Fractional parts of powers and Sturmian words

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Abstract. Let $b \geq 2$ be an integer. In terms of combinatorics on words we describe all irrational numbers $\xi > 0$ with the property that the fractional parts $\{\xi b^n\}$, $n \geq 0$, all belong to a semi-open or an open interval of length $1/b$. The length of such an interval cannot be smaller, that is, for irrational $\xi$, the fractional parts $\{\xi b^n\}$, $n \geq 0$, cannot all belong to an interval of length smaller than $1/b$.

Parties fractionnaires de puissances et mots sturmiens

Résumé. Soit $b \geq 2$ un entier. Au moyen de résultats de la combina- natoire des mots, nous caractérisons l’ensemble des nombres réels $\xi > 0$ tels que les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$, appartiennent toutes à un intervalle semi-ouvert ou ouvert de longueur $1/b$. La longueur d’un tel intervalle ne peut pas être plus petite, c’est-à-dire, quel que soit le nombre irrationnel $\xi$, aucun intervalle de longueur strictement inférieure à $1/b$ ne contient toutes les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$.

Version française abrégée

Dans tout ce qui suit, $\{\cdot\}$ désigne la fonction partie fractionnaire. Suivant la définition énoncée en 1968 par Mahler [11], un $\mathbb{Z}$-nombre est un nombre réel positif $\xi$ vérifiant $0 \leq \{\xi (3/2)^n\} < 1/2$ pour tout entier $n \geq 0$. L’ensemble des $\mathbb{Z}$- nombres est au plus dénombrable [11] et même vraisemblablement vide, mais ce problème difficile n’est à ce jour pas résolu. Plus généralement, étant donnés un nombre réel $\alpha > 1$ et un sous-intervalle $[s, t]$ de $[0, 1]$, on souhaiterait savoir s’il existe un nombre réel $\xi > 0$ vérifiant $s \leq \{\xi \alpha^n\} < t$ pour tout entier $n \geq 0$, ou bien, plus modestement, on aimerait déterminer la plus petite longueur $t - s$ pour laquelle un tel $\xi$ existe.

Cette note répond à deux objectifs : nous annonçons des résultats nouveaux obtenus pour $\alpha$ algébrique par le second auteur et nous apportons une réponse complète aux questions supra lorsque $\alpha \geq 2$ est un entier.

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Soient \( p \) et \( q \) des entiers vérifiant \( p > q \geq 2 \). Flatto, Lagarias & Pollington [9] établirent que, pour tout intervalle \( I \) de longueur strictement inférieure à \( 1/p \), il n’existe aucun nombre réel \( \xi > 0 \) vérifiant \( \{\xi(p/q)^n\} \in I \) pour tout entier \( n \geq 0 \) (cf. également [3]). Une nouvelle démonstration, plus simple, de ce résultat, ainsi que sa généralisation aux nombres algébriques réels \( > 1 \) qui ne sont ni de Pisot, ni de Salem, se trouvent dans deux travaux récents [4, 6] du second auteur.

**Théorème 1** ([4, 6]). Soit \( \alpha > 1 \) un nombre réel algébrique, et soit \( P(X) \in \mathbb{Z}[X] \) son polynôme minimal. Soit \( F(X) \) un polynôme à coefficients réels, de degré \( r \geq 0 \), et dont le coefficient dominant est positif. Supposons en outre que \( F(X) \notin \mathbb{Q}(\alpha)[X] \) si ou bien \( \alpha \) est un nombre de Pisot, ou bien \( r = 0 \) et \( \alpha \) est un nombre de Salem. Alors, les parties fractionnaires \( \{F(n)\alpha^n\}, n \geq 0 \), ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à \( 1/\ell(P^{r+1}) \).

Ici, \( \ell(P^{r+1}) \) désigne la longueur réduite du polynôme \( P(X)^{r+1} \), définition infra. Les hypothèses sur le polynôme \( F(X) \) sont nécessaires [14].

Nous supposons désormais que \( \alpha \) est un entier \( > 1 \), et choisissons de le noter \( b \).

Il découle du Théorème 1 que, pour tout irrationnel \( \xi \), la longueur de tout intervalle \( I \) contenant toutes les parties fractionnaires \( \{\xi b^n\}, n \geq 0 \), est au moins égale à \( 1/b \). Dans cette note, nous caractérisons complètement les paires \((\xi, I)\), formées d’un nombre réel irrationnel \( \xi > 0 \) et d’un intervalle \( I \), pour lesquelles \( \{\xi b^n\} \) appartient à \( I \) pour tout \( n \geq 0 \). Nous employons la terminologie de la combinatoire des mots [2, 10], et notamment la notion de suite sturmiennne.

