ON SIMULTANEOUS RATIONAL APPROXIMATION TO A p–ADIC NUMBER AND ITS INTEGRAL POWERS

NATALIA BUDARINA, YANN BUGEAUD, DETTA DICKINSON, AND HUGH O'DONNELL

Abstract. Let p be a prime number. For a positive integer n and a p–adic number ξ, let $\lambda_n(\xi)$ denote the supremum of the real numbers λ such that there are arbitrarily large positive integers q such that $||q\xi||_p, ||q\xi^2||_p, \dots, ||q\xi^n||_p$ are all less than $q^{-\lambda-1}$. Here, $||x||_p$ denotes the infimum of $|x-n|_p$ as n runs through the integers. We study the set of values taken by the function λ_n.

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1. Introduction

Throughout the present paper, p denotes a prime number and $|\cdot|_p$ the usual p–adic absolute value, normalized by $|p|_p = p^{-1}$. In 1935, in order to define his classification of p–adic numbers, Mahler [11] introduced the exponents of Diophantine approximation w_n.

Definition 1. *Let $n \geq 1$ be an integer and let ξ be a p–adic number. We denote by $w_n(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_n\xi^n + \cdots + x_1\xi + x_0|_p \leq X^{-w-1}, \qquad \max_{0\leq m\leq n} |x_m| \leq X,$$

have a solution in integers $x_0, \dots, x_n$.

The p–adic version of the Dirichlet theorem implies that $w_n(\xi) \geq n$ for every p–adic number ξ which is not algebraic of degree at most n. Furthermore, it follows from the p–adic version of the Schmidt Subspace Theorem that $w_n(\xi) = \min\{n, d-1\}$ for every positive integer n and every p–adic algebraic number ξ of degree d. Moreover, Sprindžuk [15] proved that $w_n(\xi) = n$ for every $n \geq 1$ and almost every p–adic number ξ, with respect to the Haar measure; see Section 9.3 of [5] for an overview of the known results on the exponents w_n.

Another exponent of Diophantine approximation, which measures the quality of the simultaneous rational approximation to a number and its n first integral powers, has been introduced recently [7] in the real case.

Definition 2. *Let $n \geq 1$ be an integer and let ξ be a p–adic number. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that, for arbitrarily large real numbers X, the inequalities*

$$0 < |x_0| \leq X, \qquad \max_{1\leq m\leq n} |x_0\xi^m - x_m|_p \leq X^{-\lambda-1},$$

have a solution in integers $x_0, \dots, x_n$.

The p–adic version of the Dirichlet theorem implies that $\lambda_n(\xi) \geq 1/n$ for every irrational p–adic number ξ. Furthermore, it follows from the p–adic form of the Schmidt Subspace Theorem that $\lambda_n(\xi) = \max\{1/n, 1/(d-1)\}$ for every positive integer n and every p–adic algebraic number ξ of degree d. Moreover, $\lambda_n(\xi) = 1/n$ for every $n \geq 1$ and almost every p–adic number ξ.

In the present paper, by the spectrum of a function, we mean the set of values taken by this function on the set of transcendental p–adic numbers. For $n \geq 1$, the spectrum of w_n is equal to the whole interval $[n, \infty]$, but nothing seems to be known regarding the spectrum of λ_n when $n \geq 2$. We address the following question.

Problem 1. *Let $n \geq 1$ be an integer. Is the spectrum of the function λ_n equal to $[1/n, \infty]$?*

The real analogue of Problem was recently investigated in [6]. The goal of the present paper is twofold. Firstly, we show that, for any $n \geq 1$, the spectrum of the function λ_n contains the interval $[1, \infty]$, proving thereby the exact analogue of Theorem 3.4 of [6]. Secondly, we establish the p–adic analogue of the metrical result from [4].

The notation $a \gg_d b$ means that there exists a constant $c > 0$ such that $a \geq b$ and c depends only on d. When $\gg$ is written without any subscript, it means that the constant is absolute. We write $a \asymp b$ if both $a \gg b$ and $a \ll b$ hold.

2. Main results

Our first result is a p–adic analogue of Corollary 2.3 from [6], which slightly improved an old theorem of Güting [9]. This seems to be the first result of this type for p–adic numbers.

Theorem 1. *Let $n \geq 1$ be an integer. For any real number $w \geq 2n-1$, there exist uncountably many p–adic integers ξ such that*

$$w_1(\xi) = \ldots = w_n(\xi) = w.$$

The key tool for the proof is a construction inspired by the theory of continued fractions.

Proceeding as in [8] and in [6], we combine Theorem 1 and a transference principle of Mahler [12] to get our main result on the spectra of the functions λ_n.

Theorem 2. *Let $n \geq 1$ be an integer and $\lambda \geq 1$ be a real number. There are uncountably many p–adic integers ξ, which can be constructed explicitly, such that $\lambda_n(\xi) = \lambda$. In particular, the spectrum of λ_n contains the interval $[1, \infty]$.*

It is with the help of metric Diophantine approximation that we are able to show that the spectrum of w_n is equal to $[n, \infty]$. Thus, it is meaningful to try to compute the Hausdorff dimension (for background, the reader is directed to [3]) of the set of p–adic numbers ξ with a prescribed value for $\lambda_n(\xi)$. For $n = 1$, this was done by Melničuk [13], who proved that

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_1(\xi) \geq \lambda\} = \frac{2}{1+\lambda}.$$

Actually, there is a slightly more precise result [3], namely

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_1(\xi) = \lambda\} = \frac{2}{1+\lambda}.$$

In this respect, we are able to establish the p–adic analogue of Theorem 2 from [4].

