A quantitative lower bound
for the greatest prime factor of $(ab+1)(bc+1)(ca+1)$

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À Wolfgang Schmidt, pour son soixante-dixième anniversaire

Abstract. Let $\mathcal{A}$ be a finite set of at least two triples of distinct positive integers (a, b, c) with $a > b > c$. In this note, we show that there exists a triple (a, b, c) in $\mathcal{A}$ such that the greatest prime factor of $(ab+1)(ac+1)(bc+1)$ exceeds $10^{-7} \log |\mathcal{A}| \cdot \log\log |\mathcal{A}|$, where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. This confirms a conjecture of Győry & Sárközy from [7].

2000 Mathematics Subject Classification: 11N32, 11D57.

§1 Introduction

For any integer $n \geq 2$, we denote by $P(n)$ the greatest prime factor of n. Győry, Sárközy & Stewart [8] conjectured that if $a > b > c$ are positive integers, then

$$P((ab+1)(bc+1)(ca+1)) \longrightarrow \infty$$

as a tends to infinity. Partial results have been obtained by Győry & Sárközy [7], Stewart & Tijdeman [11] and Bugeaud [3]. Very recently, Corvaja & Zannier [4] and, independently and simultaneously, Hernández & Luca [9] applied the Schmidt Subspace Theorem to give a positive answer to the above-mentioned conjecture. Actually, a stronger result is proved in [4], namely that the greatest prime factor of $(ab+1)(ac+1)$ tends to infinity as the maximum of the pairwise distinct positive integers a, b and c goes to infinity.
There are two natural extensions of such a result. First, one can search for an effective lower bound for $P((ab+1)(bc+1)(ca+1))$ in terms of $\max\{a, b, c\}$. This has been achieved, under additional assumptions on a, b and c, in [11] and in [3]. Second, given a finite set $\mathcal{A}$ of triples (a, b, c), one can aim at establishing a lower bound for $P(\prod(ab+1)(bc+1)(ca+1))$, where the product is taken over all the triples in $\mathcal{A}$, in terms of the cardinality of $\mathcal{A}$. This question has been considered in [7], where some partial results were obtained, which have motivated the following conjecture. Throughout the present paper, we denote by $|\mathcal{S}|$ the cardinality of a finite set $\mathcal{S}$.

Conjecture (Győry and Sárközy). *Let $\mathcal{A}$ be a finite set of cardinality at least two of triples (a, b, c) of pairwise distinct integers. Then there exists (a, b, c) in $\mathcal{A}$ with*

$$P((ab+1)(bc+1)(ca+1)) > \kappa \log |\mathcal{A}| \log\log |\mathcal{A}|,$$

where κ is an effectively computable positive absolute constant.

In the present work, we show that the conjecture of Győry & Sárközy holds true with the constant $\kappa = 10^{-7}$, regardless of any additional assumption on the set $\mathcal{A}$. Our main results are stated in Section 2. Section 4 is devoted to their proofs, which depend on a quantitative version of the Schmidt Subspace Theorem, due to Evertse, and recalled in Section 3. Some related questions are discussed in Section 5.

§2 Statements of the main results

For any integer $n \ge 2$ we write $\omega(n)$ for the number of distinct prime factors of n. As in [7], we first establish a lower bound for the number of distinct prime factors of $\prod (ab+1)(ac+1)(bc+1)$, where the product is taken over a finite set of triples of distinct integers.

Theorem 1. *For any finite set $\mathcal{A}$ of cardinality at least two of triples of positive integers (a,b,c) with $a > b > c$, we have*

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)(bc+1)\Big) > 10^{-6} \log|\mathcal{A}|. \tag{1}$$

By the Prime Number Theorem, Theorem 1 above enables us to confirm the conjecture of Győry & Sárközy, even with an explicit value for the constant κ.

Corollary 1. *Let $\mathcal{A}$ be a finite set of cardinality at least two of triples of positive integers (a,b,c) with $a > b > c$. There exists a triple (a,b,c) in $\mathcal{A}$ such that*

$$P\Big((ab+1)(ac+1)(bc+1)\Big) > 10^{-7} \log|\mathcal{A}| \log\log|\mathcal{A}|. \tag{2}$$

The proof of Theorem 1 requires five steps and is not a mere combination of the arguments of [4] with an effective version of the Subspace Theorem. We can summarize the argument as follows. Let (a,b,c) be a triple of positive integers with $a > b > c$ and set $u := ab+1$ and $v := ac+1$. First, we exactly follow [4] to prove that u and v satisfy linear equations of the type

$$\gamma_1 \frac{u-1}{v-1} + \gamma_2 \frac{u^2-1}{v-1} + \sum_{\substack{0\le j\le 2\\ 1\le n\le 5}} \delta_{jn} u^j v^{5-n} = 0,$$

where γ_1, γ_2 and the δ_{jn}'s are rational numbers, not all zero. We use Evertse's quantitative result to bound the number of these equations in terms of the number of distinct prime factors of uv. We would then like to prove that each of these equations can be satisfied only by finitely many pairs (u,v), but this is by no means obvious since we cannot exclude the presence of equations like $t_1 + t_2 uv + t_3 (uv)^2$, for which we have no control on the size of t_1, t_2 and t_3. Using Evertse's bound, we have an upper estimate for the number of projective solutions. To see that to each projective solution corresponds a controlled number of pairs (u,v), we apply the Subspace Theorem once again (Step 4 of the proof). We then get an explicit upper bound for the number of pairs (u,v). However, this is not sufficient to deduce an upper estimate for the number of triples (a,b,c), since $u-1$ and $v-1$ can have a very large greatest common divisor which is divisible by many small primes (see Section 5 of the paper). To conclude, we use the fact that $bc+1$ is also composed of primes from S. Our argument here rests on the existence of primitive divisors for Lucas sequences.

Remark 1. Győry & Sárközy [7] have proved that, for any positive real number ε, the right hand side of (2) cannot be replaced by $|\mathcal{A}|^{\varepsilon}$. They however think that (2) should be close to the truth.

Remark 2. By adapting arguments of Győry, Sárközy & Stewart [8], it is likely that one can prove the existence of finite sets $\mathcal{A}$ of triples (a,b,c) with $a>b>c$ such that $P((ab+1)(ac+1)) \le \kappa\,(\log|\mathcal{A}|)^{10}$ for any triple (a,b,c) in $\mathcal{A}$ and an absolute constant κ.

Remark 3. Other related quantitative questions are considered in Section 5. In particular, we show that the right-hand side of (1) cannot be replaced by $|\mathcal{A}|^{1/2+\varepsilon}$ for some $\varepsilon>0$.

Throughout this paper, we use $c_1,\ c_2,\ \ldots$ for effectively computable positive constants which are absolute. We also use the Vinogradov symbols $\ll$ and $\gg$ as well as the Landau symbols O and o with their regular meaning.

§3 Auxiliary results

We start by recalling a particular instance of a quantitative version of the Schmidt Subspace Theorem due to J.-H. Evertse [6].