**Théorème 2.** Soient \( b \geq 2 \) un entier et \( \xi \) un nombre réel irrationnel. Les parties fractionnaires \( \{\xi b^n\}, n \geq 0 \), ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à \( 1/b \). En outre, les nombres \( \{\xi b^n\}, n \geq 0 \), sont tous dans un intervalle fermé \( I \) de longueur \( 1/b \) si, et seulement si, \( \xi = g + k/(b - 1) + t_b(w) \), où \( g \) est un entier quelconque, \( k \) appartient à \( \{0, 1, \ldots, b - 2\} \) et \( w \) est un mot sturmienn sur \( \{0, 1\} \). Si tel est le cas, alors \( \xi \) est transcendant et l’intervalle \( I \) est semi-ouvert. De plus, \( I \) est ouvert sauf s’il existe un entier \( j \geq 0 \) et un mot sturmienn caractéristique \( u \) tels que \( T^j(w) = u \).

En particulier, puisqu’il existe une infinité non dénombrable de suites sturmiennes sur \( \{0, 1\} \), le Théorème 2 montre qu’il existe une infinité non dénombrable de paires \((\xi, s)\), où \( \xi \) est irrationnel et \( s \in ]0, 1 - 1/b[ \), telles que \( s < \{\xi b^n\} < s + 1/b \) pour tout \( n \geq 0 \).

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**1. Introduction**

In 1968, Mahler [11] introduced the notion of \( Z \)-numbers. These are precisely the positive real numbers \( \xi \) such that \( 0 \leq \{\xi(3/2)^n\} < 1/2 \) for all integers \( n \geq 0 \). Here and below, \( \{\cdot\} \) denotes the fractional part. The set of \( Z \)-numbers is at most countable [11], and it is widely believed it is even empty. This raises the following more general questions. Given a real number \( \alpha > 1 \) and an interval \( [s, t] \) included in \([0, 1)\), are there any positive
numbers $\xi$ such that $s \leq \{\xi \alpha^n\} < t$ for all integers $n \geq 0$? What is the smallest possible difference $t - s$ for which such positive numbers $\xi$ do exist?

The purpose of this note is twofold. Firstly, we announce several new results obtained by the second named author for algebraic numbers $\alpha$. Secondly, we give a complete answer to the above questions for rational integers $\alpha = b \geq 2$.

Let $p$ and $q$ be coprime integers with $p > q \geq 2$. Flatto, Lagarias & Pollington [9] showed that, for any interval $I$ of length strictly smaller than $1/p$, there are no $\xi > 0$ such that $\{\xi(p/q)^n\} \in I$ for all integers $n \geq 0$. Presumably, this also holds for any interval $I = [s, s + 1/p)$, with $s \in [0, 1 - 1/p]$. Actually, it was proved in [9] that this is the case for any $s$ lying in a dense subset of $[0, 1 - 1/p]$, and, later, the first named author established [3] that this is also the case for any $s$ lying in a subset of full Lebesgue measure of $[0, 1 - 1/p]$.

A new, simpler proof of the result of Flatto, Lagarias & Pollington and its generalization from rational non-integer numbers $\alpha = p/q$ to arbitrary real algebraic numbers $\alpha$ which are neither PV-numbers nor Salem numbers has been recently given by the second named author [4, 6]. Recall that an algebraic integer $\alpha > 1$ is called a PV-number (resp. Salem number) if its remaining conjugates (if any) are all inside the unit disc $|z| < 1$ (resp. in $|z| \leq 1$ with at least one conjugate lying on $|z| = 1$). To state these results, we define the reduced length of a polynomial $P(X) \in \mathbb{R}[X]$, denoted by $\ell(P)$, to be the infimum of the lengths (that is, the sums of the absolute values of the coefficients) of the polynomials $P(X) \cdot G(X)$, taken over every polynomial $G(X) \in \mathbb{R}[X]$ with either leading coefficient or constant coefficient equal to 1. It is easy to prove that $\ell(qX - p) = p$ for all integers $p > q \geq 1$ (see [4] or [13], where the reduced length of a polynomial was studied in detail).

