alpha)^2\,\mathrm{H}(\beta)^2} &\leq \frac{1}{|\operatorname{Res}(P_\alpha, P_\beta)|}\\ &\leq |\operatorname{Res}(P_\alpha, P_\beta)|_p\\ &= |a|_p^2|b|_p^2|\alpha - \beta|_p|\alpha' - \beta|_p|\alpha - \beta'|_p|\alpha' - \beta'|_p.\end{aligned} \tag{3.2}$$

Using Lemma 3, firstly part b), then part a), we get

$$\begin{aligned}&|a|_p|b|_p^2|\alpha - \beta'|_p|\alpha' - \beta'|_p\\ &\qquad\leq |a|_p|b|_p^2 \max\{1, |\alpha|_p\} \max\{1, |\alpha'|_p\} \max\{1, |\beta'|_p\}^2 \leq 1,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&|a|_p^2|b|_p^2|\alpha' - \beta|_p|\alpha - \beta'|_p|\alpha' - \beta'|_p\\ &\qquad\leq |a|_p^2|b|_p^2 \max\{1, |\alpha|_p\} \max\{1, |\alpha'|_p\}^2 \max\{1, |\beta|_p\} \max\{1, |\beta'|_p\}^2 \leq 1.\end{aligned} \tag{3.4}$$

If $|\alpha' - \beta|_p \le |\alpha - \beta|_p$, then (3.2) implies

$$|\alpha - \beta|_p \overset{(3.3)}{\ge} \frac{1}{9} \mathrm{H}(\alpha)^{-1} \mathrm{H}(\beta)^{-1} \overset{\text{L.3c}}{\ge} \frac{1}{9\sqrt{5}} |\alpha - \alpha'|_p^{-1} \mathrm{H}(\alpha)^{-2} \mathrm{H}(\beta)^{-2}$$

and

$$|\alpha - \beta|_p \overset{(3.3)}{\ge} \frac{1}{9} \mathrm{H}(\alpha)^{-1} \mathrm{H}(\beta)^{-1} \ge \frac{1}{9} \mathrm{H}(\alpha)^{-2} \mathrm{H}(\beta)^{-2},$$

which is stronger than (3.1).

If $|\alpha' - \beta|_p > |\alpha - \beta|_p$, then $|\alpha' - \beta|_p = |\alpha - \alpha'|_p$ and (3.2) implies

$$|\alpha - \beta|_p \overset{(3.3)}{\ge} \frac{1}{81} |\alpha - \alpha'|_p^{-1} \mathrm{H}(\alpha)^{-2} \mathrm{H}(\beta)^{-2} \quad \text{and}$$
$$|\alpha - \beta|_p \overset{(3.4)}{\ge} \frac{1}{81} \mathrm{H}(\alpha)^{-2} \mathrm{H}(\beta)^{-2}.$$

Putting all these estimates together proves the lemma. □

Lemma 5. *Let ξ be a p-adic number with $|\xi|_p \le 1$. Assume that there exist positive real numbers $c_1, c_2, c_3, \delta, \rho, \theta$ and a sequence $(\alpha_j)_{j\ge1}$ of quadratic numbers such that*

$$c_1 \mathrm{H}(\alpha_j)^{-\rho-1} \le |\xi - \alpha_j|_p \le c_2 \mathrm{H}(\alpha_j)^{-\delta-1} \tag{3.5}$$

and

$$\mathrm{H}(\alpha_j) \le \mathrm{H}(\alpha_{j+1}) \le c_3 \mathrm{H}(\alpha_j)^{\theta},$$

for $j \ge 1$. Set $\varepsilon = 0$ or assume that there exist $c_4 \ge 1$ and $0 < \varepsilon \le 1$ such that

$$|\alpha_j - \alpha_j'|_p \le c_4 \mathrm{H}(\alpha_j)^{-\varepsilon}, \tag{3.6}$$

for $j \ge 1$, where α_j' denotes the Galois conjugate of α_j.