Let $M_{\mathbf{Q}}$ be all the places of $\mathbf{Q}$. For $x\in\mathbf{Q}$ and $w\in M_{\mathbf{Q}}$ we put $|x|_w := |x|$ if $w=\infty$ and $|x|_w := p^{-\mathrm{ord}_p(x)}$ if w corresponds to the prime number p. When $x=0$, we set $\mathrm{ord}_p(x):=\infty$ and $|x|_w := 0$. Then the product formula

$$\prod_{w\in M_{\mathbf{Q}}} |x|_w = 1 \quad \text{holds for all } x\in\mathbf{Q}^*. \tag{3}$$

Let $N\ge 1$ be a positive integer and define the *height* of $\mathbf{x} := (x_1,\ldots,x_N)\in(\mathbf{Q})^N$ as follows. For $w\in M_{\mathbf{Q}}$ write

$$|\mathbf{x}|_w := \Big(\sum_{i=1}^{N} x_i^2\Big)^{1/2} \qquad \text{if } w=\infty,$$

and

$$|\mathbf{x}|_w := \max\{|x_1|_w,\ldots,|x_N|_w\} \qquad \text{otherwise.}$$

Then

$$\mathcal{H}(\mathbf{x}) := \prod_{w\in M_{\mathbf{Q}}} |\mathbf{x}|_w.$$

For a linear form $L(\mathbf{x}) := \sum_{i=1}^{N} a_i x_i$ with $\mathbf{a} := (a_1,\ldots,a_N)\in(\mathbf{Q})^N$, we write $\mathcal{H}(L) := \mathcal{H}(\mathbf{a})$. We now let $N\ge 1$ be a positive integer, S be a finite subset of $M_{\mathbf{Q}}$ of cardinality s containing the infinite place, and for every $w\in S$ we let $L_{1w},\ldots,L_{Nw}$ be N linearly independent linear forms in N indeterminates with coefficients in $\mathbf{Q}$ satisfying

$$\mathcal{H}(L_{iw}) \le H \qquad \text{for } i=1,\ldots,N \text{ and } w\in S. \tag{4}$$

Theorem E1. *Let* $0 < \delta < 1$ *and consider the inequality*

$$\prod_{w \in S} \prod_{i=1}^{N} \frac{|L_{iw}(\mathbf{x})|_w}{|\mathbf{x}|_w} < \Big(\prod_{w \in S} |\det(L_{1w}, \ldots, L_{Nw})|_w \Big) \cdot \mathcal{H}(\mathbf{x})^{-n-\delta}. \tag{5}$$

Then the following hold:
(i) *There exist proper linear subspaces* $T_1, \ldots, T_{t_1}$ *of* $\mathbf{Q}^N$, *with*

$$t_1 \le (2^{60N^2} \cdot \delta^{-7N})^s \tag{6}$$

such that every solution $\mathbf{x} \in \mathbf{Q}^N \backslash \{\mathbf{0}\}$ *of* (5) *satisfying* $\mathcal{H}(\mathbf{x}) \ge H$ *belongs to* $T_1 \cup \ldots \cup T_{t_1}$.
(ii) *There exist proper linear subspaces* $T'_1, \ldots, T'_{t_2}$ *of* $\mathbf{Q}^N$, *with*

$$t_2 \le (150N^4 \cdot \delta^{-1})^{Ns+1}(2 + \log\log 2H) \tag{7}$$

such that every solution $\mathbf{x} \in \mathbf{Q}^N \backslash \{\mathbf{0}\}$ *of* (5) *satisfying* $\mathcal{H}(\mathbf{x}) < H$ *belongs to* $T'_1 \cup \ldots \cup T'_{t_2}$.

We shall apply Theorem E1 above to a certain finite subset S of $M_{\mathbf{Q}}$, and certain systems of linear forms L_{iw} with $i = 1, \ldots, N$ and $w \in S$. Moreover, in our case, the points $\mathbf{x}$ for which (5) will hold will be in $(\mathbf{Z}^*)^N$. In particular, $|\mathbf{x}|_w \le 1$ will hold for all $w \in M_{\mathbf{Q}} \backslash \{\infty\}$, as well as the inequalities

$$1 \le \mathcal{H}(\mathbf{x}) \le \prod_{w \in S} |\mathbf{x}|_w, \tag{8}$$

and

$$1 \le \mathcal{H}(\mathbf{x}) \le \prod_{w \in S} |\mathbf{x}|_w \le N \cdot \max\{|x_i| \mid i = 1, \ldots, N\}. \tag{9}$$

Finally, our linear forms will have integer coefficients and will satisfy

$$\det(L_{1w}, \ldots, L_{Nw}) = \pm 1 \qquad \text{for all } w \in S. \tag{10}$$

With these conditions, the following statement is a straightforward consequence of Theorem E1 above.

Corollary E1. *Assume that* (10) *is satisfied, that* $0 < \delta < 1$, *and consider the inequality*

$$\prod_{w \in S} \prod_{i=1}^{N} |L_{iw}(\mathbf{x})|_w < N^{-\delta} \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, N\} \Big)^{-\delta}. \tag{11}$$

Then there exist proper linear subspaces $T_1, \ldots, T_{t_1}$ *of* $\mathbf{Q}^N$, *with*

$$t_1 \le (2^{60N^2} \cdot \delta^{-7N})^s \tag{12}$$

such that every solution $\mathbf{x} \in \mathbf{Z}^N \backslash \{\mathbf{0}\}$ *of* (11) *satisfying* $\mathcal{H}(\mathbf{x}) \ge H$ *belongs to* $T_1 \cup \ldots \cup T_{t_1}$.

Recall that an S-*unit* x is a non-zero rational number such that $|x|_w = 1$ for all $w \notin S$. We shall also need the following version of a Theorem of Evertse [5] on S-unit equations.

Theorem E2. *Let $a_1,\ldots,a_N$ are non-zero rational numbers. Then the equation*

$$\sum_{i=1}^{N} a_i u_i = 1 \tag{13}$$

in S-unit unknowns u_i for $i=1,\ldots,N$ and such that $\sum_{i\in I} a_i u_i \neq 0$ for each non-empty subset $I \subseteq \{1,\ldots,N\}$ has at most $(2^{35}N^2)^{N^3 s}$ solutions.

We are now ready to proceed with the proofs of our results.

§4 The proofs

The proof of Theorem 1.

We may certainly assume that $|\mathcal{A}| > e^{10^6}$, for otherwise inequality (1) is satisfied anyway. Let

$$s := \omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)(bc+1)\Big). \tag{14}$$

We need to find an upper bound of $|\mathcal{A}|$ in terms of s. We shall split our argument in several steps.

Step 1. The first system of forms.

In this part of the argument, we follow the method from [4].

We write S for the set of places consisting from the infinite place and the valuations corresponding to the primes p dividing $(ab+1)(ac+1)(bc+1)$ for some triple $(a,b,c)\in\mathcal{A}$. We assume that $a>b>c$. Clearly, S contains $s+1$ elements. We write $u := ab+1$, $v := ac+1$, and we put

$$y_1 := \frac{u-1}{v-1} = \frac{b}{c} \quad \text{and} \quad y_2 := \frac{u^2-1}{v-1} = \frac{(u+1)b}{c}.$$

Thus, $u > v \geq 4$ are positive integers which are S-units, and y_1 and y_2 are rational numbers with denominator at most c. Write

$$\frac{1}{v-1} = \frac{1}{v(1-v^{-1})} = \sum_{n\geq 1} v^{-n} = \sum_{n=1}^{5} v^{-n} + \sum_{n\geq 6} v^{-n}.$$

Thus,

$$\Big|\frac{1}{v-1} - \sum_{n=1}^{5} v^{-n}\Big| = \sum_{n\geq 6} v^{-n} = \frac{1}{v^5(v-1)} < 2v^{-6}.$$

On multiplying the above estimate by u^j-1 for $j=1,2$, we obtain

$$\Big|y_j + \sum_{n=1}^{5} v^{-n} - \sum_{n=1}^{5} u^j v^{-n}\Big| < 2u^j v^{-6}, \quad j=1,2,$$

which is equivalent to

$$\left|v^5 y_j + \sum_{n=1}^{5} v^{5-n} - \sum_{n=1}^{5} u^j v^{5-n}\right| < 2u^j v^{-1}, \quad j = 1, 2. \tag{15}$$