**Theorem 1** ([4, 6]). Let $\alpha > 1$ be a real algebraic number with minimal defining polynomial $P(X) \in \mathbb{Z}[X]$, and let $F(X)$ be a degree $r \geq 0$ real polynomial with positive leading coefficient. Suppose, in addition, that $F(X) \notin \mathbb{Q}(\alpha)[X]$ if either $\alpha$ is a PV-number or $r = 0$ and $\alpha$ is a Salem number. Then the fractional parts $\{F(n)\alpha^n\}$, $n \geq 0$, cannot all lie in an interval of length smaller than $1/\ell(P^{r+1})$.

The extra conditions on $F(X)$ in Theorem 1 that concern PV and Salem numbers $\alpha$ are necessary. This is clear for PV-numbers $\alpha$, whereas for Salem numbers $\alpha$ the necessity of the condition $\xi = F(X) \notin \mathbb{Q}(\alpha)$, where $r = \deg F = 0$, is shown in [14]. Other results related to Theorem 1 have been obtained in [1] and [5].

From now on, suppose that $\alpha > 1$ is an integer, say $\alpha = b \geq 2$. It is a PV-number, so Theorem 1 implies that, for any interval $I$ of length strictly smaller than $1/b$, there are no irrational numbers $\xi$ for which $\{\xi b^n\} \in I$ for every integer $n \geq 0$. In particular, it follows from this that $1/b$ is the smallest possible length of an interval to which all the fractional parts $\{\xi b^n\}$, $n \geq 0$, with fixed irrational $\xi$, can belong. This also raises the question whether, for an interval $I$ of length $1/b$, there exists an irrational number $\xi$ such that $\{\xi b^n\} \in I$ for all integers $n \geq 0$. In the present note, we show that there are uncountably many pairs $(\xi, I)$ with this property and describe all of them.

Note that, writing the $b$-adic expansion of $\{\xi\}$, namely, $\xi = g + x_1 b^{-1} + x_2 b^{-2} + \ldots$, where $g = \lfloor\xi\rfloor$ and $x_1, x_2, \ldots \in \{0, 1, \ldots, b - 1\}$, we have

$$\{\xi b^n\} = x_{n+1} b^{-1} + x_{n+2} b^{-2} + x_{n+2} b^{-3} + \ldots := 0.x_{n+1}x_{n+2}x_{n+3} \ldots$$
for any \( n \geq 0 \). So, in other words, we are interested in the following question: determine the smallest possible interval \( I \) to which belong all the tails of an irrational number \( \xi = g + 0.x_1x_2x_3\ldots \) (in its \( b \)-adic expansion), namely, the numbers \( 0.x_{n+1}x_{n+2}x_{n+2}\ldots \), where \( n \geq 0 \).

2. Main result

We will use the terminology from combinatorics on words (see, for instance, [2] or [10]). For an infinite word \( w \), let us denote by \( p(w, m) \) the number of distinct blocks of length \( m \) occurring in \( w \). Morse and Hedlund [12] proved that the function \( m \mapsto p(w, m) \) is either bounded, or strictly increasing. Consequently, \( w \) is not ultimately periodic (in this context usually called aperiodic) if, and only if, \( p(w, m) \geq m + 1 \) holds for every positive integer \( m \). By definition, an infinite word \( w \) is called Sturmian if we have \( p(w, m) = m + 1 \) for any positive integer \( m \). (In particular, since then \( p(w, 1) = 2 \), this implies that \( w \) is a word on an alphabet of two letters.) There are many equivalent definitions for Sturmian words, and we refer the reader to Chapter 2 from [10] or to Chapter 6 from [2]. We just recall that \( w := w_1w_2\ldots \) is a characteristic Sturmian word if, and only if, there exists an irrational number \( \beta \) (the slope) in \( (0, 1) \) such that \( w_n = \lfloor \beta(n + 1) \rfloor - \lfloor \beta(n) \rfloor \) for every positive integer \( n \).

Suppose that \( T_j \) maps the word \( w = w_1\ldots w_jw_{j+1}\ldots \) to the word \( w_{j+1}w_{j+2}\ldots \) and set \( t_b(w) := 0.w_1w_2\ldots = \sum_{j=1}^{\infty} w_jb^{-j} \). With this notation, we can state our main result.

