Combinatorics, Automata and Number Theory

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Transcendence and Diophantine approximation

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The aim of this chapter is to present several number-theoretic problems that reveal a fruitful interplay between combinatorics on words and Diophantine approximation. Finite and infinite words occur naturally in Diophantine approximation when we consider the expansion of a real number in an integer base b or its continued fraction expansion. Conversely, with an infinite word a on the finite alphabet $\{0, 1, \ldots, b-1\}$ we associate the real number ξ_a whose base-b expansion is given by a. As well, with an infinite word a on the infinite alphabet $\{1, 2, 3, \ldots\}$, we associate the real number ζ_a whose continued fraction expansion is given by a. It turns out that, if the word a enjoys certain combinatorial properties involving repetitive or symmetric patterns, then this gives interesting information on the arithmetical nature and on the Diophantine properties of the real numbers ξ_a and ζ_a.

We illustrate our results by considering the real numbers associated with two classical infinite words, the *Thue-Morse word* and the *Fibonacci word*, see Example 1.2.21 and 1.2.22. There are several ways to define them. Here, we emphasize the fact that they are fixed points of morphisms.

Consider the morphism σ defined on the set of words on the alphabet $\{0, 1\}$ by $\sigma(0) = 01$ and $\sigma(1) = 0$. Then, we have $\sigma^2(0) = 010, \sigma^3(0) = 01001, \sigma^4(0) = 01001010$, and the sequence $(\sigma^k(0))_{k\geq 0}$ converges to the Fibonacci word

$$f = 010010100100101001010\cdots . \tag{8.1}$$

Consider the morphism τ defined over the same alphabet by $\tau(0) = 01$ and $\tau(1) = 10$. Then, we have $\tau^2(0) = 0110, \tau^3(0) = 01101001$, and the sequence $(\tau^k(0))_{k\geq 0}$ converges to the Thue-Morse word

$$t = 011010011001011010010\cdots . \tag{8.2}$$

For every $n \geq 1$, we denote by f_n the nth letter of f and by t_n the nth letter of t. For an integer $b \geq 2$, we set

$$\xi_f = \sum_{n \geq 1} \frac{f_n}{b^n}$$

and

$$\xi_t = \sum_{n \geq 1} \frac{t_n}{b^n} \ .$$

We further define Fibonacci and Thue-Morse continued fractions, but, since 0 cannot be a partial quotient, we have to write our words on another alphabet than $\{0, 1\}$. We take two distinct positive integers a and b, set $f'_n = a$ if $f_n = 0$ and $f'_n = b$ otherwise, and $t'_n = a$ if $t_n = 0$ and $t'_n = b$ otherwise. Then, we define

$$\zeta_{f'} = [a, b, a, a, b, a, b, a, \ldots] = [f'_1, f'_2, f'_3, f'_4, \ldots]$$

and

$$\zeta_{t'} = [a, b, b, a, b, a, a, b, \ldots] = [t'_1, t'_2, t'_3, t'_4, \ldots] \ .$$

Among other results, we will explain how to combine combinatorial properties of the Fibonacci and Thue-Morse words with the Thue-Siegel-Roth-Schmidt method to prove that all these numbers are transcendental. Beyond transcendence, we will show that the Fibonacci continued fractions satisfy a spectacular properties regarding a classical problem in Diophantine approximation: the existence of real numbers ξ with the property that ξ and ξ^2 are uniformly simultaneously very well approximable by rational numbers of the same denominator. We will also describe an *ad hoc* construction to obtain explicit examples of pairs of real numbers that satisfy the Littlewood conjecture, which is a major open problem in simultaneous Diophantine approximation.

We use the following convention throughout this Chapter. The Greek letter ξ stands for a real number given by its base-b expansion, where b always means an integer at least equal to 2. The Greek letter ζ stands for a real number given by its continued fraction expansion. If its partial quotients take only two different values, these are denoted by a and b, which represent distinct positive integers (here, b is not assumed to be at least 2).

The results presented in this chapter are not the most general statements that can be established by the methods described here. Our goal is not to make an exhaustive review of the state-of-the-art, but rather to emphasize the ideas used. The interested reader is directed to the original papers.

8.1 The expansion of algebraic numbers in an integer base

Throughout the present section, b always denotes an integer at least equal to 2 and ξ is a real number with $0 < \xi < 1$. Recall that there exists a unique infinite word $a = a_1 a_2 \cdots$ defined over the finite set $\{0, 1, \ldots, b-1\}$, called the *base-b expansion* of ξ, such that

$$\xi = \sum_{n \geq 1} \frac{a_n}{b^n} := 0.a_1 a_2 \cdots , \tag{8.3}$$

with the additional condition that a does not terminate in an infinite string of the digit $b - 1$. Obviously, a depends on ξ and b, but we choose not to indicate this dependence.

For instance, in base 10, we have

$$3/7 = 0.(428\,571)^{\omega}$$

and

$$\pi - 3 = 0.314\,159\,265\,358\,979\,323\,846\,264\,338\,327 \cdots .$$

Conversely, if $a = a_1 a_2 \cdots$ is an infinite word defined over the finite alphabet $\{0, 1, \ldots, b-1\}$ such that a does not terminate in an infinite string of the digit $b - 1$, there exists a unique real number, denoted by ξ_a, such that

$$\xi_a := 0.a_1 a_2 \cdots .$$

This notation does not indicate in which base ξ_a is written. However, this will be clear from the context and should not cause any difficulty.

In the sequel, we will also sometimes make a slight abuse of notation and, given an infinite word a defined over the finite alphabet $\{0, 1, \ldots, b-1\}$ that could end in an infinite string of $b - 1$, we will write

$$0.a_1 a_2 \cdots$$

to denote the infinite sum

$$\sum_{n \geq 1} \frac{a_n}{b^n} .$$

We recall the following fundamental result that can be found in the classical textbook (Hardy and Wright 1985).

Theorem 8.1.1 *A real number is rational if, and only if, its base-b expansion is eventually periodic.*

8.1.1 Normal numbers and algebraic numbers

At the beginning of the 20th century, Émile Borel (Borel 1909) investigated the following question:

How does the decimal expansion of a randomly chosen real number look like?

This question leads to the notion of normality.

Definition 8.1.2 A real number ξ is called *normal to base* b if, for any positive integer n, each one of the b^n words of length n on the alphabet $\{0, 1, \ldots, b-1\}$ occurs in the base-b expansion of ξ with the same frequency $1/b^n$. A real number is called a *normal number* if it is normal to every integer base.

É. Borel (Borel 1909) proved the following fundamental result regarding normal numbers.

Theorem 8.1.3 *The set of normal numbers has full Lebesgue measure.*

Some explicit examples of real numbers that are normal to a given base are known for a long time. For instance, the number

$$0.123\,456\,789\,101\,112\,131\,415\cdots, \tag{8.4}$$

whose sequence of digits is the concatenation of the sequence of all positive integers written in base 10 and ranged in increasing order, was proved to be normal to base 10 in 1933 by D. G. Champernowne (Champernowne 1933).

In contrast, to decide whether a specific number, like e, π or

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\cdots\ ,$$

is or is not a normal number remains a challenging open problem. In this direction, the following conjecture is widely believed to be true.

Conjecture 8.1.4 Every real irrational algebraic number is a normal number.

8.1.2 Complexity of real numbers

Conjecture 8.1.4 is reputed to be out of reach. We will thus focus our attention to simpler questions. A natural way to measure the *complexity* of the real number ξ (with respect to the base b) is to count the number of distinct blocks of given length in the infinite word a defined in (8.3). For

$n \geq 1$, we set $p(n, \xi, b) = p_a(n)$ with a as above and where p_a denotes the complexity function of a. Then, we have

$$1 \leq p(n, \xi, b) \leq b^n ,$$

and both inequalities are sharp (take *e.g.*, the analogue in base b of the number given in (8.4) to show that the right-hand inequality is sharp). A weaker conjecture than Conjecture 8.1.4 reads then as follows.

Conjecture 8.1.5 For every real irrational algebraic number ξ, every positive integer n and every base b, we have $p(n, \xi, b) = b^n$.

Our aim is to prove the following lower bound for the complexity of algebraic irrational numbers, as in (Adamczewski and Bugeaud 2007a), (Adamczewski, Bugeaud, and Luca 2004).

Theorem 8.1.6 *If ξ is an algebraic irrational number, then*

$$\lim_{n \to +\infty} \frac{p(n, \xi, b)}{n} = +\infty .$$

Up to now, Theorem 8.1.6 is the main achievement regarding Conjecture 8.1.5. A notable consequence of this result is the confirmation of a conjecture suggested by A. Cobham in 1968 (Cobham 1968).

Theorem 8.1.7 *The base-b expansion of an algebraic irrational number cannot be generated by a* finite automaton.

Indeed, a classical result of A. Cobham (Cobham 1972) asserts that every infinite word a that can be generated by a finite automaton has a complexity function p_a satisfying $p_a(n) = O(n)$.

Nevertheless, we are still very far away from what is expected, and we are still unable to confirm the following conjecture.

Conjecture 8.1.8 For every algebraic irrational number ξ and every base b with $b \geq 3$, we have $p(1, \xi, b) \geq 3$.

8.1.3 Transcendence and Diophantine approximation: an introduction

In order to prove Theorem 8.1.6 , we have to show that irrational real numbers whose base-b expansion has a too low complexity are transcendental.

Throughout this section, we recall several classical results concerning the rational approximation of algebraic irrational real numbers (Liouville's inequality, Roth's theorem, Ridout's theorem) and derive from them several combinatorial transcendence criteria concerning real numbers defined by their base-b expansion. We also apply them to concrete examples.

8.1.3.1 Liouville's inequality

In 1844, J. Liouville (Liouville 1844) was the first to prove that transcendental numbers do exist. Moreover, he constructed explicit examples of such numbers. The numbers $\mathcal{L}_b$ below are usually considered as the first examples of transcendental numbers. This is however not entirely true, since the main part of Liouville's paper is devoted to the construction of transcendental continued fractions.

Theorem 8.1.9 *For every integer $b \geq 2$, the real number*

$$\mathcal{L}_b := \sum_{n=1}^{+\infty} \frac{1}{b^{n!}}$$

is transcendental.

The proof of Theorem 8.1.9 relies on the famous Liouville's inequality recalled below.

Proposition 8.1.10 *Let ξ be an algebraic number of degree $d \geq 2$. Then, there exists a positive real number c_ξ such that*

$$\left| \xi - \frac{p}{q} \right| \geq \frac{c_\xi}{q^d}$$

for every rational number p/q with $q \geq 1$.

Proof let P denote the minimal defining polynomial of ξ and set

$$c_\xi = 1/(1 + \max_{|\xi - x| < 1} |P'(x)|) \ .$$

If $|\xi - p/q| \geq 1$, then our choice of c_ξ ensures that $|\xi - p/q| \geq c_\xi / q^d$.