We let $\sigma_1, \ldots, \sigma_{15}$ denote the integers $u^j v^{5-n}$ for $j = 0, 1, 2$ and $n = 1, \ldots, 5$ in some order. We may then rewrite (15) as

$$\left|v^5 y_j + \sum_{i=1}^{15} \alpha_{ji}\sigma_i\right| < 2u^j v^{-1}, \quad j = 1, 2, \tag{16}$$

where $\alpha_{ji} \in \{0, \pm 1\}$. We now let L_{jw} be the linear forms in the 17 variables $Y_1, Y_2, X_1, \ldots, X_{15}$, where $j = 1, \ldots, 17$ and $w \in S$ defined as follows:

$$L_{j\infty} = Y_j + \sum_{i=1}^{15} \alpha_{ji} X_i, \quad L_{jw} = Y_j \text{ for } w \neq \infty, \quad j = 1, 2,$$

and $L_{jw} = X_{j-2}$ for all $j = 3, \ldots, 17$ and $w \in S$. It is easy to see that inequality (4) is satisfied with $H := 1$, and that formula (10) holds for our $N := 17$, finite set of places S, and linear forms L_{iw} with $i = 1, \ldots, 17$ and $w \in S$. We also define the vector $\mathbf{x} := (x_1, \ldots, x_{17}) \in (\mathbf{Z}^*)^{17}$ as

$$\mathbf{x} = (x_1, \ldots, x_{17}) = (cv^5 y_1, cv^5 y_2, c\sigma_1, \ldots, c\sigma_{15}).$$

It is clear that $\mathbf{x}$ is a vector whose components are non-zero integers. Inequalities (16) now yield

$$|L_{jw}(\mathbf{x})|_\infty < 2cu^j v^{-1}, \quad j = 1, 2. \tag{17}$$

The argument from [4] now shows that

$$\prod_{w \in S \setminus \{\infty\}} |L_{jw}(\mathbf{x})|_w \leq v^{-5}, \quad \text{for } j = 1, 2 \tag{18}$$

and that

$$\prod_{w \in S} |L_{jw}(\mathbf{x})|_w \leq c \quad \text{for } j = 3, \ldots, 17. \tag{19}$$

Multiplying all the above inequalities (17)–(19), we get

$$\prod_{i=1}^{17} \prod_{w \in S} |L_{iw}(\mathbf{x})|_w \leq 4c^{17} u^3 v^{-12}. \tag{20}$$

Since $u = ab + 1 < a^2$, $v = ac + 1 > ac$, we have $c^{17} u^3 v^{-12} < c^5 a^{-6} < a^{-1}$, while $\max\{|x_i| \mid i = 1, \ldots, 17\} < cu^2 v^5 < a^{15}$, and so (20) implies that

$$\prod_{i=1}^{17} \prod_{w \in S} |L_{iw}(\mathbf{x})|_w < 4 \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, 17\}\Big)^{-1/15}. \tag{21}$$

Note that only the fact that $a > \max\{b, c\}$ was used in the above argument, but not the fact that $u > v$.

Step 2. Quantitative estimates and non-degenerate Newton polygons.

Let $\mathcal{A}_1$ be the subset of those triples (a, b, c) in $\mathcal{A}$ such that

$$\max\{|x_i| \mid i = 1, \ldots, 17\} \le 4^{15\cdot 16} \cdot 17^{15} < e^{400}$$

holds. For such triples, since $a < u < \max\{|x_i| \mid i = 1, \ldots, 17\}$, we get that $a < e^{400}$, and we therefore get that

$$|\mathcal{A}_1| < e^{1200}.$$

We shall write $\mathcal{B}_1$ for the set of pairs (u, v) obtained from triples $(a, b, c) \in \mathcal{A}_1$, and therefore $|\mathcal{B}_1| < e^{1200}$.

From now on, we work only with those triples $(a, b, c) \in \mathcal{A}\backslash\mathcal{A}_1$. In this case,

$$\max\{|x_i| \mid i = 1, \ldots, 17\} > 4^{15\cdot 16} \cdot 17^{15},$$

and the above inequality implies that the inequality

$$4 \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, 17\}\Big)^{-1/15} < 17^{-1/16} \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, 17\}\Big)^{-1/16}$$

holds. With (21), we get that

$$\prod_{i=1}^{17} \prod_{w\in S} |L_{iw}(\mathbf{x})|_w < 17^{-1/16} \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, 17\}\Big)^{-1/16}, \tag{22}$$

and since $H = 1$ and $\mathcal{H}(\mathbf{x}) \ge 1$, we are entitled to apply Corollary E1 with $N = 17$, $\delta = (16)^{-1}$, and conclude that there exist proper linear subspaces $T_1, \ldots, T_{t_1}$ of $\mathbf{Q}^{17}$ with

$$t_1 < (2^{60\cdot 17^2} \cdot 16^{7\cdot 17})^{s+1} < \exp(12400(s+1)), \tag{23}$$

and such that all the solutions of inequality (22) lie in $T_1 \cup \ldots \cup T_{t_1}$.

We let T be one of the proper subspaces T_ℓ for $\ell = 1, \ldots, t_1$, and assume that $\mathbf{x} \in T$. We then have an equation of the type

$$\gamma_1 y_1 + \gamma_2 y_2 + \sum_{\substack{0\le j\le 2\\ 1\le n\le 5}} \delta_{jn} u^j v^{5-n} = 0,$$

where γ_1, γ_2 and δ_{jn} are rational numbers for $j = 0, 1, 2$ and $n = 1, \ldots, 5$ not all zero. This in turn leads to an equation of the form

$$P_T(u, v) = 0, \tag{24}$$

where

$$P_T(X, Y) := \sum_{(i,j)} \eta_{(i,j)} X^i Y^j$$

$$= \gamma_1(X-1) + \gamma_2(X^2-1) + (Y-1)\Big(\sum_{\substack{0\le j\le 2\\ 1\le n\le 5}} \delta_{jn} X^j Y^{5-n}\Big) \in \mathbf{Q}[X,Y]. \tag{25}$$

The fact that $P_T(X,Y)$ is a non-zero polynomial has been justified in [4]. Note that the vertices of the Newton polygon of $P_T(X,Y)$ (i.e., the pairs of non-negative integers (i,j) such that the monomial X^iY^j appears in $P_T(X,Y)$) are contained in $\{0 \le i \le 2,\ 0 \le j \le 5\}$, which consists of precisely 18 lattice points.

Each one of the equations (24) is an S-unit equation whose indeterminates are $M_{(i,j)} := u^iv^j$, where (i,j) is a vertex of the Newton polygon of P_T. For each one of these solutions, the equation (24) may be non-degenerate or not. If it is degenerate, then there exists a non-empty proper subset $\mathcal{D}$ of the vertices of the Newton polygon of P_T, such that $P_{T,\mathcal{D}}(u,v) = 0$ is a non-degenerate S-unit equation, where we write

$$P_{T,\mathcal{D}} := \sum_{(i,j)\in D} \eta_{(i,j)} X^iY^j.$$

Note that $\mathcal{D}$ can be chosen in at most 2^{18} ways once T is known. Omitting the dependence of the variable subset $\mathcal{D}$, it follows that up to multiplying the upper bound on t_1 shown at (23) by the factor $2^{18} < \exp(13)$, we may assume that each one of the equations (24) is non-degenerate. Assume now that the Newton polygon of P_T has exactly $m \le 18$ monomials (note that $m \ge 2$), and let them be $M_\mu := X^{i_\mu}Y^{j_\mu}$ for $\mu = 1,\ldots,m$. By Theorem E2, it follows that there exist solutions $(u^{(\lambda)}, v^{(\lambda)})$ with λ in a finite set Λ_T of cardinality at most

$$|\Lambda_T| \le (2^{35}(m-1)^2)^{(m-1)^3(s+1)} \le (2^{35}\cdot 17^2)^{17^3(s+1)} < \exp(150000(s+1)), \tag{26}$$

and such that for any other solution (u,v) of equation (24) whose components are S-units of equation there exists an S-unit ζ and $\lambda \in \Lambda$ such that $M_\mu(u,v) = M_\mu(u^{(\lambda)}, v^{(\lambda)})\zeta$ holds for all $\mu = 1,\ldots,m$. Eliminating ζ and taking logarithms, these last equations are seen to imply that

$$(i_\mu - i_1)\log u - (j_\mu - j_1)\log v = (i_\mu - i_1)\log u^{(\lambda)} - (j_\mu - j_1)\log v^{(\lambda)} \qquad \text{for } \mu = 2,\ldots,m. \tag{27}$$

Since all the data in (27) is fixed except for the pair (u,v), it follows that the only solution of the system of equations (27) is $(u,v) = (u^{(\lambda)}, v^{(\lambda)})$, *except* for the case when the Newton polygon of P_T is degenerate, i.e., when all the points (i_μ, j_μ) for $\mu = 1,\ldots,m$ are collinear.