Then we have

$$\delta \le w_2^*(\xi) \le \rho$$

if

$$(\rho - 1)(\delta - 1 + \varepsilon) \ge 2\theta(2 - \varepsilon). \tag{3.7}$$

Furthermore, we have

$$\delta \le w_2^*(\xi) \le \rho \quad \textit{and} \quad w_2(\xi) \ge w_2^*(\xi) + \varepsilon,$$

if

$$\varepsilon > 0 \quad \textit{and} \quad (\delta - 2 + \varepsilon)(\delta - 1 + \varepsilon) \ge 2\theta(2 - \varepsilon). \tag{3.8}$$

Finally, if (3.8) *holds and the limit*

$$\lim_{j\to\infty} \frac{\log |\alpha_j - \alpha_j'|_p}{\log \mathrm{H}(\alpha_j)} = -\chi \tag{3.9}$$

exists, then we have

$$\delta \le w_2^*(\xi) \le \rho \quad \textit{and} \quad w_2(\xi) - w_2^*(\xi) \in [\varepsilon, \chi].$$

Proof. Set $Q_j = \mathrm{H}(\alpha_j)$ for $j \geq 1$. Let $\alpha \in \mathbb{Q}_p$ be a rational or a quadratic number not in $(\alpha_j)_{j\geq 1}$ and define the integer j by

$$Q_{j-1} < \big(c_5 \,\mathrm{H}(\alpha)\big)^{2/(\varepsilon+\delta-1)} \leq Q_j,$$

where

$$c_5 = 10(c_2 c_4)^{1/2}. \tag{3.10}$$

We assume that $\mathrm{H}(\alpha)$ and hence j, is sufficiently large. By Lemma 4, we have

$$\begin{aligned}
|\alpha - \alpha_j|_p &\geq \frac{1}{81}|\alpha_j - \alpha_j'|_p^{-1}\,\mathrm{H}(\alpha_j)^{-2}\,\mathrm{H}(\alpha)^{-2}\\
&\overset{(3.6)}{\geq} c_4^{-1}\,\mathrm{H}(\alpha_j)^{\varepsilon}\,\mathrm{H}(\alpha_j)^{-2} c_5^2 Q_j^{-\varepsilon-\delta+1}\\
&= \frac{1}{81}c_4^{-1}c_5^2 Q_j^{-1-\delta} \overset{(3.10)}{>} c_2\,\mathrm{H}(\alpha_j)^{-\delta-1}.
\end{aligned}$$

Consequently, we get $|\alpha - \alpha_j|_p > |\xi - \alpha_j|_p$, hence $|\xi - \alpha|_p = |\alpha - \alpha_j|_p$. Since

$$\mathrm{H}(\alpha_j) = Q_j \leq c_3 Q_{j-1}^{\theta} < c_3\big(c_5\,\mathrm{H}(\alpha)\big)^{2\theta/(\varepsilon+\delta-1)},$$

we conclude that

$$\begin{aligned}
|\xi - \alpha|_p = |\alpha - \alpha_j|_p &\geq \frac{1}{81}c_4^{-1}\,\mathrm{H}(\alpha_j)^{-2+\varepsilon}\,\mathrm{H}(\alpha)^{-2}\\
&> \frac{1}{81}c_4^{-1}c_3^{-2+\varepsilon}c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)}\,\mathrm{H}(\alpha)^{-2-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)},
\end{aligned} \tag{3.11}$$

thus

$$w_2^*(\xi) \leq \max\{\rho, 1 + 2\theta(2-\varepsilon)/(\varepsilon+\delta-1)\},$$

so (3.7) ensures that $w_2^*(\xi) \leq \rho$, while $w_2^*(\xi) \geq \delta$ is guaranteed by (3.5).

If $(\delta - 2 + \varepsilon)(\delta - 1 + \varepsilon) \geq 2\theta(2-\varepsilon)$ also holds, then it follows from (3.11) that

$$|\xi - \alpha|_p > \frac{1}{81}c_4^{-1}c_3^{-2+\varepsilon}c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)}\,\mathrm{H}(\alpha)^{-\delta-\varepsilon}.$$