Let us now assume that $|\xi - p/q| < 1$. Since P is the minimal polynomial of ξ, it does not vanish at p/q and $q^d P(p/q)$ is a non-zero integer. Consequently,

$$|P(p/q)| \geq \frac{1}{q^d} \cdot$$

Since $|\xi - p/q| < 1$, Rolle's theorem implies the existence of a real number t in $[p/q - 1, p/q + 1]$ such that

$$|P(p/q)| = |P(\xi) - P(p/q)| = \left| \xi - \frac{p}{q} \right| \times |P'(t)| \ .$$

We thus have

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c_\xi}{q^d} ,$$

which ends the proof. □

Proof [Proof of Theorem 8.1.9] Let $d \geq 2$ be an integer. Let j be a positive integer with $j \geq d$ and set

$$\frac{p_j}{b^{j!}} := \sum_{n=1}^{j} \frac{1}{b^{n!}} .$$

Observe that

$$\left|\mathcal{L}_b - \frac{p_j}{b^{j!}}\right| = \sum_{n>j} \frac{1}{b^{n!}} < \frac{2}{b^{(j+1)!}} < \frac{1}{(b^{j!})^d} .$$

It then follows from Proposition 8.1.10 that $\mathcal{L}_b$ cannot be algebraic of degree less than d. Since d is arbitrary, $\mathcal{L}_b$ is transcendental. □

8.1.3.2 Roth's theorem

The following famous improvement of Liouville's inequality was established by K. F. Roth (Roth 1955).

Theorem 8.1.11 *Let ξ be a real algebraic number and ε be a positive real number. Then, there are only a finite number of rationals p/q such that $q \geq 1$ and*

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}} .$$

We give a first application of Roth's theorem to transcendence.

Corollary 8.1.12 *For any integer $b \geq 2$, the real number*

$$\sum_{n=1}^{+\infty} \frac{1}{b^{3^n}}$$

is transcendental.

Proof Use the same argument as in the proof of Theorem 8.1.9. □

Actually, the same proof shows that the real number

$$\sum_{n=1}^{+\infty} \frac{1}{b^{\lfloor d^n \rfloor}}$$

is transcendental for any real number $d > 2$, but gives no information on the arithmetical nature of the real number

$$\sum_{n=1}^{+\infty} \frac{1}{b^{2^n}} \ ,$$

which will be considered in Corollary 8.1.18 below.

Up to now, we have just truncated the infinite series to construct our very good rational approximants, taking advantage of the existence of very long strings of 0 in the base-b expansion of the real numbers involved. We describe below a (slightly) more involved consequence of Roth's theorem. Roughly speaking, instead of just truncating, we truncate and complete by periodicity.

We consider the Fibonacci word f defined at the beginning of this chapter. Unlike the base-b expansions of the number $\mathcal{L}_b$ (defined in Theorem 8.1.9) and of the number defined in Corollary 8.1.12, the word f contains no occurrence of more than two consecutive 0's. However, its combinatorial structure can be used to reveal more hidden good rational approximations by mean of which we will derive the following result.

Theorem 8.1.13 *The real number*

$$\xi_f := \sum_{n\geq 1} \frac{f_n}{b^n} = 0.010\,010\,100\,100\,101\,001\,010\cdots \ .$$

is transcendental.

Before proving Theorem 8.1.13, we need the following result. Let $(F_j)_{j\geq 0}$ denote the Fibonacci sequence, that is, the sequence starting with $F_0 = 0$, $F_1 = 1$, and satisfying the recurrence relation $F_{j+2} = F_{j+1} + F_j$, for every $j \geq 0$.

Lemma 8.1.14 *For every integer $j \geq 4$, the finite word*

$$f_1 f_2 \dots f_{F_j} f_1 f_2 \dots f_{F_j} f_1 f_2 \dots f_{F_{j-1}-2}$$

is a prefix of f.

Proof For every integer $j \geq 2$, set $w_j := f_1 f_2 \dots f_{F_j}$. Then we recall the following fundamental relation:

$$w_{j+1} = w_j w_{j-1}, \tag{8.5}$$

valid for $j \geq 3$. If a finite word $u := u_1 u_2 \cdots u_r$ has length larger than 2, we set $h(u) := u_1 u_2 \cdots u_{r-2}$.

We now prove by induction that, for every integer $j \geq 3$,

$$h(w_j w_{j-1}) = h(w_{j-1} w_j). \tag{8.6}$$

For $j = 3$ the result follows from the two obvious equalities $w_3 w_2 = aba$ and $w_2 w_3 = aab$. Let us assume now that Equality (8.6) holds for an integer $j \geq 3$. We infer from (8.5) that

$$h(w_{j+1} w_j) = h(w_j w_{j-1} w_j) = w_j h(w_{j-1} w_j) \ .$$

By assumption, we get that

$$h(w_{j-1} w_j) = h(w_j w_{j-1}) \ .$$

Using again Equation (8.5) we obtain that

$$h(w_{j+1} w_j) = w_j h(w_j w_{j-1}) = w_j h(w_{j+1}) = h(w_j w_{j+1}) \ ,$$

as claimed.

We now end the proof of the lemma. Let $j \geq 4$ and note that by definition $h(w_{j+2})$ is a prefix of f. On the other hand, we infer from (8.5) and (8.6) that

$$h(w_{j+2}) = h(w_{j+1} w_j) = h(w_j w_{j-1} w_j) = h(w_j w_j w_{j-1}) \ ,$$

which ends the proof, since w_{j-1} has at least two letters. □

We are now ready to prove Theorem 8.1.13.

Proof For every integer $j \geq 4$, let us consider the rational number ρ_j defined by

$$\rho_j := 0.(f_1 f_2 \dots f_{F_j})^\omega \ .$$

Thus,

$$\begin{aligned} \rho_j &= \frac{f_1}{b} + \frac{f_2}{b^2} + \cdots + \frac{f_{F_j}}{b^{F_j}} + \frac{f_1}{b^{F_j+1}} + \frac{f_2}{b^{F_j+2}} + \cdots + \frac{f_{F_j}}{b^{2F_j}} + \cdots \\ &= \left(\frac{f_1}{b} + \frac{f_2}{b^2} + \cdots + \frac{f_{F_j}}{b^{F_j}} \right) \frac{b^{F_j}}{b^{F_j} - 1} \end{aligned}$$

and there exists an integer p_j such that

$$\rho_j = \frac{p_j}{b^{F_j} - 1} \ .$$

Now, Lemma 8.1.14 tells us that ρ_j is a very good approximation to ξ_f. Indeed, the first $F_{j+2} - 2$ digits in the base-b expansion of ξ_f and of ρ_j are the same. Furthermore, ρ_j is distinct from ξ_f since the latter is irrational as a consequence of Theorem 8.1.1 and the fact that f is aperiodic. We thus obtain that

$$0 < |\xi_f - \rho_j| < \sum_{h \geq -1} b^{-F_{j+2}-h} \leq b^{-F_{j+2}+2}. \qquad (8.7)$$

On the other hand, an easy induction shows that $F_{j+2} \geq 1.5\, F_{j+1}$ for every $j \geq 2$. Consequently, we infer from (8.7) that

$$0 < \left| \xi_f - \frac{p_j}{b^{F_j} - 1} \right| < \frac{b^2}{(b^{F_j} - 1)^{2.25}} \ ,$$

and it follows from Roth's theorem that ξ_f is transcendental. □

8.1.3.3 Repetitions in infinite words

The key observation in the previous proof is that the infinite word f begins in infinitely many 'more-than-squares'. Let us now formalise what has been done above. For any positive integer ℓ, we write u^ℓ for the word

$$\underbrace{u \cdots u}_{\ell \text{ times}}$$

(ℓ times repeated concatenation of the word u). More generally, for any positive real number x, we denote by u^x the word $u^{\lfloor x \rfloor} u'$, where u' is the prefix of u of length $\lceil (x - \lfloor x \rfloor)|U| \rceil$. We recall that $\lfloor y \rfloor$ and $\lceil y \rceil$ denote the floor and ceiling functions. A *repetition* of the form u^x, with $x > 2$, is called an *overlap*. A repetition of the form u^x, with $x > 1$, is called a *stammering*. For instance the word $ababab = (ab)^3$ is a cube that contains the overlap $ababa = (ab)^{2+1/2}$. The word $1234567891 = (123456789)^{1+1/9}$ is a stammering which is overlap-free. For more on repetitions, see also Section 11.2.2.

Definition 8.1.15 Let $w > 1$ and $c \geq 0$ be real numbers. We say that an infinite word a defined over a finite or an infinite alphabet satisfies Condition $(*)_{w,c}$ if a is not eventually periodic and if there exist two sequences of finite words $(u_j)_{j\geq 1}$, $(v_j)_{j \geq 1}$ such that:

(i) For every $j \geq 1$, the word $u_j v_j^w$ is a prefix of a,

(ii) The sequence $(|u_j|/|v_j|)_{j\geq 1}$ is bounded from above by c,

(iii) The sequence $(|v_j|)_{j \geq 1}$ is strictly increasing.

Let a be an infinite word defined over the alphabet $\{0, 1, \ldots, b-1\}$ and satisfying Condition $(*)_{w,c}$ for some w and c. By definition of Condition

$(*)_{w,c}$, we stress that

$$\xi_a = 0.u_j \underbrace{v_j \dots v_j}_{\lfloor w \rfloor \text{ times}} v'_j \cdots ,$$

for every $j \geq 1$, where v'_j is the prefix of v_j of length $\lceil (w - \lfloor w \rfloor)|v_j| \rceil$.

We derived above the transcendence of ξ_f by proving that the Fibonacci word f satisfies Condition $(*)_{2.25,0}$. Actually, the proof of Theorem 8.1.13 leads to the following combinatorial transcendence criterion.

Proposition 8.1.16 *If an infinite word a defined over the finite alphabet* $\{0, 1, \dots, b-1\}$ *satisfies Condition* $(*)_{w,0}$ *for some* $w > 2$*, then the real number* ξ_a *is transcendental.*

We leave the proof to the reader.

8.1.3.4 A p-adic Roth theorem

A disadvantage of the use of Roth's theorem in this context is that we need, in order to apply Proposition 8.1.16, that the repetitions occur at the very beginning (otherwise, we would have to assume that w is much larger than 2). We present here an idea of S. Ferenczi and C. Mauduit (Ferenczi and Mauduit 1997) to get a stronger transcendence criterion. It relies on the following non-Archimedean extension of Roth's theorem proved by D. Ridout (Ridout 1957).

In the sequel of this chapter, for every prime number ℓ, the ℓ-adic absolute value $|\cdot|_\ell$ is normalised such that $|\ell|_\ell = \ell^{-1}$.

Theorem 8.1.17 *Let* ξ *be an algebraic number and* ε *be a positive real number. Let* S *be a finite set of distinct prime numbers. Then there are only a finite number of rationals* p/q *such that* $q \geq 1$ *and*

$$\left(\prod_{\ell \in S} |p|_\ell \cdot |q|_\ell \right) \cdot \left| \xi - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}} .$$

We point out a first consequence of Ridout's theorem.

Corollary 8.1.18 *For every integer* $b \geq 2$*, the real number*

$$\mathcal{K}_b := \sum_{n=1}^{+\infty} \frac{1}{b^{2^n}}$$

is transcendental.

Proof Let j be a positive integer and set

$$\rho_j := \sum_{n=1}^{j} \frac{1}{b^{2^n}} .$$

There exists an integer p_j such that $\rho_j = p_j/q_j$ with $q_j = b^{2^j}$. Observe that

$$|\mathcal{K}_b - \rho_j| = \sum_{n>j} \frac{1}{b^{2^n}} < \frac{2}{b^{2^{j+1}}} = \frac{2}{(q_j)^2} ,$$

and let S be the set of prime divisors of b. Then, an easy computation gives that

$$\left(\prod_{\ell \in S} |q_j|_\ell \cdot |p_j|_\ell \right) \cdot |\mathcal{K}_b - p_j/q_j| < \frac{2}{(q_j)^3} ,$$

and Theorem 8.1.17 implies that $\mathcal{K}_b$ is transcendental. □

As shown in (Ferenczi and Mauduit 1997), Ridout's theorem yields the following improvement of Proposition 8.1.16.