**Theorem 2.** Let \( b \geq 2 \) be an integer and \( \xi \) be an irrational real number. Then the numbers \( \{\xi b^n\}, n \geq 0 \), cannot all lie in an interval of length strictly smaller than \( 1/b \). On the other hand, the real numbers \( \{\xi b^n\}, n \geq 0 \), are all lying in a closed interval \( I \) of length \( 1/b \) if, and only if, \( \xi = g + k/(b - 1) + t_b(w) \), where \( g \) is an arbitrary integer, \( k \) is in \( \{0, 1, \ldots, b - 2\} \), and \( w \) is a Sturmian word on \( \{0, 1\} \). If this is the case, then \( \xi \) is transcendental and the interval \( I \) is semi-open. Moreover, it is open, unless there exists an integer \( j \geq 1 \) such that \( T_j(w) = u \) is a characteristic Sturmian word.

In particular, since there are uncountably many Sturmian sequences on \( \{0, 1\} \), Theorem 2 shows that there are uncountably many pairs \((\xi, s)\), where \( \xi \) is irrational and \( s \in (0, 1 - 1/b) \), such that \( s < \{\xi b^n\} < s + 1/b \) for every \( n \geq 0 \).

At the end of the paper [7] the following problem is posed: prove that, for any real numbers \( \xi \) and \( \nu \) with \( \xi > 0 \), the numbers \( \lfloor \xi 2^n + \nu \rfloor \) are composite for infinitely many \( n \in \mathbb{N} \). Observe that if we have \( 0 \leq \lfloor \xi 2^{n-1} + (\nu - 1)/2 \rfloor < 1/2 \), then the number \( \lfloor \xi 2^n + \nu - 1 \rfloor \) is even and so \( \lfloor \xi 2^n + \nu \rfloor \) is odd. Thus, since there are uncountably many Sturmian words on the alphabet \( \{0, 1\} \), it follows from Theorem 2 that there do exist uncountably many pairs \((\xi, \nu)\) for which \( \lfloor \xi 2^n + \nu \rfloor \) is odd for every positive integer \( n \).

3. Proof of Theorem 2

Before giving the proof of Theorem 2, we gather in an auxiliary lemma results from Proposition 2.1.3, Theorem 2.1.5 and Proposition 2.1.22 of Chapter 2 of [10].
Lemma. Let \( w \) be an infinite aperiodic word on \( \{0,1\} \). Then, \( w \) is Sturmian if, and only if, for any finite word \( v \), at least one of the words \( 0v0 \) and \( 1v1 \) is not a factor of \( w \). Moreover, \( w \) is Sturmian characteristic if, and only if, both \( 0w \) and \( 1w \) are Sturmian.

Let us write \( \xi \) in the form \( g + t_b(x) \), where \( g = \lfloor \xi \rfloor \) is an integer and \( t_b(x) = x_1b^{-1} + x_2b^{-2} + x_3b^{-3} + \ldots = 0.x_1x_2x_3 \ldots \) is the \( b \)-adic expansion of \( \{ \xi \} = \xi - g \). As above, \( \{ \xi b^n \} = 0.x_{n+1}x_{n+2} \ldots \). In particular, since \( \xi \) is irrational, this implies that \( x_{n+i}b^{-n-i} < \{ \xi b^n \} \). Thus, if there exist \( i, j \geq 0 \), satisfying \( x_{j+1} - x_{i+1} \geq 2 \), then we get

\[
\{ \xi b^j \} - \{ \xi b^i \} = x_{j+1}b^{-n} - x_{i+1}b^{-n} - b^{-n} \geq 2/b - 1/b = 1/b.
\]

Consequently, we can assume without loss of generality that \( x_1, x_2, \ldots \in \{k,k+1\} \), where \( k = 0,1,\ldots,b-2 \). Thus, we can write \( \xi \) in the form \( g + k/(b-1) + t_b(w) \), where \( w = w_1w_2 \ldots \) is a word on the alphabet \( \{0,1\} \) and \( t_b(w) = w_1b^{-1} + w_2b^{-2} + w_3b^{-3} + \ldots = 0.w_1w_2w_3 \ldots \). Now, we have

\[
\{ \xi b^n \} - k/(b-1) = 0.w_{n+1}w_{n+2} \ldots = w_{n+1}b^{-1} + w_{n+2}b^{-2} + \ldots.
\]