Let $\mathcal{A}_2$ be the set of triples $(a,b,c) \in \mathcal{A}\backslash\mathcal{A}_1$ with $a > b > c$ and such that the corresponding pair (u,v) is a non-degenerate solution of an equation of the type $P_T(u,v) = 0$, where the Newton polygon of P_T is non-degenerate, and let $\mathcal{B}_2$ be the set of pairs (u,v) which arise from $(a,b,c) \in \mathcal{A}_2$. The above argument together with estimates (23) and (26) shows that

$$|\mathcal{B}_2| \le 2^{18}\cdot t_1 \cdot \max\{|\Lambda_T| \mid T = T_1,\ldots,T_{t_1}\}$$

$$< \exp(13 + 12400(s+1) + 150000(s+1)) < \exp(170000(s+1)). \tag{28}$$

From now on, we shall assume that $(a,b,c) \in \mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2)$, and therefore that the Nexton polygon of P_T is degenerate. Let (i_1,j_1) and (i_2,j_2) be two distinct vertices of the Newton polygon of P_T, and write $i_0 := i_2 - i_1$ and $j_0 := j_2 - j_1$. Note that $(i_0, j_0) \neq (0,0)$. Then any solution (u,v) of the equation $P_T(u,v) = 0$ satisfies $u^{i_0}v^{j_0} = K_\lambda$, where K_λ is a rational number belonging to a set of finite set of cardinality $|\Lambda_T|$ of such. Note that $|i_0| \leq 2$ and $|j_0| \leq 5$.

Step 3. Exploiting the symmetry.

As we pointed out at Step 1, the fact that $u > v$ is not used in the argument leading to the conclusion that inequality (20) holds. Thus, interchanging u and v anywhere in the first two steps, we conclude that there exists a subset $\mathcal{A}_3 \in \mathcal{A}\backslash\mathcal{A}_1$ such that if we write $\mathcal{B}_3$ for the set of all pairs (u,v) arising from triples $(a,b,c) \in \mathcal{A}_3$, then

$$\begin{aligned} |\mathcal{B}_3| &\leq 2^{18} \cdot t_1' \cdot \max\{|\Lambda_{T'}'| \mid T' = T_1', \ldots, T_{t_1'}'\} \\ &< \exp(13 + 12400(s+1) + 150000(s+1)) < \exp(170000(s+1)), \end{aligned} \tag{29}$$

and that if $(a,b,c) \in \mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_3)$, then there exist a proper subspace T' of $\mathbf{Q}^m$, a subset $\mathcal{D}'$ of the vertices of the Newton polygon of $P_{T'}$, integers $(i_0', j_0') \neq (0,0)$ with $|i_0'| \leq 5$ and $|j_0'| \leq 2$, and rational numbers $K_{\lambda'}$ in a set $|\Lambda_T'|$ of cardinality bounded by the expression appearing in the right hand side of (26), and such that $u^{i_0'}v^{j_0'} = K_\lambda'$. In the above inequality, t_1', T', $\mathcal{D}'$ and $\Lambda_{T'}'$ have the same meaning as t_1, T and Λ_T, respectively, when u and v are interchanged.

If the vector (i_0, j_0) is not parallel to (i_0', j_0'), then the system of equations $u^{i_0}v^{j_0} = K_\lambda$ and $u^{i_0'}v^{j_0'} = K_\lambda'$ has a unique solution (u,v) once K_λ and $K_{\lambda'}$ are fixed. Let $\mathcal{A}_4$ be the subset of $\mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$ formed by those triples (a,b,c) such that the corresponding vectors (i_0, j_0) and (i_0', j_0') are not parallel, and let $\mathcal{B}_4$ be the set of pairs (u,v) arising from triples $(a,b,c) \in \mathcal{A}_4$. The above argument and estimates (23) and (26) show that

$$\begin{aligned} |\mathcal{B}_4| &\leq 2^{2\cdot 18} \cdot t_1 \cdot t_1' \cdot \left(\max\{|\Lambda_T|, |\Lambda_{T'}'| \mid T = T_1, \ldots, T_{t_1},\ T' = T_1', \ldots, T_{t_1'}'\}\right)^2 \\ &< \exp(340000(s+1)). \end{aligned} \tag{30}$$

From now on, we assume that $(a,b,c) \in \mathcal{A}\backslash \cup_{k=1}^4 \mathcal{A}_k$. In this case, (i_0, j_0) and (i_0', j_0') are parallel, and since $|i_0| \leq 2$, $|j_0'| \leq 2$, $|j_0| \leq 5$ and $|i_0'| \leq 5$, it follows that we may assume that $\max\{|i_0|, |j_0|\} \leq 2$. Moreover, since $(i_0, j_0) \neq (0,0)$, by the symmetry, and up to changing the signs of both i_0 and j_0, and cancelling their greatest common divisor, if needed, we may assume that $i_0 = 1$, and that $j_0 \in \{0, \pm 1, \pm 2\}$.

Step 4. The second system of forms.

We now assume that $(a,b,c) \in \mathcal{A}\backslash \cup_{k=1}^4 \mathcal{A}_k$, that the subspace $T \in \{T_1, \ldots, T_{t_1}\}$, the subset $\mathcal{D}$, the index j_0 in $\{0, \pm 1, \pm 2\}$, (note that j_0 depends only on T and $\mathcal{D}$), and the number $K := K_\lambda$ for $\lambda \in \Lambda_T$ are fixed, and that $uv^{j_0} = K$.

Case 1. *Assume that* $j_0 \geq 0$.

We multiply both sides of inequality (15) for $j = 1$ by c and we rewrite it as

$$\left| cv^5 y_1 - \sum_{4-j_0 < n \leq 4} cv^n + \sum_{0 \leq n < j_0} cuv^n + \sum_{0 \leq n \leq 4 - j_0} (uv^{j_0} - 1)cv^n \right| < 2cuv^{-1}. \tag{31}$$

We write $N_1 := 6 + j_0$, note that $N_1 \leq 8$, and consider the linear forms in N_1 variables $(X_1, \ldots, X_{N_1})$ given by

$$L_{1\infty} := X_1 - \sum_{1<n\leq 1+j_0} X_n + \sum_{1+j_0<n\leq 1+2j_0} X_n + \sum_{1+2j_0<n\leq 6+j_0} (K-1)X_n,$$

$$L_{1w} = X_1, \quad w \in S\backslash\{\infty\}, \tag{32}$$

and $L_{nw} = X_n$ for all $n = 2, \ldots, N_1$ and $w \in S$. Let $\mathbf{x} := (x_1, \ldots, x_{N_1})$ be the vector given by $x_1 := cv^5 y_1$, $x_n = cv^{n+3-j_0}$ for all $n \in \{2, \ldots, 1 + j_0\}$, $x_n := cuv^{n-2-j_0}$ for $n \in \{2+j_0, \ldots, 1+2j_0\}$, and $x_n := cv^{n-2-2j_0}$ for $n \in \{2+2j_0, \ldots, N_1\}$. Note that $\mathbf{x} \in (\mathbf{Z}^*)^{N_1}$. A similar calculation as in Step 1 shows that

$$|L_{1\infty}(\mathbf{x})|_\infty < 2cuv^{-1}, \tag{33}$$

that

$$\prod_{w\in S\backslash\{\infty\}} |L_{1w}(\mathbf{x})|_w \leq v^{-5}, \tag{34}$$

and that

$$\prod_{w\in S} |L_{jw}(\mathbf{x})|_w \leq c \qquad \text{for } j = 2, \ldots, N_1. \tag{35}$$