Proposition 8.1.19 *If an infinite word* a *defined over* $\{0, 1, \ldots, b-1\}$ *satisfies Condition* $(*)_{w,c}$ *for some* $w > 2$ *and some* $c \geq 0$*, then the associated real number* ξ_a *is transcendental.*

An interesting consequence of this combinatorial transcendence criterion, pointed out in (Ferenczi and Mauduit 1997), is that every real number with a *Sturmian* base-b expansion is transcendental. Such a result cannot be proved by using Proposition 8.1.16.

Proof Let a be an infinite word defined over $\{0, 1, \ldots, b-1\}$ and satisfying Condition $(*)_{w,c}$ for some $w > 2$ and some $c \geq 0$. Then, for every $j \geq 1$, the real number

$$\xi_a = 0.u_j \underbrace{v_j \cdots v_j}_{\lfloor w \rfloor \text{ times}} v'_j \cdots ,$$

where v'_j is the prefix of v_j of length $\lceil (w - \lfloor w \rfloor)|v_j| \rceil$, is very close to the rational number

$$0.u_j(v_j)^\omega ,$$

obtained from ξ_a by truncating its base-b expansion and completing by periodicity. Precisely, as shown by an easy computation, there exist integers p_j, r_j and s_j such that

$$\left| \xi - \frac{p_j}{b^{r_j}(b^{s_j} - 1)} \right| < \frac{2}{b^{r_j + ws_j}}, \tag{8.8}$$

where $r_j = |u_j|$ and $s_j = |v_j|$.

Note that the rational approximations to ξ_a that we have obtained are very specific: their denominators have a possibly very large part composed of a finite number of fixed prime numbers (namely, the prime divisors of b). More precisely, if S denotes the set of prime divisors of b, we have

$$\prod_{\ell \in S} |b^{r_j}(b^{s_j} - 1)|_\ell = \frac{1}{b^{r_j}} .$$

We thus infer from (8.8) that, setting $q_j := b^{r_j}(b^{s_j} - 1)$, we have

$$\left(\prod_{\ell \in S} |p_j|_\ell \cdot |q_j|_\ell \right) \cdot \left| \xi_a - \frac{p_j}{q_j} \right| < \frac{2}{b^{2r_j + ws_j}}, \tag{8.9}$$

for every positive integer j.

Set $\varepsilon := (w-2)/2(c+1)$. Since $r_j \le cs_j$, we obtain that

$$s_j/(r_j + s_j) \ge 1/(c+1) .$$

Combining

$$\frac{2}{b^{2r_j + ws_j}} \le \frac{2}{b^{(r_j+s_j)(2+(w-2)/(c+1))}} < \frac{2}{q_j^{2+2\varepsilon}}$$

and (8.9), we deduce that

$$\left(\prod_{\ell \in S} |p_j|_\ell \cdot |q_j|_\ell \right) \cdot \left| \xi_a - \frac{p_j}{q_j} \right| < \frac{1}{q_j^{2+\varepsilon}} ,$$

for every integer j large enough. Since ε is positive, Theorem 8.1.17 implies that the real number ξ_a is transcendental, concluding the proof. □

8.1.4 The Schmidt subspace theorem

A formidable multidimensional generalization of the Roth and Ridout theorems is known as the Schmidt subspace theorem (Schmidt 1980b). We state below without proof a simplified p-adic version of this result that will be enough to derive our main result regarding the complexity of algebraic irrational real numbers, namely, Theorem 8.1.6. This result will also play a key role in Section 8.3.

Theorem 8.1.20 *Let $m \ge 2$ be an integer and ε be a positive real number. Let S be a finite set of distinct prime numbers. Let $L_1, \ldots, L_m$ be m linearly* *independent linear forms with real algebraic coefficients. Then, the set of solutions* $\mathbf{x} = (x_1, \dots, x_m)$ *in* $\mathbb{Z}^m$ *to the inequality*

$$\left(\prod_{i=1}^{m} \prod_{\ell \in S} |x_i|_\ell\right) \cdot \prod_{i=1}^{m} |L_i(\mathbf{x})| \leq (\max\{|x_1|, \dots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^m$.

Before going on, we show how Roth's theorem follows from Theorem 8.1.20. Let ξ be a real algebraic number and ε be a positive real number. Consider the two independent linear forms $\xi X - Y$ and X. Theorem 8.1.20 implies that all the integer solutions (p, q) to

$$|q| \cdot |q\xi - p| < |q|^{-\varepsilon} \tag{8.10}$$

are contained in a finite union of proper subspaces of $\mathbb{Q}^2$. There thus is a finite set of equations $x_1 X + y_1 Y = 0, \dots, x_t X + y_t Y = 0$ such that, for every solution (p, q) to (8.10), there exists an integer k with $x_k p + y_k q = 0$. If ξ is irrational, this means that there are only finitely many rational solutions to $|\xi - p/q| < |q|^{-2-\varepsilon}$, which is Roth's theorem.

The following combinatorial transcendence criterion was proved by mean of Theorem 8.1.20 by B. Adamczewski, Y. Bugeaud and F. Luca (Adamczewski, Bugeaud, and Luca 2004) .

Proposition 8.1.21 *If an infinite word a defined over* $\{0, 1, \dots, b-1\}$ *satisfies Condition* $(*)_{w,c}$ *for some* $w > 1$ *and some* $c \geq 0$*, then the real number* ξ_a *is transcendental.*

The strategy to prove this result is the same as for the other criteria, but, in addition, we will take advantage of the specific shape of the factors $b^{s_j} - 1$ in the denominators of the good approximations to ξ_a. In this new criterion, it is not needed anymore that squares occur in order to prove the transcendence of our number. Only occurrences of stammerings are enough, provided that they do not occur too far from the beginning. This difference turns out to be the key for applications.

Before proving Proposition 8.1.21, let us quote a first consequence.

Note that the combinatorial structure of the Thue-Morse word t is quite different from that of the Fibonacci word. Indeed, a well-known property of t is that it is overlap-free and thus cannot satisfy Condition $(*)_{w,c}$ for some $w > 2$. The following result, first proved by K. Mahler in 1929 (Mahler 1929) by mean of a totally different approach, is also a straightforward consequence of Proposition 8.1.21.

Theorem 8.1.22 *The real number*

$$\xi_t := 0.011\,010\,011\,001\,011\,010\,010\,110\cdots$$

is transcendental.

Proof First, we recall that t is aperiodic. Note that t begins with the word 011. Consequently, for every positive integer j, the word $\tau^j(011)$ is also a prefix of t. Set $u_j := \tau^j(0)$ and $v_j := \tau^j(1)$. Then, for every positive integer j, the word t begins with $u_j v_j^2$. Furthermore, a simple computation shows that

$$|u_j| = |v_j| = 2^j \ .$$

This proves that t satifies Condition $(*)_{2,1}$. In view of Proposition 8.1.21, the theorem is established. □

Proof [Proof of Proposition 8.1.21] Let a be an infinite word defined over $\{0, 1, \ldots, b-1\}$ and satisfying Condition $(*)_{w,c}$ for some $w > 1$ and some $c \geq 0$. We assume that ξ_a is algebraic, and we will reach a contradiction.

We consider, as in the proof of Proposition 8.1.19, the integers p_j, r_j, s_j and the set S of prime divisors of b.

Consider the three linearly independent linear forms with real algebraic coefficients:

$$\begin{aligned} L_1(X_1, X_2, X_3) = &\ \ \xi_a X_1 - \xi_a X_2 - X_3, \\ L_2(X_1, X_2, X_3) = &\ \ X_1, \\ L_3(X_1, X_2, X_3) = &\ \ X_2. \end{aligned}$$

For $j \geq 1$, evaluating them on the integer triple $\mathbf{x}_j := (b^{r_j+s_j}, b^{r_j}, p_j)$, we obtain that

$$\prod_{i=1}^{3} |L_i(\mathbf{x}_j)| \leq 2\, b^{2r_j+s_j-(w-1)s_j}. \tag{8.11}$$

On the other hand, we get that

$$\prod_{i=1}^{3} \prod_{\ell \in S} |x_i|_\ell \leq \prod_{\ell \in S} |x_1|_\ell \cdot \prod_{\ell \in S} |x_2|_\ell = b^{-2r_j-s_j}. \tag{8.12}$$

Combining (8.11) and (8.12), we get that

$$\left(\prod_{i=1}^{3} \prod_{\ell \in S} |x_i|_\ell \right) \cdot \prod_{i=1}^{3} |L_i(\mathbf{x})| \leq\ 2\,(b^{r_j+s_j})^{-(w-1)s_j/(r_j+s_j)}\ .$$

Set $\varepsilon := (w-1)/2(c+1)$. Since by assumption a satisfies Condition $(*)_{w,c}$, we obtain

$$\left(\prod_{i=1}^{3}\prod_{\ell\in S}|x_i|_\ell\right)\cdot\prod_{i=1}^{3}|L_i(\mathbf{x}_j)|\le\left(\max\{b^{r_j+s_j},b^{r_j},p_j\}\right)^{-\varepsilon}\ ,$$

if the integer j is sufficiently large.

We then infer from Theorem 8.1.20 that all points $\mathbf{x}_j$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{J}$ such that

$$z_1 b^{r_j+s_j} + z_2 b^{r_j} + z_3 p_j = 0, \tag{8.13}$$

for every j in $\mathcal{J}$. Recall that $p_j/b^{r_j+s_j}$ tends to ξ when j tends to infinity. Dividing (8.13) by $b^{r_j+s_j}$ and letting j tend to infinity along $\mathcal{J}$, we get that ξ_a is a rational number. Since by assumption a satisfies Condition $(*)_{w,c}$, it is not eventually periodic. This provides a contradiction, ending the proof. □

8.1.4.1 Proof of Theorem 8.1.6

We are now ready to finish the proof of Theorem 8.1.6.

Proof [Proof of Theorem 8.1.6] Let $a = a_1a_2\cdots$ be a non-eventually periodic infinite word defined over the finite alphabet $\{0, 1, \ldots, b-1\}$. We assume that there exists an integer $\kappa \geq 2$ such that its complexity function p_a satisfies

$$p_a(n) \leq \kappa n \quad \text{for infinitely many integers } n \geq 1\ ,$$

and we shall derive that a satisfies Condition $(*)_{w,c}$ for some $w > 1$ and some $c \geq 0$. In view of Proposition 8.1.21, we will obtain that the real number ξ_a is transcendental, concluding the proof.

Let n_k be an integer with $p_a(n_k) \leq \kappa n_k$. Denote by $a(\ell)$ the prefix of a of length ℓ. By the pigeonhole principle, there exists (at least) one word m_k of length n_k which has (at least) two occurrences in $a((\kappa+1)n_k)$. Thus, there are (possibly empty) words b_k, c_k, d_k and e_k, such that

$$a((\kappa+1)n_k) = b_k m_k d_k e_k = b_k c_k m_k e_k \text{ and } |c_k| \geq 1\ .$$

We observe that $|b_k| \leq \kappa n_k$. We have to distinguish three cases:

(i) $|c_k| > |m_k|$,

(ii) $\lceil |m_k|/3\rceil \leq |c_k| \leq |m_k|$,

(iii) $1 \leq |c_k| < \lceil |m_k|/3\rceil$.