This means that the best algebraic approximants of degree at most two to ξ are in the sequence $(\alpha_j)_{j\geq 1}$. Assume that α is a quadratic number not in $(\alpha_j)_{j\geq 1}$ and let $P_\alpha(X) = a_\alpha(X-\alpha)(X-\alpha')$ be its minimal polynomial over $\mathbb{Z}$. Without loss of generality, we assume that $|\xi - \alpha|_p \leq |\xi - \alpha'|_p$, thus $|\xi - \alpha'|_p \geq |\alpha - \alpha'|_p$ and

$$\begin{aligned}
|P_\alpha(\xi)|_p = |a_\alpha|_p|\xi - \alpha|_p|\xi - \alpha'|_p &\geq |a_\alpha(\alpha - \alpha')|_p|\xi - \alpha|_p\\
&> \frac{1}{200}c_4^{-1}c_3^{-2+\varepsilon}c_5^{-2\theta(2-\varepsilon)/(\varepsilon+\delta-1)}\,\mathrm{H}(\alpha)^{-\delta-\varepsilon-1},
\end{aligned} \tag{3.12}$$

where we have used that $|a_\alpha(\alpha - \alpha')|_p = |\operatorname{Disc}(P_\alpha)|_p^{1/2} \geq (\mathrm{H}(\alpha)\sqrt{5})^{-1}$.

Denoting by $P_j(X)$ the minimal polynomial of α_j over $\mathbb{Z}$ and using

$$|\xi - \alpha_j'|_p \le \max\{|\xi - \alpha_j|_p, |\alpha_j - \alpha_j'|_p\} \overset{(3.6)}{\le} c_4 \, \mathrm{H}(\alpha_j)^{-\varepsilon} \quad \text{and} \quad |a_j|_p \le 1,$$

we deduce that

$$|P_j(\xi)|_p = |a_j(\xi - \alpha_j)(\xi - \alpha_j')|_p \le c_4 \, \mathrm{H}(\alpha_j)^{-\varepsilon} |\xi - \alpha_j|_p \le c_2 c_4 \, \mathrm{H}(\alpha_j)^{-\delta-\varepsilon-1}.$$

This shows that $w_2(\xi) \ge w_2^*(\xi) + \varepsilon \ge \delta + \varepsilon$.

It remains for us to prove that $|P_j(\xi)|_p$ cannot be too small when the limit in (3.9) exists. It follows from $|\xi|_p \le 1$ and (3.5) that $|\alpha_j|_p \le 1$ and, by (3.6), we get $|\alpha_j'|_p \le 1$ as well. This shows that $|a_j|_p = 1$, for otherwise $P_j(X)$ would not be minimal. If (3.9) holds, then we have the reverse inequalities

$$|\xi - \alpha_j'|_p = \max\{|\xi - \alpha_j|_p, |\alpha_j - \alpha_j'|_p\} \gg \mathrm{H}(\alpha_j)^{-\chi}$$

and

$$|P_j(\xi)|_p = |a_j(\xi - \alpha_j)(\xi - \alpha_j')|_p \gg \mathrm{H}(\alpha_j)^{-\chi} |\xi - \alpha_j|_p,$$

which together with (3.12) yields $w_2(\xi) \le w_2^*(\xi) + \chi$. □

4 Proofs of main theorems

Proof of Theorem 1. Let $j \ge 2$ be an integer. Define the quadratic number $\xi_{w,j}$ as in (2.13) and denote by $(p_n/q_n)_{n\ge1}$ the sequence of convergents to ξ_w. By (1.4), we have

$$\left| \xi_w - \frac{p_{\lfloor w^{j+1} \rfloor - 1}}{q_{\lfloor w^{j+1} \rfloor - 1}} \right|_p = p^{-\sum_{i=1}^{\lfloor w^{j+1} \rfloor - 1} a_{i,w}} p^{-b-3j-5}$$

and

$$\left| \xi_{w,j} - \frac{p_{\lfloor w^{j+1} \rfloor - 1}}{q_{\lfloor w^{j+1} \rfloor - 1}} \right|_p = p^{-\sum_{i=1}^{\lfloor w^{j+1} \rfloor - 1} a_{i,w}} p^{-b-3j-\varepsilon_j},$$

thus the real number $E_2(j)$ defined by

$$|\xi_w - \xi_{w,j}|_p = p^{-\sum_{i=1}^{\lfloor w^{j+1} \rfloor - 1} a_{i,w}} p^{-b-3j-\varepsilon_j} = p^{-(b+3j+3+E_2(j))w^{j+1}}$$

is in absolute value less than 10 when j is large enough. The same holds for the real numbers $E_3(j), E_4(j)$ defined below.