Multiplying all the above inequalities, we get

$$\prod_{j=1}^{N_1} \prod_{w\in S} |L_{jw}|_w < 2c^{6+j_0} uv^{-6}, \tag{36}$$

and since $v > ac$, $a > c$ and $u < a^2$ we get

$$2c^{6+j_0} uv^{-6} < 2c^{j_0} ua^{-6} < 2c^{j_0} a^{-4} < 2a^{-2}. \tag{37}$$

It is clear that $\max\{|x_i|\ i = 1, \ldots, N_1\} < cv^5 u < a^{13}$, and therefore the above inequalities lead to the conclusion that

$$\prod_{w\in S} |L_{jw}(\mathbf{x})|_w < 2 \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, N_1\}\Big)^{-2/13}. \tag{38}$$

We now note that the inequality

$$2 \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, N_1\}\Big)^{-2/13} < 8^{-1/7} \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, N_1\}\Big)^{-1/7}$$

is satisfied whenever

$$\max\{|x_i| \mid i = 1, \ldots, N_1\} > 2^{7\cdot 13} \cdot 8^{13}$$

and that if the above inequality is not satisfied, then since $a < v < cv^5 y_1 \leq \max\{|x_i| \mid i = 1, \ldots, N_1\}$, we get that $a < 2^{7\cdot 13} \cdot 8^{13} < e^{400}$ and such triples (a, b, c) have already been accounted for in $\mathcal{A}_1$. Thus, we may assume that the inequality

$$\prod_{w\in S} |L_{jw}(\mathbf{x})|_w < 8^{-1/7} \cdot \Big(\max\{|x_i| \mid i = 1, \ldots, N_1\}\Big)^{-1/7} \tag{39}$$

is satisfied. In particular, inequality (11) is satisfied for our system of forms with $\delta = 1/7$ because $N_1 = 6 + j_0 \leq 8$. It is clear that formula (10) holds for our system of linear forms. Moreover, note that since $x_1 = cv^5 y_1 \geq cv^4(u-1) > 9cv^2(u-1)$ (because $v = ac+1 > 3$), and since one of the numbers x_n for $n = 2, \ldots, N_1$ equals c, we get that

$$\mathcal{H}(\mathbf{x}) > 9cv^2(u-1) \cdot c^{-1} > 9v^2(u-1).$$

On the other hand, since the coefficients of our linear forms are integers of absolute value at most $K-1$, we get

$$H(L_{iw}) \leq (N_1(K-1)^2)^{1/2} \leq 8^{1/2}(K-1) := H$$

and

$$H = 8^{1/2}(K-1) < 3uv^2 < 9v^2(u-1) < \mathcal{H}(\mathbf{x}).$$

We are therefore entitled to apply Corollary E1, and deduce that there exist proper subspaces $W_1, \ldots, W_{t_2}$ of $\mathbf{Q}^{N_1}$ and with

$$t_2 \leq (2^{60N_1^2} \cdot 7^{7N_1})^{s+1} \leq (2^{60 \cdot 8^2} \cdot 7^{7 \cdot 8})^{s+1} < \exp(2800(s+1)) \tag{40}$$

and such that $\mathbf{x} \in W_1 \cup \ldots \cup W_{t_2}$. Let W be one of these subspaces. Imposing that $\mathbf{x} \in W$, and simplifying c, it follows that there exists a rational number γ_W and a polynomial $P_W(X,Y) \in \mathbf{Q}[X,Y]$ consisting only of the monomials Y^n for $n = 0, \ldots, 4$ and XY^j for $j = 0, \ldots, j_0 - 1$, not both zero, such that the equation

$$\gamma_W \cdot \frac{v^5(u-1)}{v-1} + P_W(u,v) = 0. \tag{41}$$

The fact that the rational function appearing in the left hand side of (41) is not identically zero can be justified by the argument from [4].

We now look at the solutions (u,v) arising from (41), and which further have the property that $uv^{j_0} = K$. Assume first that $j_0 = 0$. In this case $u = K > 1$ is fixed, $P_W(u,v) = P_W(v)$ does not depend on u, and since not both γ_W and P_W are zero, equation (41) leads to a non-trivial polynomial equation in v of degree at most 5, so that v can take at most 5 values. Assume now that $j_0 > 0$. If $\gamma_W = 0$, then either $\partial P_W/\partial X = 0$, i.e., $P_W(u,v)$ does not depend on u, and then (41) leads to a non-trivial polynomial equation in v of degree at most 4, and hence, v can assume at most 4 values, or $\partial P_W/\partial X \neq 0$, in which case equation (41) gives $u = R(v)$ where $R(v)$ is a non-zero rational function in v whose denominator has degree $\leq j_0 - 1$ and whose numerator has degree ≤ 4. Since $u = K/v^{j_0}$, the equation $R(v) = K/v^{j_0}$ leads to a non-trivial polynomial equation in v of degree $\leq 4 + j_0 \leq 6$, and so v can take at most 6 values in this instance.

Finally, assume that $\gamma_W \neq 0$. In this case, we may assume that $\gamma_W = 1$. Then equation (41) can be rewritten as

$$v^5 u - v^5 + (v-1)P_1(v) + u(v-1)P_2(v) = 0,$$

where $P_2(v)$ is of degree $\le j_0-1$ and $P_1(v)$ is of degree at most 4. Thus,

$$u(v^5+(v-1)P_2(v)) = v^5-(v-1)P_1(v),$$

and the polynomial $v^5+(v-1)P_2(v)$ is non-zero because the degree of $(v-1)P_2(v)$ is at most $j_0\le 2$. Thus, we get the equation

$$\frac{v^5-(v-1)P_1(v)}{v^5+(v-1)P_2(v)} = \frac{K}{v^{j_0}}. \tag{42}$$

If P_2 is non-zero, we may then write $v^5+(v-1)P_2(v)=v^kQ(v)$, where $k\le j_0-1<j_0$ and Q is such that $Q(0)\ne 0$. It is then easy to see (by comparing the orders at which v divides the denominators of the rational functions appearing in the two sides of (42)), that (42) leads to a non-trivial polynomial equation in v of degree at most $5+j_0\le 7$, and therefore v can take at most 7 values. Finally, when $P_2=0$, equation (41) becomes

$$\frac{v^5-(v-1)P_1(v)}{v^5} = \frac{K}{v^{j_0}},$$

which can be rewritten as

$$v^{5-j_0}(v^{j_0}-K) = (v-1)P_2(v), \tag{43}$$

which together with the fact that $K>1$ and $j_0>0$ implies that $v-1$ is coprime to both v^{5-j_0} and to $v^{j_0}-K$ (as polynomials in $\mathbf{Q}[v]$), and therefore equation (43) is a non-trivial polynomial equation in v of degree at most 5, and therefore v can take at most 5 values.
Let therefore $\mathcal{A}_5$ be the set of triples (a,b,c) in $\mathcal{A}\backslash\cup_{k=1}^4\mathcal{A}_k$ for which $j_0\ge 0$. The preceeding argument together with estimates (23), (26) and (41), shows that if we write $\mathcal{B}_5$ for the set of pairs (u,v) arising from triples $(a,b,c)\in\mathcal{A}_5$, then

$$\begin{aligned}|\mathcal{B}_5| &< 2^{18}\cdot 7\cdot \exp(12400(s+1))\cdot\exp(150000(s+1))\cdot\exp(2800(s+1))\\ &< \exp(170000(s+1)).\end{aligned} \tag{44}$$