(i). Under this assumption, there exists a word f_k such that

$$a((\kappa+1)n_k) = b_k m_k f_k m_k e_k \ .$$

Since $|e_k| \leq (\kappa - 1)|m_k|$, the word $b_k(m_k f_k)^s$ with $s := 1 + 1/\kappa$ is a prefix of a. Furthermore, we observe that

$$|m_k f_k| \geq |m_k| \geq \frac{|b_k|}{\kappa} \ .$$

(ii). Under this assumption, there exist two words f_k and g_k such that

$$a((\kappa+1)n_k) = b_k m_k^{1/3} f_k m_k^{1/3} f_k g_k \ .$$

Thus, the word $b_k(m_k^{1/3} f_k)^2$ is a prefix of a. Furthermore, we observe that

$$|m_k^{1/3} f_k| \geq \frac{|m_k|}{3} \geq \frac{|b_k|}{3\kappa} \ .$$

(iii). In this case, c_k is clearly a prefix of m_k and m_k is a prefix of $c_k m_k$. Consequently, c_k^t is a prefix of m_k, where t is the integer part of $|m_k|/|c_k|$. Observe that $t \geq 3$. Setting $s = \lfloor t/2 \rfloor$, we see that $b_k(c_k^s)^2$ is a prefix of a and

$$|c_k^s| \geq \frac{|m_k|}{4} \geq \frac{|b_k|}{4\kappa} \ .$$

In each of the three cases above, we have proved that there are finite words u_k, v_k and a positive real number w such that $u_k v_k^w$ is a prefix of a and:

- $|u_k| \leq \kappa n_k$,
- $|v_k| \geq n_k/4$,
- $w \geq 1 + 1/\kappa > 1$.

Consequently, the sequence $(|u_k|/|v_k|)_{k\geq 1}$ is bounded from above by 4κ. Furthermore, it follows from the lower bound $|v_k| \geq n_k/4$ that we can assume without loss of generality that the sequence $(|v_k|)_{k\geq 1}$ is increasing. This implies that the infinite word a satisfies Condition $(*)_{1+1/\kappa, 4\kappa}$, concluding the proof. □

8.2 Basics from continued fractions

We collect in this section several basic results from the theory of continued fractions that will be useful in the rest of this chapter. These results are stated without proofs and we refer the reader to classical monographs on continued fractions, such as (Perron 1929), (Khintchine 1963), (Rockett and Szüsz 1992), (Lang 1995), (Schmidt 1980b), (Bugeaud 2004a) for more details.

8.2.1 Notations

Every rational number that is not an integer has a unique *continued fraction expansion*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where a_0 is an integer and a_i, $i \geq 1$, are positive integers with $a_n \geq 2$. As well, every irrational real number ζ has a unique continued fraction expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}$$

where a_0 is an integer and a_i, $i \geq 1$, are positive integers. For short, we will write $[a_0, a_1, \ldots, a_n]$ to denote a finite continued fraction and $[a_0, a_1, \ldots]$ for an infinite continued fraction.

For instance, we have:

$$\frac{77\,708\,431}{2\,640\,858} = [29, 2, 2, 1, 5, 1, 4, 1, 1, 2, 1, 6, 1, 10, 2, 2, 3],$$

$$\sqrt{2} = [1, 2, 2, 2, 2, 2, 2, 2, \ldots],$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots],$$

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 12, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, \ldots].$$

The integers $a_0, a_1, \ldots$ are called the *partial quotients*. For $n \geq 1$, the rational number $p_n/q_n := [a_0, a_1, \ldots, a_n]$ is called the nth *convergent* to ζ. Setting

$$p_{-1} = 1, p_0 = a_0, q_{-1} = 0, q_0 = 1$$

the integers p_n and q_n satisfy, for every non-negative integer n, the fundamental relations

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = a_{n+1}q_n + q_{n-1}. \tag{8.14}$$

The sequence $(p_n/q_n)_{n\geq 0}$ converges to ζ. More precisely, we have

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|\zeta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \tag{8.15}$$

for every positive integer n.

Note also that to any sequence $(a_n)_{n\geq 1}$ of positive integers corresponds a unique real irrational number ζ such that

$$\zeta = [0, a_1, a_2, \ldots] \ .$$

8.2.2 Speed of convergence

It follows from inequalities (8.15) that two real numbers having the same first n partial quotients are close to each other. However, they cannot be too close if their $(n + 1)$th partial quotients are different.

Lemma 8.2.1 *Let $\zeta = [a_0, a_1, \ldots]$ and $\eta = [b_0, b_1, \ldots]$ be real numbers. Let us assume that there exists a positive integer n such that $a_j = b_j$ for $j = 0, \ldots, n$. Then,*

$$|\zeta - \eta| \leq q_n^{-2} \ ,$$

where q_n denotes the nth convergent to ζ. Furthermore, if the partial quotients of ζ and η are bounded by M, and if $a_{n+1} \neq b_{n+1}$, then

$$|\zeta - \eta| \geq \frac{1}{(M + 2)^3 q_n^2} \ .$$

A proof of Lemma 8.2.1 is given in (Adamczewski and Bugeaud 2006a).

8.2.3 Growth of convergents and continuants

Next lemma is an easy consequence of the recurrence relation satisfied by the denominators of the convergents to a real number.

Lemma 8.2.2 *Let* $(a_i)_{i\geq 1}$ *be a sequence of positive integers and let* n *be a positive integer. If* $p_n/q_n = [0, a_1, \ldots, a_n]$ *and* $M = \max\{a_1, \ldots, a_n\}$*, then*

$$2^{(n-1)/2} \leq q_n \leq (M+1)^n \ .$$

Given positive integers $a_1, \ldots, a_m$, we denote by $K_m(a_1, \ldots, a_m)$ the denominator of the rational $[0, a_1, \ldots, a_m]$ written in lowest terms. This quantity is called the *continuant* associated with the sequence $a_1, \ldots, a_m$. If $a = a_1 a_2 \cdots a_m$ denotes a finite word defined over the set of positive integers, we also write $K_m(a)$ for the continuant $K_m(a_1, a_2, \ldots, a_m)$, when the context is clear enough to avoid a possible confusion.

Lemma 8.2.3 *Let* $a_1, \ldots, a_m$ *be positive integers and let* k *be an integer such that* $1 \leq k \leq m-1$. *Then,*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

and

$$K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) \leq K_m(a_1, \ldots, a_m)$$

$$\leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) \ .$$

This lemma is proved in (Cusick and Flahive 1989). As we will see in the sequel of this chapter, the formalism of continuants is often very convenient to estimate the size of denominators of convergents.

8.2.4 The mirror formula

The *mirror formula*, which can be established by induction on n using the recurrence relations (8.14) giving the sequence $(q_n)_{n\geq 1}$, is sometimes omitted from classical textbooks.

Lemma 8.2.4 *Let* $\zeta = [a_0, a_1, \ldots]$ *be a real number and* $(p_n/q_n)_{n\geq 0}$ *be the sequence of convergents to* ζ. *Then,*

$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1]$$

for every positive integer n.

However, this is a very useful auxiliary result, as will become clear in the next sections.

8.2.5 The Euler–Lagrange theorem

It is easily seen that if the continued fraction expansion of ζ is ultimately periodic, then ζ is a quadratic number. The converse is also true.

Theorem 8.2.5 *The continued fraction expansion of a real number is eventually periodic if, and only if, it is a quadratic irrational number.*

In contrast, very little is known on the continued fraction expansion of an algebraic real number of degree at least 3.

8.3 Transcendental continued fractions

All along this section, we use the following notation. If $a = a_1 a_2 \cdots$ is an infinite word defined over the set of positive integers, we denote by ζ_a the associated continued fraction, that is,

$$\zeta_a := [0, a_1, a_2, \ldots] .$$

It is widely believed that the continued fraction expansion of any irrational algebraic number ζ either is eventually periodic (and, according to Theorem 8.2.5, this is the case if, and only if, ζ is a quadratic irrational), or it contains arbitrarily large partial quotients. Apparently, this problem was first considered by A. Ya. Khintchine (Khintchine 1963). Some speculations about the randomness of the continued fraction expansion of algebraic numbers of degree at least three have later been made by several authors. However, one shall admit that our knowledge on this topic is very limited.

In this section, we use the Schmidt subspace theorem (Theorem 8.1.20) to prove the transcendence of families of continued fractions involving periodic or symmetric patterns.

8.3.1 The Fibonacci continued fraction

We first prove that continued fractions beginning in arbitrarily large squares are either quadratic or transcendental.

Proposition 8.3.1 *Let a be an infinite word whose letters are positive integers. If a satisfies Condition $(*)_{w,0}$ for some $w \geq 2$, then the real number ζ_a is transcendental.*

Let a and b be distinct positive integers and let

$$f' := f'_1 f'_2 f'_3 \cdots = abaababaabaabab \cdots$$

be the Fibonacci word defined over the alphabet $\{a, b\}$ as in the introduction of this chapter. Recall that we already proved in Section 8.1.3.2 that f (and thus f') satisfies Condition $(*)_{2.25,0}$. As a direct consequence of Proposition 8.3.1, we obtain the following result.

Theorem 8.3.2 *The real number*

$$\zeta_{f'} := [f'_1, f'_2, \ldots]$$

is transcendental.

Proof [Proof of Proposition 8.3.1] Let $a := a_1 a_2 \cdots$ be an infinite word defined over the infinite alphabet $\{1, 2, \ldots\}$ and satisfying Condition $(*)_{w,0}$ for some $w \geq 2$. Assume that the parameter $w \geq 2$ is fixed, as well as the sequence $(v_j)_{j\geq 1}$ occurring in the definition of Condition $(*)_{w,0}$. Set also $s_j = |v_j|$, for every $j \geq 1$. We want to prove that the real number

$$\zeta_a := [0, a_1, a_2, \ldots]$$

is transcendental.

By definition of Condition $(*)_{w,0}$, the sequence a is aperiodic and it follows from the Euler–Lagrange theorem (Theorem 8.2.5) that ζ_a is not a quadratic number. Furthermore, ζ_a is irrational since the sequence a is infinite.

From now on, we assume that ζ_a is algebraic of degree at least three and we aim at deriving a contradiction. Throughout this proof, the constants implied by $\ll$ are independent of j.

Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to ζ_a. We infer from (8.15) that

$$|q_{s_j}\zeta_a - p_{s_j}| \leq q_{s_j}^{-1} \tag{8.16}$$

and

$$|q_{s_j-1}\zeta_a - p_{s_j-1}| \leq q_{s_j}^{-1}. \tag{8.17}$$

The key fact for the proof of Proposition 8.3.1 is the observation that ζ_a admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for every positive integer j, we define the sequence $(b_k^{(j)})_{k\geq 1}$ by

$$b_{h+ks_j}^{(j)} = a_h \text{ for } 1 \leq h \leq s_j \text{ and } k \geq 0.$$

The sequence $(b_k^{(j)})_{k\geq 1}$ is purely periodic with period s_j. Set

$$\alpha_j = [0, b_1^{(j)}, b_2^{(j)}, \ldots]$$

and observe that

$$\alpha_j = [0, a_1, \ldots, a_{s_j}, 1/\alpha_j] = \frac{p_{s_j}/\alpha_j + p_{s_j-1}}{q_{s_j}/\alpha_j + q_{s_j-1}} \ .$$

Thus, α_j is a root of the quadratic polynomial

$$P_j(X) := q_{s_j-1}X^2 + (q_{s_j} - p_{s_j-1})X - p_{s_j} \ .$$

By Rolle's theorem and Lemma 8.2.1, for every positive integer j, we have

$$|P_j(\zeta_a)| = |P_j(\zeta_a) - P_j(\alpha_j)| \ll q_{s_j} \, |\zeta_a - \alpha_j| \ll q_{s_j} \, q_{2s_j}^{-2}, \tag{8.18}$$

since the first $2s_j$ partial quotients of ζ_a and α_j are the same. Furthermore, with the notation of Section 8.2.3, we have $q_{s_j} = K_{s_j}(v_j)$ and $q_{2s_j} = K_{2s_j}(v_j v_j)$. Then, we infer from Lemma 8.2.3 that

$$q_{2s_j} \geq q_{s_j}^2.$$

By (8.18), this gives

$$|P_j(\zeta_a)| \ll \frac{1}{q_{s_j}^3}. \tag{8.19}$$

Consider now the three linearly independent linear forms:

$$\begin{aligned} &L_1(X_1, X_2, X_3) = \zeta_a^2 X_1 + \zeta_a X_2 - X_3, \\ &L_2(X_1, X_2, X_3) = X_1, \\ &L_3(X_1, X_2, X_3) = X_3. \end{aligned}$$

Evaluating them on the triple $(q_{s_j-1}, q_{s_j} - p_{s_j-1}, p_{s_j})$, we infer from (8.19) that

$$\prod_{1 \leq i \leq 3} |L_i(q_{s_j-1}, q_{s_j} - p_{s_j-1}, p_{s_j})| \ll \frac{1}{\max\{q_{s_j-1}, q_{s_j} - p_{s_j-1}, p_{s_j}\}}. \tag{8.20}$$

It then follows from Theorem 8.1.20 that the points $(q_{s_j-1}, q_{s_j} - p_{s_j-1}, p_{s_j})$ with $j \geq 1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (x_1, x_2, x_3) and an infinite set of distinct positive integers $\mathcal{J}_1$ such that

$$x_1 q_{s_j-1} + x_2(q_{s_j} - p_{s_j-1}) + x_3 p_{s_j} = 0, \tag{8.21}$$

for every j in $\mathcal{J}_1$. Observe that $(x_1, x_2) \neq (0, 0)$, since (x_1, x_2, x_3) is a non-zero triple. Dividing (8.21) by q_{s_j}, we obtain

$$x_1 \frac{q_{s_j-1}}{q_{s_j}} + x_2 \left(1 - \frac{p_{s_j-1}}{q_{s_j-1}} \cdot \frac{q_{s_j-1}}{q_{s_j}}\right) + x_3 \frac{p_{s_j}}{q_{s_j}} = 0. \tag{8.22}$$

By letting j tend to infinity along $\mathcal{J}_1$ in (8.22), we get that

$$\lim_{\mathcal{J}_1 \ni j \to +\infty} \frac{q_{s_j-1}}{q_{s_j}} = -\frac{x_2 + x_3\zeta_a}{x_1 - x_2\zeta_a} =: \alpha \ .$$

By definition of α and Equality (8.22), we observe that

$$\left| \alpha - \frac{q_{s_j-1}}{q_{s_j}} \right| = \left| \frac{x_2 + x_3\zeta_a}{x_1 - x_2\zeta_a} - \frac{x_2 + x_3 p_{s_j}/q_{s_j}}{x_1 - x_2 p_{s_j-1}/q_{s_j-1}} \right|,$$

for every j in $\mathcal{J}_1$. As a consequence of (8.16) and (8.17), we get that

$$\left| \alpha - \frac{q_{s_j-1}}{q_{s_j}} \right| \ll \frac{1}{q_{s_j}^2}, \tag{8.23}$$

for every j in $\mathcal{J}_1$. Since q_{s_j-1} and q_{s_j} are coprime and s_j tends to infinity when j tends to infinity along $\mathcal{J}_1$, the real number α is irrational.

Consider now the three linearly independent linear forms:

$$\begin{aligned} L_1'(Y_1, Y_2, Y_3) &= \alpha Y_1 - Y_2, \\ L_2'(Y_1, Y_2, Y_3) &= \zeta_a Y_1 - Y_3, \\ L_3'(Y_1, Y_2, Y_3) &= Y_1. \end{aligned}$$

Evaluating them on the triple $(q_{s_j}, q_{s_j-1}, p_{s_j})$ with $j \in \mathcal{J}_1$, we infer from (8.16) and (8.23) that

$$\prod_{1 \le j \le 3} |L_j'(q_{s_j}, q_{s_j-1}, p_{s_j})| \ll q_{s_j}^{-1} \ .$$

It then follows from Theorem 8.1.20 that the points $(q_{s_j}, q_{s_j-1}, p_{s_j})$ with $j \in \mathcal{J}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{J}_2$, included in $\mathcal{J}_1$, such that

$$y_1 q_{s_j} + y_2 q_{s_j-1} + y_3 p_{s_j} = 0, \tag{8.24}$$

for every j in $\mathcal{J}_2$. Dividing (8.24) by q_{s_j} and letting j tend to infinity along $\mathcal{J}_2$, we get

$$y_1 + y_2\alpha + y_3\zeta_a = 0. \tag{8.25}$$

To obtain another equation linking ζ_a and α, we consider the three linearly independent linear forms:

$$\begin{aligned} L_1''(Z_1, Z_2, Z_3) &= \alpha Z_1 - Z_2, \\ L_2''(Z_1, Z_2, Z_3) &= \zeta_a Z_2 - Z_3, \\ L_3''(Z_1, Z_2, Z_3) &= Z_1. \end{aligned}$$

Evaluating them on the triple $(q_{s_j}, q_{s_j-1}, p_{s_j-1})$ with $j \in \mathcal{J}_1$, we infer from (8.17) and (8.23) that

$$\prod_{j=1}^{3} |L''_j(q_{s_j}, q_{s_j-1}, p_{s_j-1})| \ll q_{s_j}^{-1} .$$

It then follows from Theorem 8.1.20 that the points $(q_{s_j}, q_{s_j-1}, p_{s_j-1})$ with $j \in \mathcal{J}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{J}_3$, included in $\mathcal{J}_1$, such that

$$z_1 q_{s_j} + z_2 q_{s_j-1} + z_3 p_{s_j-1} = 0, \tag{8.26}$$

for every j in $\mathcal{J}_3$. Dividing (8.26) by q_{s_j-1} and letting j tend to infinity along $\mathcal{J}_3$, we get

$$\frac{z_1}{\alpha} + z_2 + z_3 \zeta_a = 0. \tag{8.27}$$

We infer from (8.25) and (8.27) that

$$(z_3\zeta_a + z_2)(y_3\zeta_a + y_1) = y_2 z_1. \tag{8.28}$$

If $y_3 z_3 = 0$, then (8.25) and (8.27) yield that α is rational, which is a contradiction. Consequently, $y_3 z_3 \neq 0$ and we infer from (8.28) that ζ_a is a quadratic real number, which is again a contradiction. This completes the proof the proposition. □

8.3.2 The Thue-Morse continued fraction

Let a and b be distinct positive integers and let

$$t' := t'_1 t'_2 t'_3 \cdots = abbabaabbaababba \cdots$$

be the Thue-Morse word defined over the alphabet $\{a, b\}$ as in the introduction of this chapter. M. Queffélec (Queffélec 1998) showed that the Thue-Morse continued fraction $\zeta_{t'}$ is transcendental. To prove this result, she used an extension of Roth's theorem to approximation by quadratic numbers, worked out by W. M. Schmidt and which is a consequence of Theorem 8.1.20, combined with the fact that t satisfies Condition $(*)_{1.6,0}$ and that the subshift associated with t is uniquely ergodic.

The purpose of this subsection is to present an alternative and shorter proof of her result, using the fact that t begins in arbitrarily large *palindromes*. Actually, the following combinatorial transcendence criterion obtained in (Adamczewski and Bugeaud 2007c) (see also the paper (Adamczewski and Bugeaud 2007d)) relies, once again, on the Schmidt subspace theorem.

Proposition 8.3.3 *Let a be an infinite word whose letters are positive integers. If a begins with arbitrarily long palindromes, then the real number ζ_a is either quadratic or transcendental.*

As a consequence of Proposition 8.3.3, we obtain the following result.

Theorem 8.3.4 *The real number*

$$\zeta_{t'} := [t'_1, t'_2, \ldots]$$

is transcendental.

Proof We first recall that the word t (and thus t') is not eventually periodic. The Euler-Lagrange theorem (Theorem 8.2.5) implies that $\zeta_{t'}$ is not a quadratic irrational number. Note that $\tau^2(0) = 0110$ and $\tau^2(1) = 1001$ are palindromes. Now, observe that the Thue-Morse word t begins with the palindrome 0110. Since for every positive integer j, the word $\tau^j(0)$ is a prefix of t, we obtain that, for every positive integer n, the prefix of length 4^n of t (and thus of t') is a palindrome. In view of Proposition 8.3.3, this concludes the proof. □

Proof [Proof of Proposition 8.3.3] Let $a = a_1 a_2 \cdots$ be an infinite word satisfying the assumptions of the proposition and set

$$\zeta_a := [0, a_1, a_2, \ldots] .$$

Let us denote by $(n_j)_{j \geq 1}$ the increasing sequence of all lengths of prefixes of a that are palindromes. Let us also denote by p_n/q_n the nth convergent to ζ_a.

In the sequel, we assume that ζ_a is algebraic and our aim is to prove that ζ_a is a quadratic irrational number. Note that ζ_a is irrational since it has an infinite continued fraction expansion.

Let $j \geq 1$ be an integer. Since by assumption the word $a_1 a_2 \cdots a_{n_j}$ is a palindrome, we infer from Lemma 8.2.4 that

$$\frac{q_{n_j - 1}}{q_{n_j}} = \frac{p_{n_j}}{q_{n_j}} ,$$

that is,

$$q_{n_j - 1} = p_{n_j} .$$

It then follows from (8.15) that

$$|q_{n_j} \zeta_a - q_{n_j - 1}| < \frac{1}{q_{n_j}} \tag{8.29}$$

and

$$|q_{n_j-1}\zeta_a - p_{n_j-1}| < \frac{1}{q_{n_j}}. \tag{8.30}$$

Consider now the three linearly independent linear forms:

$$\begin{aligned} L_1(X_1, X_2, X_3) &= \zeta_a X_1 - X_2, \\ L_2(X_1, X_2, X_3) &= \zeta_a X_2 - X_3, \\ L_3(X_1, X_2, X_3) &= X_1. \end{aligned}$$

Evaluating them on the triple $(q_{n_j}, q_{n_j-1}, p_{n_j-1})$, we infer from (8.29) and (8.30) that

$$\prod_{i=1}^{3} |L_i(q_{n_j}, q_{n_j-1}, p_{n_j-1})| < \frac{1}{\max\{q_{n_j}, q_{n_j-1}, p_{n_j-1}\}}.$$

It then follows from Theorem 8.1.20 that the points $(q_{n_j}, q_{n_j-1}, p_{n_j-1})$, $j \geq 1$, lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{J}$ such that

$$z_1 q_{n_j} + z_2 q_{n_j-1} + z_3 p_{n_j-1} = 0, \tag{8.31}$$

for every j in $\mathcal{J}$. Dividing (8.31) by q_{n_j}, this gives

$$z_1 + z_2 \frac{q_{n_j-1}}{q_{n_j}} + z_3 \left(\frac{p_{n_j-1}}{q_{n_j-1}} \cdot \frac{q_{n_j-1}}{q_{n_j}} \right) = 0 \ .$$

Letting j tend to infinity along $\mathcal{J}$, we infer from (8.29) and (8.30) that

$$z_1 + z_2\zeta_a + z_3\zeta_a^2 = 0.$$

Since (z_1, z_2, z_3) is a non-zero triple, this implies that ζ_a is a quadratic or a rational number. Since we already observed that ζ_a is irrational, this ends the proof. □

8.4 Simultaneous rational approximations to a real number and its square

All along this section, φ will denote the Golden Ratio.