Since \( \xi \) is irrational, the complexity function of the infinite word \( w := w_1w_2 \ldots \) is strictly increasing. This implies that, for any \( m \geq 1 \), there exists (at least) one block \( w_m \) of \( m \) letters such that both \( 0w_m \) and \( 1w_m \) are subblocks of \( w \). In other words, there exist integers \( u = u(m) \) and \( v = v(m) \) such that \( \{ \xi b^u \} - k/(b-1) = 0.0w_mw^u \) and \( \{ \xi b^v \} - k/(b-1) = 0.1w_mw^v \). Hence \( \{ \xi b^u \} - \{ \xi b^v \} > b^{-1} - b^{-m} \). By taking \( m \) sufficiently large, we conclude that no interval of length strictly smaller than \( 1/b \) can contain all the \( \{ \xi b^n \} \) with \( n \geq 0 \). (Taking \( \xi = 0.101010 \ldots \) or simply \( \xi = 1 \) shows that the assumption ‘\( \xi \) is irrational’ is necessary.)

Let us now prove the second statement. Assume that \( w \) is Sturmian. By the lemma, for any finite word \( v \), the words \( 0v0 \) and \( 1v1 \) cannot be both factors of \( w \). Consequently, the difference between any two numbers \( \{ \xi b^i \} \) and \( \{ \xi b^j \} \) is bounded above in absolute value by \( 1/b \). The inequality is strict, since \( w \) is aperiodic. Thus, we have shown that, for \( w \) Sturmian, there exists a semi-open interval of length \( 1/b \) that contains all the \( \{ \xi b^n \} \), where \( n \geq 0 \). Furthermore, it follows from [8] that \( \xi \) is transcendental.

Assume now that \( w \) is neither Sturmian, nor ultimately periodic. Then, by the lemma, there exists a finite word \( u \) such that both \( 0u0 \) and \( 1u1 \) are factors of \( w \). Arguing as above, we see that the difference between corresponding fractional parts is greater than \( 1/b \). This shows that, for such \( w \), there does not exist a closed interval of length \( 1/b \) containing all fractional parts \( \{ \xi b^n \} \), \( n \geq 0 \), and proves the second part.

Finally, let \( w \) be an infinite Sturmian word. The fact that the numbers \( 0.w_{n+1}w_{n+2} \ldots \) all belong to a closed interval of length \( 1/b \) can be expressed in the form

\[
t_b(0u) \leq t_b(T^n w) \leq t_b(1u) = t_b(0u) + b^{-1},
\]

where \( u \) is a word on \( \{0,1\} \) and where \( n \) runs through every non-negative integer. For simplicity (and according to the lexicographical order of words), we can write this inequality in the form

\[
0u \leq T^n w \leq 1u, \quad \text{for any } n \geq 0.
\]
Evidently, all \( t_0(T^n w) \) belong to an open interval of length \( 1/b \), unless there is a \( h \geq 0 \) such that \( T^h w = 0 u \) or \( 1 u \). Assume that \( T^h w = 0 u \). Then we have

\[
0 u < T^n w < 1 u, \quad \text{for any } n \geq h + 1,
\]

that is,

\[
0 u < T^n u < 1 u, \quad \text{for any } n \geq 0.
\]

The case \( T^h w = 1 u \) leads to the same inequalities. These are strict, since \( w \) is aperiodic.

Let us prove now that \( u \) is Sturmian characteristic. In view of the lemma, it is sufficient to show that both \( 0 u \) and \( 1 u \) are Sturmian. Observe that \( u \) is aperiodic, since \( u = T^{h+1}(w) \). Assume that \( p(0 u, m) \geq m + 2 \) for some \( m \). The first part of Theorem 2 implies that \( 0 u \) is a limit point of the sequence \( u, T^1 u, T^2 u, \ldots \), hence, we get that \( p(T^n u, m) \geq m + 2 \) for some \( n \). This yields

\[
m + 1 = p(w, m) \geq p(T^n u, m) \geq m + 2,
\]

a contradiction. Consequently, \( 0 u \) is Sturmian, and so is \( 1 u \), by a similar argument.

We thus conclude that the numbers \( t_0(T^n w), n \geq 0 \), all belong to an open interval of length \( 1/b \), unless there are an integer \( h \geq 0 \) and a characteristic Sturmian word \( u \) such that \( T^h w = 0 u \) or \( 1 u \). So \( u = T^j(w) \) with \( j = h + 1 \geq 1 \). The proof of Theorem 2 is completed.

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