Case 2. *Assume that* $j_0\in\{-1,-2\}$.
In this case, we replace j_0 by $-j_0$ and we assume again that T, $\mathcal{D}$ and $K:=K_\lambda$ for $\lambda\in\Lambda_T$ are fixed, and that $u/v^{j_0}=K$. It then follows easily that there exist fixed positive integers α and β, which are S-units, and another positive integer ρ, which is also an S-unit, and such that $u=\alpha\rho^{j_0}$ and $v=\beta\rho$. We multiply again both sides of inequality (15) by c, and we rewrite it as

$$\left|cv^5y_1-\sum_{n=0}^{j_0-1}c\beta^n\rho^n+\sum_{n=0}^{4-j_0}(\alpha-\beta^{j_0})c\beta^n\rho^{n+j_0}+\sum_{n=5-j_0}^{4}c\alpha\beta^n\rho^{n+j_0}\right| < 2cuv^{-1}.$$

We let $N_1:=6+j_0$, $K_1:=\alpha-\beta^{j_0}$ and consider the linear forms in N_1 unknowns $X_1,\ldots,X_{N_1}$ given by

$$L_{1\infty}:=X_1-\sum_{n=2}^{j_0+1}X_n+\sum_{n=j_0+2}^{6}K_1X_n+\sum_{n=7}^{6+j_0}X_n,\quad L_{1w}:=X_1,\ w\in S\backslash\{\infty\},$$

and $L_{jw} = X_j$ for $j = 2, \ldots, N_1$ and $w \in S$. Note that $K_1 \neq 0$, for otherwise we get that $u - 1 = (\beta\rho)^{j_0} - 1 = v^{j_0} - 1$ leading either to $u = v$ if $j_0 = 1$, which is impossible, or to $u = v^2$ and $a \leq \gcd(v^2 - 1, v - 1) = v - 1 = u^{1/2} - 1 < u^{1/2}$, which is again impossible. We let $\mathbf{x} := (x_1, \ldots, x_{N_1})$ to be the obvious vector with nonzero integer components given by $x_1 = cv^5y_1$, $x_n = c(\beta\rho)^{n-2}$ when $n \in \{2, \ldots, j_0 + 1\}$, $x_n = c\beta^{n-2-j_0}\rho^{n-2}$ when $n \in \{2 + j_0, 6\}$, and $x_n = c\alpha\beta^{n-2-j_0}\rho^{n-2}$ when $n \in \{7, \ldots, 6 + j_0\}$. Computations similar to the ones employed in the previous case show that inequality (38) holds for our forms, and since we are assuming that $(a, b, c) \notin \mathcal{A}_1$, we get that inequality (39) is satisfied. Moreover, it is clear that we can take

$$H := 8^{1/2}|\alpha - \beta^{j_0}| < 3uv^{j_0} \leq 3uv^2$$

and one checks, like in the previous case, that our vector $\mathbf{x}$ has the property that $\mathcal{H}(\mathbf{x}) > H$. Finally, since (10) is also satisfied, we conclude, as in the previous case, that there exist proper subspaces $W_1', \ldots, W_{t_2'}'$ of $\mathbf{Q}^{N_1}$ and with

$$t_2' \leq (2^{60N_1^2} \cdot 7^{7N_1})^{s+1} \leq (2^{60\cdot 8^2} \cdot 7^{7\cdot 8})^{s+1} < \exp(2800(s+1)), \tag{45}$$

and such that $\mathbf{x} \in W_1' \cup \ldots \cup W_{t_2'}'$. Let W' be one of these subspaces. Imposing that $\mathbf{x} \in W'$, and simplifying c, it follows that there exists a rational number $\gamma_{W'}$ and a polynomial $P_{W'} \in \mathbf{Q}[\rho]$ consisting only of the monomials ρ^n for $n = 0, \ldots, 4 + j_0$, not both zero, such that the equation

$$\gamma_{W'} \cdot \frac{\beta^5\rho^5(\alpha\rho^{j_0} - 1)}{\beta\rho - 1} + P_{W'}(\rho) = 0. \tag{46}$$

The fact that the rational function appearing in the left hand side of (46) is not identically zero is almost clear. Indeed, say if $\gamma_{W'} = 0$, then this is obviously so because $P_{W'}$ is not the zero polynomial, while when $\gamma_{W'} \neq 0$, this follows from the fact that $\beta\rho - 1$ does not divide $\rho^5(\alpha\rho^{j_0} - 1)$ in $\mathbf{Q}[\rho]$, which holds because $\alpha \neq \beta^{j_0}$. Clearly, each one of the equations (46) is a non-trivial polynomial equation in ρ of degree at most $5 + j_0 \leq 7$, and therefore ρ can take at most 7 values.

Thus, if we let $\mathcal{A}_6$ be the set of triples (a, b, c) in $\mathcal{A} \setminus \cup_{k=1}^5 \mathcal{A}_k$ for which our initial value of j_0 was negative, then the preceeding argument together with estimates (23), (26) and (45), shows that if we write $\mathcal{B}_6$ for the set of pairs (u, v) arising from triples $(a, b, c) \in \mathcal{A}_6$, then

$$\begin{aligned} |\mathcal{B}_6| &< 2^{18} \cdot 7 \cdot \exp(12400(s+1)) \cdot \exp(150000(s+1)) \cdot \exp(2800(s+1)) \\ &< \exp(170000(s+1)). \end{aligned} \tag{47}$$

The conclusion is that all pairs (u, v) obtained from all $(a, b, c) \in \mathcal{A}$ form a finite set $\mathcal{B} := \cup_{k=1}^6 \mathcal{B}_k$, whose cardinality is, by (28), (29), (30), (44) and (47), at most

$$\begin{aligned} |\mathcal{B}| &\leq \sum_{k=1}^{6} |\mathcal{B}_k| \\ &< \exp(1200) + 4 \cdot \exp(170000(s+1)) + \exp(340000(s+1)) \end{aligned}$$

$$< 6 \cdot \exp(340000(s+1)) < \exp(341000(s+1)). \tag{48}$$

Step 5. Some Pell equations.

Let B denote the upper bound on $|\mathcal{B}|$ appearing in the right hand side of (48) and let (u,v) be an element of $\mathcal{B}$. Write $D := \gcd(u-1, v-1)$, $b_1 := (u-1)/D$, $c_1 := (v-1)/D$ and $\rho := D/a$. Write $d_1 := b_1 c_1$, and note that d_1 is fixed. It is then clear that ρ is an integer, that $b = b_1\rho$, $c = c_1\rho$, and that $bc + 1 = d_1\rho^2 + 1$. We now finally exploit the fact that $w := bc + 1$ is an S-unit. Write $w := d_2 z^2$, where d_2 is square-free. It is clear that d_2 can be chosen in at most 2^s ways. Fixing d_2, it follows that ρ and z are related via the Pell equation

$$d_2 z^2 - d_1 \rho^2 = 1, \tag{49}$$

and that moreover z is an S-unit. It is now clear that not both d_1 and d_2 can be perfect squares. It is then well-known that all the positive integer solutions (z,ρ) of the above equation have the property that z is a member of a Lucas or a Lehmer sequence. That is, if (z_0, ρ_0) denotes the smallest solution in positive integers of equation (49), and if we write

$$\lambda := \sqrt{d_2} z_0 + \sqrt{d_1}\rho_0 \quad \text{and} \quad \mu := \sqrt{d_2} z_0 - \sqrt{d_1}\rho_0$$