8.4.1 Uniform Diophantine approximation

A fundamental result in Diophantine approximation was obtained by P. G. L. Dirichlet in 1842 (Dirichlet 1842).

Theorem 8.4.1 *For every real number ξ and every real number $X > 1$, the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\leq X^{-1}, \\ |x_0| &\leq X, \end{aligned} \tag{8.32}$$

has a non-zero solution $(x_0, x_1) \in \mathbb{Z}^2$.

Proof Let t denote the smallest integer greater than or equal to $X - 1$. If ξ is the rational a/b, with a and b integers and $1 \leq b \leq t$, it is sufficient to set $x_1 = a$ and $x_0 = b$. Otherwise, the $t + 2$ points 0, $\{\xi\}, \ldots, \{t\xi\}$, and 1 are pairwise distinct and they divide the interval $[0, 1]$ into $t + 1$ subintervals. By the pigeonhole principle, at least one of these has its length at most equal to $1/(t + 1)$. This means that there exist integers k, ℓ and m_k, m_ℓ with $0 \leq k < \ell \leq t$ and

$$|(\ell\xi - m_\ell) - (k\xi - m_k)| \leq \frac{1}{t+1} \leq \frac{1}{X} \, .$$

We get (8.32) by setting $x_1 := m_\ell - m_k$ and $x_0 := \ell - k$, and by noticing that x_0 satisfies $1 \leq x_0 \leq t \leq X$. □

Theorem 8.4.1 implies that every irrational real number is approximable at order at least 2 by rationals, a statement that also follows from (8.15). However, Theorem 8.4.1 gives a stronger result, in the sense that it asserts that the system (8.32) has a solution for every real number $X > 1$, while (8.15) only implies that (8.32) has a solution for arbitrarily large values of X. In Diophantine approximation, a statement like Theorem 8.4.1 is called *uniform*.

Obviously, the quality of approximation strongly depends on whether we are interested in a uniform statement or in a statement valid only for arbitrarily large X. Indeed, for any $w > 1$, there clearly exist real numbers ξ for which, for arbitrarily large values of X, the equation

$$|x_0\xi - x_1| \leq X^{-w}$$

has a solution in integers x_0 and x_1 with $1 \leq x_0 \leq X$. In contrast, it was proved by A. Ya. Khintchine (Khintchine 1926) that there is no irrational real number ξ satisfying a stronger form of Theorem 8.4.1 in which the exponent of X in (8.32) is less than -1.

In the case of rational approximation, these questions are quite well understood, essentially thanks to the theory of continued fractions.

It is a notorious fact that questions of simultaneous Diophantine approximation are in general much more difficult when the quantities we approximate are dependent. A classical example is provided by the simultaneous rational approximation of the first n powers of a transcendental number by rational numbers of the same denominator. In the sequel, we will focus on the 2-dimensional case, that is, on the uniform simultaneous approximation to a real number and its square.

In this framework, Dirichlet's theorem can be extended as follows.

Theorem 8.4.2 *For every real number ξ and every real number $X > 1$, the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\leq X^{-1/2}, \\ |x_0\xi^2 - x_2| &\leq X^{-1/2}, \\ |x_0| &\leq X, \end{aligned}$$

has a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$.

We omit the proof which follows the same lines as that of Theorem 8.4.1.

It was proved by I. Kubilyus (Kubilius 1949) that, for almost every real number ξ with respect to the Lebesgue measure, the exponent $-1/2$ in the above statement cannot be lowered. It was also expected that this exponent cannot be lowered for a real number that is neither rational nor quadratic. In this direction, a first limitation was obtained by H. Davenport and W. M. Schmidt (Davenport and Schmidt 1968).

Theorem 8.4.3 *Let ξ be a real number that is neither rational nor quadratic. Then, there exists a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\leq cX^{-1/\varphi}, \\ |x_0\xi^2 - x_2| &\leq cX^{-1/\varphi}, \\ |x_0| &\leq X, \end{aligned}$$

has no solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for arbitrarily large real numbers X.

Note that $1/\varphi = 0.618\ldots$ is larger than $1/2$.

8.4.2 Extremal numbers

As we just mentioned, it was expected for a long time that the constant $-1/\varphi$ in Theorem 8.4.3 could be replaced by $-1/2$. This is actually not the case, as was proved by D. Roy (Roy 2004) (see also (Roy 2003a)).

Theorem 8.4.4 *There exist a real number ξ which is neither rational nor quadratic and a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\le cX^{-1/\varphi}, \\ |x_0\xi^2 - x_2| &\le cX^{-1/\varphi}, \\ |x_0| &\le X, \end{aligned} \tag{8.33}$$

has a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$ for every real number $X > 1$.

Such a result is quite surprising, since the volume of the convex body defined by (8.33) tends rapidly to zero as X grows to infinity.

Any real number ξ satisfying an exceptional Diophantine condition as in Theorem 8.4.4 was termed by D. Roy an *extremal number.* He proved that the set of extremal numbers is countable. Furthermore, he also gave some explicit example of extremal numbers. As we will see in the sequel, if a and b denote two distinct positive integers, the real number $\zeta_{f'}$ defined in the introduction of this chapter is an extremal number.

8.4.3 Simultaneous rational approximations, continued fractions and palindromes

The crucial point for the proof of Theorem 8.4.4 is a surprising connection between simultaneous rational approximation, continued fractions and palindromes. This is the aim of this section to describe this connection.

Let $\zeta = [0, a_1, a_2, \ldots]$ be a real number and let p_n/q_n denote the nth convergent to ζ. Let us assume that the word $a_1 \cdots a_n$ is a palindrome. As we already observed in the proof of Proposition 8.3.3, Lemma 8.2.4 then implies that

$$p_n = q_{n-1} \ .$$

On the other hand, we infer from (8.15) that

$$\left|\zeta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \ .$$

Since $0 < \zeta < 1$, $a_1 = a_n$ and $q_n \le (a_n + 1)q_{n-1}$, we obtain that

$$\begin{aligned} \left|\zeta^2 - \frac{p_{n-1}}{q_n}\right| &\le \left|\zeta^2 - \frac{p_{n-1}}{q_{n-1}} \times \frac{p_n}{q_n}\right| \\ &\le \left|\zeta + \frac{p_{n-1}}{q_{n-1}}\right| \times \left|\zeta - \frac{p_n}{q_n}\right| + \frac{1}{q_n q_{n-1}} \\ &\le 2\left|\zeta - \frac{p_n}{q_n}\right| + \frac{1}{q_n q_{n-1}} < \frac{a_1 + 3}{q_n^2}\cdot \end{aligned}$$

To sum up, if the word $a_1 a_2 \cdots a_n$ is a palindrome, then

$$|q_n \zeta - p_n| < \frac{1}{q_n} \quad \text{and} \quad |q_n \zeta^2 - p_{n-1}| < \frac{a_1 + 3}{q_n} \cdot \tag{8.34}$$

In other words, palindromes provide very good simultaneous rational approximations to ζ and ζ^2.

An essential aspect of the question we consider here is that it is a problem of uniform approximation. Let us assume that the infinite word $a = a_1 a_2 \cdots$ begins with arbitrarily long palindromes, and denote by $(n_j)_{j \geq 1}$ the increasing sequence formed by the lengths of prefixes of a that are palindromes. Set

$$\zeta_a := [0, a_1, a_2, \ldots] \ .$$

If the sequence $(n_j)_{j \geq 1}$ increases sufficiently slowly to ensure the existence of a positive real number c_1 and a real number τ such that $q_{n_{j+1}} \leq c_1 q_{n_j}^{\tau}$ for every large j, then Inequalities (8.34) ensure that, for every real number X large enough, there exists a positive real number c_2 such that the system of inequalities

$$|x_0| \leq X, \ |x_0 \zeta_a - x_1| \leq c_2 X^{-1/\tau}, \ |x_0 \zeta_a^2 - x_2| \leq c_2 X^{-1/\tau}, \tag{8.35}$$

has a non-zero solution $(x_0, x_1, x_2) \in \mathbb{Z}^3$. Indeed, given a sufficiently large real number X, there exists an integer i such that $q_{n_i} \leq X < q_{n_i+1}$ and the triple $(q_{n_i}, p_{n_i}, p_{n_i - 1})$ provides a non-zero solution to the system (8.35).

Consequently, if the continued fraction expansion of a real number ζ_a begins with many palindromes, then ζ_a and ζ_a^2 are uniformly very well simultaneously approximated by rationals. In view of Theorem 8.4.1, this observation is only interesting if there exist infinite words a for which the associated exponent τ is less than 2.

8.4.4 Fibonacci word and palindromes

Let a and b be distinct positive integers. As in Section 8.3.1, we consider the Fibonacci word

$$f' := abaababaabaabab \cdots$$

defined over the alphabet $\{a, b\}$. As in Section 8.1.3.2, $(F_j)_{j \geq 0}$ denotes the Fibonacci sequence.

In this section, we prove that many prefixes of f' are palindromes.

Proposition 8.4.5 *For every integer $j \geq 1$, the prefix of f' of length*

$$n_j := F_{j+3} - 2, \tag{8.36}$$

is a palindrome.

Proof For every integer $j \geq 2$ we set $w_j = f'_1 f'_2 \cdots f'_{F_j}$ and we recall that $w_2 = a$, $w_3 = ab$ and $w_{j+2} = w_{j+1} w_j$. One can show by an easy induction that, for every integer $j \geq 2$, the word w_{2j} ends with ba while w_{2j+1} ends with ab. Furthermore, the length of w_j is equal to F_j for every $j \geq 2$.

Let $j \geq 1$ and φ_j denote the prefix of f' of length n_j. Observe that $\varphi_1 = a$, $\varphi_2 = aba$, $\varphi_3 = abaaba$. We then obtain by induction that

$$\varphi_j = \varphi_{j-1} ba \varphi_{j-2}, \text{ for every even integer } j \geq 4,$$

while

$$\varphi_j = \varphi_{j-1} ab \varphi_{j-2}, \text{ for every odd integer } j \geq 3.$$

Then, we observe that

$$\varphi_j = \varphi_{j-2} ab \varphi_{j-3} ba \varphi_{j-2}, \text{ for every even integer } j \geq 4, \tag{8.37}$$

while

$$\varphi_j = \varphi_{j-2} ba \varphi_{j-3} ab \varphi_{j-2}, \text{ for every odd integer } j \geq 5. \tag{8.38}$$

Since φ_1, φ_2 and φ_3 are palindromes, we deduce again by induction that the word φ_j is a palindrome for every positive integer j. This ends the proof. □

8.4.5 Proof of Theorem 8.4.4

We are now ready to complete the proof of Theorem 8.4.4.