It follows from (2.3) that the distance between $\xi_{w,j}$ and its Galois conjugate $\xi_{w,j}'$ satisfies

$$|\xi_{w,j} - \xi_{w,j}'|_p = p^{-\sum_{i=1}^{\lfloor w^j \rfloor} a_{i,w}} p^{b+3j+\varepsilon_j} = p^{-(b+3j+E_3(j))w^j}.$$

Furthermore, we deduce from (2.15), (2.14) and (2.16) that

$$\mathrm{H}(\xi_{w,j}) = p^{(b+3j+E_4(j))w^j}.$$

We are now in condition to apply Lemma 5. Let $\iota > 0$ be given and set

$$\delta = w - 1 - \iota, \quad \rho = w - 1 + \iota, \quad \theta = w + \iota, \quad \varepsilon = 1 - \iota.$$

The condition $(w-1)(w-2) > 2w$ holds because $w > (5+\sqrt{17})/2$. For $j > j(\iota)$ large enough, the conditions of Lemma 5 are satisfied. Thus, we have $w_2^*(\xi_w) \in (w-1-\iota, w-1+\iota)$ and $w_2(\xi_w) - w_2^*(\xi_w) \geq 1-\iota$. Since ι is arbitrary, we deduce that $w_2^*(\xi_w) = w-1$ and $w_2(\xi_w) = w$. The proof of the theorem is complete. □

Proof of Theorem 2. Let $j \geq 2$ be an integer. Define the quadratic number $\xi_{w,\eta,j}$ as in (2.18) and denote by $(p_n/q_n)_{n\geq 1}$ the sequence of convergents to $\xi_{w,\eta}$. The real numbers $\widetilde{E}_3(j), \widetilde{E}_4(j), \widetilde{E}_5(j)$ occurring below are all in absolute value less than $10 + \frac{1}{\eta}$ when j is large enough. By (1.4), we get

$$\begin{aligned} |\xi_{w,\eta} - \xi_{w,\eta,j}|_p &= \max\left\{ \left|\xi_{w,\eta} - \frac{p_{\lfloor w^{j+1}\rfloor -1}}{q_{\lfloor w^{j+1}\rfloor -1}}\right|_p, \left|\xi_{w,\eta,j} - \frac{p_{\lfloor w^{j+1}\rfloor -1}}{q_{\lfloor w^{j+1}\rfloor -1}}\right|_p \right\} \\ &= p^{-(b+4j+4+\widetilde{E}_3(j))w^{j+1}}. \end{aligned}$$

For the distance between $\xi_{w,\eta,j}$ and its Galois conjugate $\xi'_{w,\eta,j}$, we deduce from (2.3) and (2.21) the estimate

$$|\xi_{w,\eta,j} - \xi'_{w,\eta,j}|_p = p^{-(b+4j+\widetilde{E}_4(j))w^j}.$$

Furthermore, it follows from (2.19), (2.20) and (2.21) that

$$\mathrm{H}(\xi_{w,\eta,j}) = p^{(b+4j+\widetilde{E}_5(j))(1+\eta/2)w^j}.$$

Let $\iota \in (0, 0.01)$ be given such that $\iota < 2/(2+\eta)$, and set

$$\delta = \frac{w}{1+\frac{\eta}{2}} - 1 - \iota, \quad \rho = \frac{w}{1+\frac{\eta}{2}} - 1 + \iota, \quad \theta = w + \iota, \quad \varepsilon = \frac{1}{1+\frac{\eta}{2}} - \iota.$$

We check that, for $w \in [16, +\infty)$ and $\eta \in [0, \sqrt{w}/4]$, the inequalities (3.8) hold. For $j > j(\iota)$ large enough, the conditions of Lemma 5 are satisfied. Thus, we have

$$w_2^*(\xi_{w,\eta}) \in \left[\frac{2w}{2+\eta} - 1 - \iota, \frac{2w}{2+\eta} - 1 + \iota\right]$$

and

$$w_2(\xi_{w,\eta}) - w_2^*(\xi_{w,\eta}) \in \left[\frac{2}{2+\eta} - \iota, \frac{2}{2+\eta}\right].$$

Since ι is arbitrary, we deduce that $w_2^*(\xi_{w,\eta}) = \frac{2w-2-\eta}{2+\eta}$ and $w_2(\xi_{w,\eta}) = \frac{2w-\eta}{2+\eta}$. This proves the theorem. □

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