then any solution in positive integers of equation (49) must have

$$z = \frac{\lambda^t + \mu^t}{\lambda + \mu} \cdot z_0 \tag{50}$$

for some odd positive integer t, except when $d_2 = 1$, case in which the same formula holds but with an arbitrary positive integer t not necessarily odd. The set of possible values of z given by (50) forms a *Lehmer* sequence $(z_t)_{t \geq 0}$, where t is allowed to take only odd values if $d_2 > 1$. A result of Morgan Ward [12] says that if $t > 18$, then z_t has *primitive divisors*, i.e., for such t there exists a prime number $p | z_t$ such that p does not divide z_ℓ for any positive integer $\ell < t$. It now follows that if we want that $z = z_t$ is an S-unit, then t can take at most $s + 18$ values. This shows that the triple (u, v, w) can take at most

$$2^s (s+18) B < \exp(s + 18s) \cdot B < \exp(342000(s+1)) \tag{51}$$

values. Finally, note that if the triple (u,v,w) is given, then (a,b,c) is uniquely determined, because $a^2 = \dfrac{(u-1)(v-1)}{w-1}$, and a is positive. Thus,

$$|\mathcal{A}| + 1 < 1 + \exp(342000(s+1)) < \exp(350000(s+1)). \tag{52}$$

We further remark that $s \geq 2$. Indeed, if $s = 1$ and $\mathcal{A}$ is non-empty, it follows there exists a prime number p, positive integers $i > j$, and positive integers $a > b > c$, such that $ab + 1 = u = p^i$, $ac + 1 = v = p^j$. Thus,

$$a \leq \gcd(u-1, v-1) = \gcd(p^i - 1, p^j - 1) = p^{\gcd(i,j)} - 1 \leq p^{i/2} < p^{i/2} = u^{1/2} < a,$$

which is a contradiction. Thus, $s \geq 2$, therefore $s+1 \leq 3s/2$. Hence,

$$|\mathcal{A}|+1 < \exp(350000(s+1)) < \exp(350000 \cdot 3s/2) < \exp(6 \cdot 10^5 s),$$

and therefore

$$s > c_1 \log(|\mathcal{A}|+1),$$

with $c_1 := 6^{-1} \cdot 10^{-5}$, which is a stronger result than what is claimed by our Theorem. The proof of Theorem 1 is therefore complete.

The Proof of Corollary 1.

Since $s \geq 2$ whenever $\mathcal{A}$ is non-empty, we may assume that $P := \max\{P((ab+1)(ac+1)(bc+1)) \mid (a,b,c) \in \mathcal{A}\} \geq 3$. We may therefore assume that $\log(|\mathcal{A}|+1) > 2 \cdot 10^6$, for otherwise the lower bound appearing in the right hand side of inequality (2) is smaller than 3. Let m be the smallest integer larger than or equal to $\frac{1}{6 \cdot 10^5} \cdot \log(|\mathcal{A}|+1)$. Note that since $\log(|\mathcal{A}|+1) > 2 \cdot 10^6$, it follows that $m \geq 4$. Let p_m be the mth prime number. From the above proof of Theorem 1, we know that $s \geq m$, therefore $P \geq p_m > m \log m$, where the last inequality is well-known (see [10], for example). We now show that

$$m \geq \log^{1/15}(|\mathcal{A}|+1).$$

Indeed, this inequality is implied by

$$\frac{1}{6 \cdot 10^5} \cdot \log(|\mathcal{A}|+1) > \log^{1/15}(|\mathcal{A}|+1),$$

which is equivalent to

$$\log(|\mathcal{A}|+1) > (6 \cdot 10^5)^{15/14},$$

and this last inequality is satisfied when $\log(|\mathcal{A}|+1) > 2 \cdot 10^6$. Thus,

$$P > m \log m > \frac{1}{6 \cdot 10^5} \cdot \log(|\mathcal{A}|+1) \cdot \log\Big(\frac{\log(|\mathcal{A}|+1)}{6 \cdot 10^5}\Big)$$

$$> \frac{1}{6 \cdot 15} \cdot \frac{1}{10^5} \cdot \log(|\mathcal{A}|+1) \cdot \log\log(|\mathcal{A}|+1) > c_2 \cdot \log(|\mathcal{A}|+1) \cdot \log\log(|\mathcal{A}|+1),$$

with $c_2 := 9^{-1} \cdot 10^{-6}$, which is a stronger inequality than the one asserted by Corollary 1. The proof of Corollary 1 is therefore complete.

§5 Other quantitative aspects

As we have mentioned in the Introduction, it is shown in [4] that $P((ab+1)(ac+1))$ tends to infinity over all the triples of distinct positive integers (a,b,c) with $a > b > c$. One could ask whether there exists a quantitative lower bound for the number of distinct prime factors of the expressions $(ab+1)(ac+1)$, where (a,b,c) is a triple of distinct positive integers with

$a > b > c$ ranging in a finite set $\mathcal{A}$ of such. More precisely, one can address the following question:

Question 1. *Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ with $\lim_{n\to\infty} f(n) = \infty$, and such that if $\mathcal{A}$ is any non-empty set of triples of distinct positive integers (a, b, c) with $a > b > c$, then the inequality*

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)\Big) > f(|\mathcal{A}|) \tag{53}$$

holds?

The answer to above question is no. In order to show this, we recall a result on the distribution of primes in arithmetic progressions. In what follows, for positive integers $1 \le a < d$ with $\gcd(a, d) = 1$ and for a large positive real number x we write $\pi(x; d, a)$ for the number of prime numbers $p \le x$ with $p \equiv a \pmod d$. We also write $\pi(x)$ for the number of prime numbers $p \le x$. The following Theorem on the distribution of primes in arithmetic progressions with large moduli follows from Theorem 9 in [2] by partial integration.

Theorem BFI. *For any positive constant B and any $\varepsilon > 0$, there exists a positive constant $C := C(B)$ depending on B such that if x is a large real number, and Q and R are positive integers with $R < x^{1/10-\varepsilon}$ and $QR < x/\log^C x$, then*

$$\sum_{r=1}^{R}\Big|\sum_{q=1}^{Q}\Big(\pi(x, qr, 1) - \frac{\pi(x)}{\phi(qr)}\Big)\Big| \ll \frac{x}{\log^B x}. \tag{54}$$

We let $c_3 > 1$ to be any fixed constant, we let x be a large positive real number, we put $z := c_3 \log\log x$, and $R := \prod_{p\le z} p$. We note that by the Prime Number Theorem, we have that the inequality

$$R = \exp(c_3(1 + o(1)) \log\log x) < \log^{2c_3} x \tag{55}$$

holds for large values of x. In particular, the inequality $R < x^{1/10-\varepsilon}$ holds say with $\varepsilon := 1/20$ when $x > x(c_3)$.