Proof [Proof of Theorem 8.4.4] Let a and b be distinct positive integers. We are going to prove that the real number $\zeta_{f'}$ defined in Theorem 8.3.2 is an extremal number.

Note first that, as a consequence of Theorem 8.4.4, the real number $\zeta_{f'}$ is neither rational nor quadratic. Let $(n_j)_{j\geq 1}$ be the sequence of positive integers defined in (8.36). Set also $Q_j = q_{n_j}$, where p_n/q_n denotes the nth convergent to $\zeta_{f'}$.

In view of Proposition 8.4.5 and Inequalities (8.34), there exists a positive real number c_1 such that the system

$$\begin{aligned} |x_0 \zeta_{f'} - x_1| &\leq c_1 Q_j^{-1}, \\ |x_0 \zeta_{f'}^2 - x_2| &\leq c_1 Q_j^{-1}, \\ |x_0| &\leq Q_j, \end{aligned} \tag{8.39}$$

has a non-zero integer solution $(x_0^{(j)}, x_1^{(j)}, x_2^{(j)})$ for every positive integer

$j \geq 4$. More precisely, we have $(x_0^{(j)}, x_1^{(j)}, x_2^{(j)}) = (q_{n_j}, p_{n_j}, p_{n_j-1})$. In particular, $x_0^{(j)} = Q_j$.

We are now going to prove the existence of a positive real number c such that

$$Q_{j+1} \leq c\, Q_j^{\varphi} \tag{8.40}$$

for every positive integer j.

We argue by induction. We infer from (8.37), (8.38) and Lemma 8.2.3 that there exist two positive real numbers c_2 and c_3 such that

$$c_2 < \frac{Q_{j+1}}{Q_j Q_{j-1}} < c_3, \tag{8.41}$$

for every integer $j \geq 2$. Without loss of generality, we can assume that

$$c_3 \geq \left\{ \frac{(c_2 Q_1)^{\varphi}}{Q_2}, \frac{(c_2 Q_2)^{1/\varphi}}{Q_1} \right\}. \tag{8.42}$$

Set $c_4 := c_2^{\varphi}/c_3$ and $c_5 := c_3^{\varphi}/c_2$. We will prove by induction that

$$c_4\, Q_{j-1}^{\varphi} \leq Q_j \leq c_5\, Q_{j-1}^{\varphi} \tag{8.43}$$

for every integer $j \geq 2$. For $j = 2$, this follows from (8.41) and (8.42). Let us assume that (8.43) is satisfied for an integer $j \geq 2$. By (8.41), we obtain that

$$c_2\, Q_j^{\varphi} \left(Q_j^{1-\varphi} Q_{j-1} \right) < Q_{j+1} < c_3\, Q_j^{\varphi} \left(Q_j^{1-\varphi} Q_{j-1} \right) .$$

Since $\varphi(\varphi - 1) = 1$, it follows that

$$c_2\, Q_j^{\varphi} \left(Q_j Q_{j-1}^{-\varphi} \right)^{1-\varphi} < Q_{j+1} < c_3\, Q_j^{\varphi} \left(Q_j Q_{j-1}^{-\varphi} \right)^{1-\varphi} .$$

We then deduce from (8.43) that

$$\left(c_2 c_5^{1-\varphi} \right) Q_j^{\varphi} < Q_{j+1} < \left(c_3 c_4^{1-\varphi} \right) Q_j^{\varphi} .$$

By definition of c_4 and c_5, and since $\varphi(\varphi - 1) = 1$, this gives

$$c_4\, Q_j^{\varphi} < Q_{j+1} < c_5\, Q_j^{\varphi} ,$$

which proves that (8.43) is true for $j + 1$. Consequently, Inequality (8.40) is established.

Let X be a sufficiently large real number. There exists an integer $j \geq 4$

such that $Q_j \le X < Q_{j+1}$. We infer from (8.39) and (8.40) that

$$\begin{aligned}\max\left\{|x_0^{(j)}\zeta_{f'} - x_1^{(j)}|,\ |x_0^{(j)}\zeta_{f'}^2 - x_2^{(j)}|\right\} &< c_1 Q_j^{-1} \\ &\le c_1 X^{-\log Q_j/\log X} \\ &\le c_1 X^{-\log Q_j/\log Q_{j+1}} \\ &\le c_6 X^{-1/\varphi},\end{aligned}$$

for a suitable positive real number c_6.

This proves that $\zeta_{f'}$ is an extremal number, concluding the proof. □

8.4.6 Palindromic density of an infinite word

For an infinite word $a = a_1 a_2 \cdots$ let denote by $n_1 < n_2 < \ldots$ the increasing (finite or infinite) sequence of all the lengths of the prefixes of a that are palindromes. We define the *palindromic density* of a, denoted by $d_p(a)$, by setting $d_p(a) = 0$ if only a finite number of prefixes of a are palindromes and, otherwise, by setting

$$d_p(a) := \left(\limsup_{j\to+\infty} \frac{n_{j+1}}{n_j}\right)^{-1} .$$

Clearly, for every infinite word a we have

$$0 \le d_p(a) \le 1 .$$

Furthermore, if $a = uu\cdots$ is a periodic word, then either $d_p(a) = 0$ or $d_p(a) = 1$, and the latter holds if, and only if, there exist two (possibly empty) palindromes v and w such that $u = vw$. On the other hand, an eventually periodic word that begins with arbitrarily long palindromes is purely periodic. Thus, the palindromic density of an eventually periodic word is either maximal or minimal.

S. Fischler (Fischler 2006) proved that the Fibonacci word has the highest palindromic density among aperiodic infinite words. We state his result without proof.

Theorem 8.4.6 *Let a be a non-eventually periodic word. Then,*

$$d_p(a) \le \frac{1}{\varphi} ,$$

where φ is the Golden Ratio. Furthermore, the bound is sharp and reached by the Fibonacci word.

This explains *a posteriori* why the Fibonacci continued fraction was a good candidate to be an extremal number.

8.5 Explicit examples for the Littlewood conjecture

As we have already seen, the theory of continued fractions ensures that, for every real number ξ, there exist infinitely many positive integers q such that

$$q \cdot \|q\xi\| < 1, \tag{8.44}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. In particular, for all pairs (α, β) of real numbers, there exist infinitely many positive integers q such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| < 1 \ .$$

In this section we consider the *Littlewood conjecture* (Littlewood 1968), a famous open problem in simultaneous Diophantine approximation. It claims that in fact, for any given pair (α, β) of real numbers, a slightly stronger result holds, namely

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0. \tag{8.45}$$

We will see how the theory of continued fractions and combinatorics on words can be combined to construct a large class of explicit pairs satisfying this conjecture. In the sequel, we denote by $\mathbf{L}$ the set of pairs of real numbers satisfying Littlewood's conjecture, that is,

$$\mathbf{L} := \left\{ (\alpha, \beta) \in \mathbb{R}^2 \mid \inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0 \right\} \ .$$

8.5.1 Two useful remarks

Let us denote by

$$\mathbf{Bad} := \left\{ \xi \in \mathbb{R} \mid \inf_{q \geq 1} q \cdot \|q\xi\| > 0 \right\}$$

the set of *badly approximable* real numbers. A straightforward consequence of Inequalities (8.15) is that a real number belongs to **Bad** if, and only if, the sequence of partial quotients in its continued fraction expansion is bounded.

Our first remark is a trivial observation: If α or β has unbounded partial quotients, then the pair (α, β) satisfies the Littlewood conjecture. Combined with a classical theorem of É. Borel (Borel 1909), a notable consequence of this fact is the following proposition.

Proposition 8.5.1 *The set* $\mathbf{L}$ *has full Lebesgue measure.*

Our second remark is concerned with pairs of linearly dependent numbers.

Proposition 8.5.2 *Let α and β be two real numbers such that* 1*,* α*, and* β *are linearly dependent over* $\mathbb{Q}$*. Then, the pair* (α, β) *belongs to* $\mathbf{L}$*.*

Proof Let (α, β) be a pair of real numbers such that 1, α, and β are linearly dependent over $\mathbb{Q}$. If α or β is rational, then (α, β) belongs to the set $\mathbf{L}$. Thus, we assume that α and β are irrational.

Let q be a positive integer such that

$$\|q\alpha\| < \frac{1}{q} \ .$$

By assumption, there exist integers a, b and c not all zeros, such that

$$a\alpha + b\beta + c = 0 \ .$$

Since α and β are irrational, a and b are both non-zero. Thus, $\|qa\alpha\| = \|qb\beta\| < a/q$. This gives

$$\|qab\alpha\| < \frac{ab}{q} \quad \text{and} \quad \|qab\beta\| < \frac{a^2}{q} \ .$$

Setting $Q = qab$, we then obtain

$$Q \cdot \|Q\alpha\| \cdot \|Q\beta\| < \frac{a^5 b^3}{Q} \ .$$

Since q can be taken arbitrarily large, the pair (α, β) belongs to $\mathbf{L}$, concluding the proof. □

From now on, we say that (α, β) is a *non-trivial pair* of real numbers if the following conditions hold:

(i) α and β both belong to **Bad**,

(ii) 1, α and β are linearly independent over $\mathbb{Q}$.

In view of the remarks above, it is natural to focus our attention on non-trivial pairs and to ask whether there are examples of non-trivial pairs (α, β) satisfying Littlewood's conjecture.

8.5.2 The problem of explicit examples

Recently, in a important paper, M. Einsiedler, A. Katok and E. Lindenstrauss (Einsiedler, Katok, and Lindenstrauss 2006) used an approach based on the theory of dynamical systems to prove the following outstanding result regarding Littlewood's conjecture.

Theorem 8.5.3 *The complement of* $\mathbf{L}$ *in* $\mathbb{R}^2$ *has Hausdorff dimension zero.*

Since **Bad** has full Hausdorff dimension, Theorem 8.5.3 implies that non-trivial examples for the Littlewood conjecture do exist. In particular, for every real number α in **Bad**, there are many non-trivial pairs (α, β) in **L**. Unfortunately, this result says nothing about the following simple question:

Given a real number α in **Bad**, *can we construct explicitly a real number β such that (α, β) is a non-trivial pair satisfying the Littlewood conjecture?*

The aim of this section is to answer this question. We will use an elementary construction based on the theory of continued fractions.

Let $\alpha := [0, a_1, a_2, \ldots]$ be a real number whose partial quotients are bounded, say by an integer $M \geq 2$. With any increasing sequence of positive integers $n = (n_i)_{i\geq 1}$ and any sequence $t = (t_i)_{i\geq 1}$ taken its values in $\{M+1, M+2\}$, we associate a real number $\beta_{n,t}$ as follows. For every positive integer j, let us denote by u_j the prefix of length j of the infinite word $a_1 a_2 \cdots$ and let $\widetilde{u_j}$ be the *mirror* of u_j. Then, we set

$$\beta_{n,t} := [0, \widetilde{u_{n_1}}, t_1, \widetilde{u_{n_2}}, t_2, \widetilde{u_{n_3}}, t_3, \ldots] \, .$$

We will show that if the sequence n increases sufficiently rapidly, then the pair $(\alpha, \beta_{n,t})$ provides a non-trivial example for the Littlewood conjecture. More precisely, we will prove the following result established in (Adamczewski and Bugeaud 2006a).

Theorem 8.5.4 *Let ε be a positive real number with $\varepsilon < 1$. Keeping the previous notation and under the additional assumption that*

$$\liminf_{i \to +\infty} \frac{n_{i+1}}{n_i} > \frac{4 \log(M+3)}{\varepsilon \log 2}, \tag{8.46}$$

the pair $(\alpha, \beta_{n,t})$ is a non-trivial pair satisfying

$$q \cdot \|q\alpha\| \cdot \|q\beta_{n,t}\| \leq \frac{1}{q^{1-\varepsilon}} \cdot$$

In particular, the pair $(\alpha, \beta_{n,t})$ belongs to **L**.

Proof We keep the notation of the theorem. Let $(p_j/q_j)_{j\geq 1}$ denote the sequence of convergents to α and let $(r_j/s_j)_{j\geq 1}$ denote the sequence of convergents to $\beta_{n,t}$. Set $m_j = n_1 + n_2 + \ldots + n_j + (j-1)$.

By Lemma 8.2.4, we have

$$\frac{s_{m_j - 1}}{s_{m_j}} = [0, a_1, \ldots, a_{n_j}, t_{j-1}, a_1, \ldots, a_{n_{j-1}}, t_{j-2}, a_1, \ldots, t_1, a_1, \ldots, a_{n_1}] \, .$$

Thus, Lemma 8.2.1 implies that

$$\|s_{m_j} \alpha\| \leq s_{m_j} \, q_{n_j}^{-2}. \tag{8.47}$$

On the other hand, (8.15) ensures that

$$s_{m_j} \cdot \|s_{m_j}\beta_{n,t}\| < 1 \ . \tag{8.48}$$

Combining (8.47) and (8.48), it remains to prove that

$$s_{m_j}\, q_{n_j}^{-2} < \frac{1}{(s_{m_j})^{1-\varepsilon}} \ ,$$

that is,

$$q_{n_j} > (s_{m_j})^{1-\varepsilon/2}. \tag{8.49}$$

In order to prove (8.49), we will use the formalism of continuants introduced in Subsection 8.2.3 combined with the following simple idea: If the sequence n increases very quickly, then the word $a_1a_2\cdots a_{n_j}$ is much longer than the word $t_{j-1}a_1\cdots a_{n_{j-1}}t_{j-2}\cdots t_1a_1\cdots a_{n_1}$. Since all these integers are bounded by $M+2$, we obtain that the integer $q_{n_j} = K(a_1, a_2, \ldots, a_{n_j})$ is much larger than

$$K_j := K(t_{j-1}, a_1, \ldots, a_{n_{j-1}}, t_{j-2}, \ldots, t_1, a_1, \ldots, a_{n_1}) \ .$$

Furthermore, since Lemma 8.2.3 implies that

$$q_{n_j}K_j \le s_{m_j} \le 2q_{n_j}K_j, \tag{8.50}$$

we will get the desired result.

Let us now give more details on this computation. Since the partial quotients of $\beta_{n,t}$ are bounded by $M+2$, Lemma 8.2.2 gives that

$$K_j < (M+3)^{m_{j-1}+1}$$

and, also,

$$s_{m_j} \ge 2^{(m_j-1)/2} \ .$$

Consequently,

$$K_j \le \frac{1}{2}(s_{m_j})^{\delta_j},$$

where

$$\delta_j := \frac{m_{j-1}+1}{m_j-1}\cdot\frac{2\log(M+3)}{\log 2} + \frac{2}{m_j-1} \ .$$

On the other hand, an easy computation starting from Inequality (8.46) shows that

$$\liminf_{j\to+\infty}\frac{m_j}{m_{j-1}} > \frac{4\log(M+3)}{\varepsilon\log 2} \ .$$

This implies that $\delta_j < \varepsilon/2$ for every integer j large enough, and, consequently, that

$$K_j < \frac{1}{2} \cdot (s_{m_j})^{\varepsilon/2},$$

for every integer j large enough. Inequality (8.49) then follows from (8.50).

To end the proof, it now remains to prove that 1, α and $\beta_{n,t}$ are linearly independent over $\mathbb{Q}$. We assume that they are dependent and we aim at deriving a contradiction. In the rest of this proof, the constants implied by the symbols $\gg$ and $\ll$ do not depend on the positive integer j.

By assumption, there exists a non-zero triple of integers (a, b, c) such that

$$a\alpha + b\beta_{n,t} + c = 0 \ .$$

Thus,

$$\|s_{m_j} a\alpha\| = \|s_{m_j} b\beta_{n,t}\| \le |b| \cdot \|s_{m_j}\beta_{n,t}\| \ll \frac{1}{s_{m_j}} \ll \frac{1}{q_{n_j} K_j} \cdot \qquad (8.51)$$

We infer from Lemma 8.2.2 that

$$|s_{m_j}\alpha - s_{m_j-1}| \gg \frac{s_{m_j}}{q_{n_j}^2} = \frac{K_j}{q_{n_j}} \ .$$

Note that, for every integer j large enough, we have

$$|s_{m_j} a\alpha - s_{m_j-1} a| < \frac{1}{2},$$

thus,

$$\|s_{m_j} a\alpha\| = |s_{m_j} a\alpha - s_{m_j-1} a| = |a| \cdot |s_{m_j}\alpha - s_{m_j-1}| \gg \frac{K_j}{q_{n_j}} \ .$$

Since K_j tends to infinity with j, this contradicts (8.51) if j is large enough. This concludes the proof. □

8.6 Exercises and open problems

Exercise 8.1 Prove that there is no irrational real number ξ satisfying a stronger form of Theorem 8.4.1 in which the exponent of X in (8.32) is less than -1. You may use the theory of continued fractions.

Exercise 8.2 Prove that the pair $(\sqrt{2}, e)$ satisfies the Littlewood conjecture. You may use the continued fraction expansion of e.

Exercise 8.3 (Open problem) Prove that the pair $(\sqrt{2}, \sqrt{3})$ satisfies the Littlewood conjecture.

Exercise 8.4 Let α be an irrational real number whose continued fraction expansion begins with arbitrarily large palindromes. Prove that the Littlewood conjecture is true for the pair $(\alpha, 1/\alpha)$ and that, moreover, we have

$$\liminf_{q \to +\infty} q^2 \cdot \|q\alpha\| \cdot \|q/\alpha\| < +\infty \ .$$

Exercise 8.5 (Open problem) The base-b expansion of an algebraic irrational number cannot be generated by a morphism. To this end, it would be sufficient to establish that, if ξ is an algebraic irrational number, then

$$\limsup_{n \to +\infty} \frac{p(n, \xi, b)}{n^2} = +\infty$$

holds for every integer base $b \geq 2$.

Exercise 8.6 (Open problem) Prove that

$$\lim_{n \to +\infty} p(n, \pi, b) - n = +\infty$$

holds for every integer base $b \geq 2$.

Exercise 8.7 (Open problem) The base-b expansion of an algebraic irrational number cannot begin with arbitrarily large palindromes.

Exercise 8.8 (Open problem) The continued fraction expansion of an algebraic irrational number of degree ≥ 3 cannot be generated by a finite automaton.

8.7 Notes

Section8.1

Many other examples of normal numbers with respect to a given integer base have been worked out (see for instance (Copeland and Erdős 1946) and (Bailey and Crandall 2002)). In contrast, no natural example of a normal number seems to be known.

Conjecture 8.1.4 is sometimes attributed to É. Borel after he suggested that $\sqrt{2}$ could be normal with respect to the base 10 (Borel 1950).

A consequence of Conjecture 8.1.4 would be that the digits 0 and 1 occur with the same frequency in the binary expansion of any algebraic number. In (Bailey, Borwein, Crandall, et al. 2004) the authors proved that, an algebraic real number α of degree d being given, there exists a positive c such that, for every sufficiently large integer N, there are at least $cN^{1/d}$ non-zero digits among the first N digits of the binary expansion of α.

A famous open problem of K. Mahler (Mahler 1984) asks whether there are irrational algebraic numbers in the triadic Cantor set. This corresponds to a special instance of Conjecture 8.1.8.

As a complement to Theorem 8.1.6, Y. Bugeaud and J.-H. Evertse (Bugeaud and Evertse 2008) established that, if ξ is an algebraic irrational number, then

$$\limsup_{n\to+\infty} \frac{p(n,\xi,b)}{n(\log n)^{0.09}} = +\infty$$

holds for every integer base $b \geq 2$.

Partial results in the direction of Cobham's conjecture (Theorem 8.1.7) were obtained in (Loxton and van der Poorten 1988). A classical result of G. Christol (Christol 1979) is related to Theorem 8.1.7: given an integer q that is a power of a prime number p, a Laurent power series $\sum_{n=-k}^{\infty} a_n T^n \in \mathbb{F}_q((T))$ is algebraic over the field $\mathbb{F}_q(T)$ if, and only if, the infinite word $a_0 a_1 \cdots$ is *p-automatic* (see also the paper (Christol, Kamae, Mendès France, et al. 1980)). More references about automatic sequences and automatic real numbers can be found in the monograph (Allouche and Shallit 2003).

A recent application of Proposition 8.1.21 to repetitive patterns that should occur in the binary expansion of algebraic numbers is given in (Adamczewski and Rampersad 2008).

It was also recently observed in (Adamczewski 2009) that the transcendence of real numbers whose base-b expansion is a Sturmian word can be obtained by combining Roth's theorem with some results from (Berthé, Holton, and Zamboni 2006). As a consequence of classical Diophantine results, it follows that the complexity of the number e (and of many other classical transcendental numbers) satisfies $\lim_{n\to+\infty} p(n,e,b)-n = +\infty$. In contrast, the best lower bound for the complexity of π seems to be $p(n,\pi,b) \geq n+1$ for every positive integer n, as follows from (Morse and Hedlund 1938).

The first p-adic version of the Schmidt subspace theorem is due H. P. Schlickewei (Schlickewei 1976). A survey of recent applications of the Schmidt subspace theorem can be found in (Bilu 2008). See also (Waldschmidt 2006) or (Waldschmidt 2008) for a survey of known results about base-b expansions and continued fraction expansions of algebraic numbers.

Section8.2

A survey of recent works involving the mirror formula can be found in (Adamczewski and Allouche 2007).

Section8.3

Proposition 8.3.1 is a special instance of the main result proved in (Adamczewski and Bugeaud 2005). See also (Adamczewski and Bugeaud 2005), (Adamczewski, Bugeaud, and Davison 2006) and (Adamczewski and Bugeaud 2007b) for more general transcendence results regarding continued fractions involving repetitive patterns. These results extend in particular those obtained in (Baker 1962), (Queffélec 1998), (Queffélec 2000) and (Allouche, Davison, Queffélec, et al. 2001).

Some generalizations of Proposition 8.3.3 can be found in (Adamczewski and Bugeaud 2007c). In contrast, there are only few results about transcendental numbers whose base-b expansion involves some symmetric pattern (see (Adamczewski and Bugeaud 2006b)).

Section8.4

Theorem 8.4.4 also leads to some results (see (Roy 2003b)) related to a famous conjecture due to E. Wirsing concerning the approximation of real numbers by algebraic numbers of bounded degree (Wirsing 1960).

It was proved in (Bugeaud and Laurent 2005) that ξ and ξ^2 are uniformly simultaneously very well approximated by rational numbers when the real number ξ belongs to a large class of Sturmian continued fractions.

Section8.5

A classical result regarding the Littlewood conjecture is that any pair of algebraic numbers lying in a same cubic number field satisfies the Littlewood conjecture (Cassels and Swinnerton-Dyer 1955). Note that a weaker result than Theorem 8.5.3 was obtained previously by different techniques in (Pollington and Velani 2000).