We let $B := 2c_3$, and $C := C(B)$, and since $x^{3/4}R < x^{3/4}\log^{2c_3} x < x/\log^C x$ holds for for sufficiently large values of x, it follows that we are entitled to apply Theorem BFI above twice, once with $Q = Q_1 := \lfloor x^{2/3}\rfloor$ and once with $Q = Q_2 := \lfloor x^{3/4}\rfloor$, and use the absolute value inequality, to conclude that the estimate

$$\Big|\sum_{Q_1<q\le Q_2}\Big(\pi(x; qR, 1) - \frac{\pi(x)}{\phi(qR)}\Big)\Big| \ll \frac{x}{\log^B x} \tag{56}$$

holds. We now show that there exists $q \in [Q_1, Q_2]$ such that $\pi(x; qR, 1) \ge 2$. Assume that this is not so. In this case, $\pi(x; qR, 1) \le 1$ holds for all $q \in [Q_1, Q_2]$, and therefore

$$\Big|\sum_{Q_1<q\le Q_2}\Big(\frac{\pi(x)}{\phi(qR)} - \pi(x; qR, 1)\Big)\Big| \ge \sum_{Q_1<q\le Q_2}\frac{\pi(x)}{qR} - Q_2 > \frac{\pi(x)}{R}\sum_{Q_1<q\le Q_2}\frac{1}{q} - x^{3/4}. \tag{57}$$

Clearly, the estimate

$$\sum_{Q_1<q\le Q_2}\frac{1}{q}=\log\Big(\frac{Q_2}{Q_1}\Big)+o(1)=\frac{1}{12}\cdot\log x+o(1)>c_4\log x$$

holds for large values of x, where one can take $c_4:=1/13$, and since $\pi(x)>x/\log x$ holds for all $x>17$ (see [10]), the above inequality together with (55) implies that

$$\frac{\pi(x)}{R}\sum_{Q_1<q\le Q_2}\frac{1}{q}\ge c_4\cdot\frac{x}{\log^{2c_3-1}x}.$$

With (57), we get that the inequality

$$\Big|\sum_{Q_1<q\le Q_2}\Big(\frac{\pi(x)}{\phi(qR)}-\pi(x;qR,1)\Big)\Big|\ge c_4\cdot\frac{x}{\log^{2c_3-1}x}-x^{3/4}\ge c_5\cdot\frac{x}{\log^{2c_3-1}x}\qquad(58)$$

holds for large values of x, where one can take $c_5:=1/14$, but inequality (58) contradicts inequality (56) for large values of x.

Thus, we have shown that there exists $q\in[Q_1,Q_2]$ such that $\pi(x;qR,1)\ge 2$. Let $v<u\le x$ be two prime numbers which are congruent to 1 modulo qR. For every divisor d of R we let $a:=qR/d$, $b:=(u-1)/a$ and $c:=(v-1)/a$. Note that $a\ge q\ge\lfloor x^{2/3}\rfloor>x^{1/2}\ge u^{1/2}$, and therefore $a>b>c$. Let $\mathcal{A}$ be the set of all the above triples. It is clear that

$$|\mathcal{A}|=\tau(R)=2^{\pi(z)}>\exp\Big(c_6\cdot\frac{\log\log x}{\log\log\log x}\Big),\qquad(59)$$

where the constant c_6 can be taken to be any positive constant smaller than $c_3\log 2$, and the above inequality holds for sufficiently large values of x. However, note that $(ab+1)(ac+1)=uv$ holds for all triples (a,b,c) of $\mathcal{A}$. Thus,

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}}(ab+1)(ac+1)\Big)=2,\qquad(60)$$

and now inequalities (59) and (60) show that the answer to the above Question 1 is indeed negative.

Győry & Sárközy [7] raised also the question of finding examples of finite sets $\mathcal{A}$ of triples of distinct positive integers (a,b,c) such that the quantity

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}}(ab+1)(ac+1)(bc+1)\Big)$$

is small with respect to $|\mathcal{A}|$. The trivial construction obtained by letting $\mathcal{A}$ be the set of all triples of distinct positive integers (a,b,c) with $\max\{a,b,c\}<x$ shows that the inequality

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}}(ab+1)(ac+1)(bc+1)\Big)\ll\frac{|\mathcal{A}|^{2/3}}{\log|\mathcal{A}|}\qquad(61)$$

holds for infinitely many finite sets $\mathcal{A}$ whose cardinalities tend to infinity. Our next result improves upon the above estimate.

Proposition 1. *Let $\varepsilon > 0$ be any fixed positive real number. There are infinitely many finite sets $\mathcal{A}$ of triples (a, b, c) of distinct positive integers whose cardinalities tend to infinity and such that for each one of these sets the inequality*

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)(bc+1)\Big) \ll |\mathcal{A}|^{1/2+\varepsilon} \tag{62}$$

is satisfied. The constant understood in $\ll$ above depends at most on ε.

For the proof of Proposition 1 above, we need a result concerning the distribution of smooth numbers in arithmetic progressions. Let x be a large positive real number. For any positive integer $y \le x$, we write $\Psi(x, y)$ for the number of positive integers $n \le x$ with $P(n) \le y$. For positive integers $1 \le r < q$ with $\gcd(r, q) = 1$ we write $\Psi(x, y; q, r)$ for the number of numbers $n \le x$ with $P(n) \le y$ and such that $n \equiv r \pmod q$. The following result is due to Balog & Pomerance [1].

Theorem BP. *Let $\varepsilon > 0$ be an arbitrarily small positive real number. The estimate*

$$\Psi(x, y; q, r) = \frac{x}{q} \cdot (w \log(w + 1))^{-w} \cdot e^{O(w)} \tag{63}$$

holds uniformly under the conditions

$$x \ge 2,\ \exp(\log\log x)^2) \le y \le x^{2/3-\varepsilon},\ 1 \le q \le (\min\{y, x/y\})^{4/3-\varepsilon},\ \gcd(r, q) = 1, \tag{64}$$

where $w := \log x/\log y$.

We let $\varepsilon > 0$ be a sufficiently small positive real number, x be a large positive real number, and we put $I := [x^{2/3-\varepsilon/2}/2, x^{2/3-\varepsilon/2}]$. We also let $r := 1$, q to be an arbitrary integer in I, and $y := x^{1/2}$. It is clear that the inequality

$$\exp((\log\log x)^2) \le y \le x^{2/3-\varepsilon/2}$$

is satisfied if x is sufficiently large and $\varepsilon < 1/12$. Note also that

$$(\min\{y, x/y\})^{4/3-\varepsilon/2} = x^{2/3-\varepsilon/4} > q$$

holds for all $q \in I$. Thus, all conditions (64) are satisfied, and by (11) with $w = \log x/\log y = 2$, we get that

$$\Psi(x, y; q, 1) \gg \frac{x}{q} \gg x^{1/3+\varepsilon/2}. \tag{65}$$

We now take $\mathcal{A}$ to be the set of all triples (a, b, c), where $a := q \in I$, $b := (u-1)/a$, $c := (v-1)/a$, where $v < u \le x$ are positive integers with $P(uv) \le y$ both in the arithmetic progression $1 \pmod q$. We observe that since $q > x^{2/3-\varepsilon/2}/2 > x^{1/2}$ holds whenever $\varepsilon < 1/12$ and x is sufficiently large, if follows that all such triples are distinct. Thus,

$$|\mathcal{A}| \ge |I \cap \mathbf{N}| \cdot \binom{\Psi(x, y; q, 1)}{2} \gg |I \cap \mathbf{N}| \cdot \Psi(x, y; q, 1)^2 \gg x^{4/3+\varepsilon/2}. \tag{66}$$

We now note that

$$bc+1 \le \left(2x^{1/3+\varepsilon/2}\right)^2 + 1 \le 5x^{2/3+\varepsilon},$$

and since $P(uv) \le x^{1/2}$, it follows that for large x we have

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)(bc+1)\Big) \le \pi(5x^{2/3+\varepsilon}) \ll \frac{x^{2/3+\varepsilon}}{\log x}. \tag{67}$$

Finally, note that inequality (66) implies that

$$|\mathcal{A}|^{1/2+\varepsilon} \gg x^{(4/3+\varepsilon/2)(1/2+\varepsilon)} > x^{2/3+\varepsilon},$$

which together with (67) shows that the inequality

$$\omega\Big(\prod_{(a,b,c)\in\mathcal{A}} (ab+1)(ac+1)(bc+1)\Big) < |\mathcal{A}|^{1/2+\varepsilon}$$

holds for our sets $\mathcal{A}$ and large values of x, which completes the proof of Proposition 1.

Acknowledgements

This work was completed while the second author visited the University Louis Pasteur in Strasbourg, France in July 2003. He thanks the people of the Mathematical Institute for their warm hospitality. Both authors were supported by the joint project ECOS Nord – CONACyT, France-Mexico M02-M01.

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