Theorem 3. *Let $n \geq 2$ be an integer. Let $\lambda > n-1$ be a real number. Then,*

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)}.$$

For $n = 2$ and $1/2 \leq \lambda \leq 1$, it is expected that

$$\dim\{\xi \in \mathbb{Q}_p : \lambda_2(\xi) = \lambda\} = \frac{2-\lambda}{1+\lambda},$$

in analogy with the real case [1, 16]. We plan to investigate this problem in a subsequent work.

3. p–adic continued fractions

This section was inspired by [10].

Set

$$p_{-1} = 1, q_{-1} = 0, p_0 = 1, q_0 = 1.$$

Let $\mathbf{v} = (v_n)_{n\geq 1}$ be a sequence of positive integers and set

$$p_n = p^{v_n} p_{n-2} + p_{n-1}, \quad q_n = p^{v_n} q_{n-2} + q_{n-1}, \quad (n \geq 1).$$

A rapid calculation shows that

$$q_1 = 1, q_2 = p^{v_2} + 1, q_3 = p^{v_3} + p^{v_2} + 1, q_4 = p^{v_2+v_4} + p^{v_4} + p^{v_3} + p^{v_2} + 1,$$

and

$$\frac{p_n}{q_n} = \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{\dots + p^{v_n}}}}.$$

The reader may note the differences between these continued fractions and the classical continued fraction algorithm for real numbers. In the latter case, the convergents p_n/q_n are given by the recurrences $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$, where the partial quotients a_n are positive integers.

Observe that

$$\left| \frac{p_1}{q_1} - \frac{p_0}{q_0} \right|_p = p^{-v_1}$$

and that, for $n \geq 2$, we have

$$\begin{aligned} \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|_p &= \left| \frac{(p^{v_n} p_{n-2} + p_{n-1}) q_{n-1} - (p^{v_n} q_{n-2} + q_{n-1}) p_{n-1}}{q_n q_{n-1}} \right|_p \\ &= p^{-v_n} \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} \right|_p, \end{aligned}$$

since p does not divide $q_n q_{n-1} q_{n-2}$.

Consequently, for $n \geq 0$ and $k \geq 1$, we have

$$\left| \frac{p_{n+k}}{q_{n+k}} - \frac{p_n}{q_n} \right|_p = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p = p^{-v_{n+1}-v_n-\ldots-v_1}, \tag{1}$$

since $v_1, v_2, \ldots$ are positive. Here, we have used that

$$|a+b|_p = \max\{|a|_p, |b|_p\}$$

holds for every p–adic numbers a and b such that $|a|_p \neq |b|_p$. This fact will be repeatedly used in the course of the proof of Theorem 1.

Equalities (1) show that the sequence $(p_n/q_n)_{n\geq 1}$ converges p–adically. Let $\xi_{\mathbf{v}}$ denote its limit. It follows from (1) that

$$\left| \xi_{\mathbf{v}} - \frac{p_n}{q_n} \right|_p \leq p^{-v_{n+1}-v_n-\ldots-v_1}. \tag{2}$$

If

$$\left| \xi_{\mathbf{v}} - \frac{p_n}{q_n} \right|_p < p^{-v_{n+1}-v_n-\ldots-v_1},$$

then, by (1), we get

$$\left| \xi_{\mathbf{v}} - \frac{p_{n+1}}{q_{n+1}} \right|_p = \max\left\{ \left| \xi_{\mathbf{v}} - \frac{p_n}{q_n} \right|_p, \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p \right\} = p^{-v_{n+1}-v_n-\ldots-v_1},$$

a contradiction with (2) since $v_{n+2} \geq 1$. Consequently, we have proved that

$$\left| \xi_{\mathbf{v}} - \frac{p_n}{q_n} \right|_p = p^{-v_{n+1}-v_n-\ldots-v_1}, \quad (n \geq 1). \tag{3}$$

4. Proof of Theorem 1

Let $w > 1$ be a real number. Set $v_1 = \lceil w \rceil$ and $v_2 = \lceil w^2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. For $n \geq 3$, let v_n be the integer such that

$$v_n + v_{n-2} + \cdots + v_{\varepsilon(n)} = \lceil w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} \rceil,$$

where $\varepsilon(n) = 2$ if n is even and $\varepsilon(n) = 1$ otherwise. Let $\xi = \xi_{\mathbf{v}}$ be the p–adic number constructed by the algorithm described in Section 3 applied with $\mathbf{v} = (v_n)_{n\geq 1}$.

To shorten the notation, for $n \geq 1$, we set

$$u_n = v_n + v_{n-2} + \cdots + v_{\varepsilon(n)}.$$

Note that

$$u_n \geq u_{n-1}, \quad n \geq 2. \tag{4}$$

Observe that

$$\begin{aligned} u_n &\leq w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} + 1 \\ &\leq w(w^{n-1} + w^{n-3} + \cdots + w^{\varepsilon(n-1)}) + w + 1 \\ &\leq w u_{n-1} + w + 1, \end{aligned} \tag{5}$$

and

$$\begin{aligned} u_n &\geq w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} \\ &\geq w(w^{n-1} + w^{n-3} + \cdots + w^{\varepsilon(n-1)}) \\ &\geq w(u_{n-1} - 1) = wu_{n-1} - w. \end{aligned} \tag{6}$$

We begin with an easy lemma.

Lemma 1. *Using the above notation, we have*

$$q_j \geq p^{u_j}, \quad j \geq 2.$$

and there exists C_1, depending only on p and w, such that

$$q_j \leq C_1 p^{u_j}, \quad j \geq 1.$$

Proof. The first statement of the lemma is straightforward, since $q_j \geq p^{v_j} q_{j-2}$ for $j \geq 2$. For the second, we first check inductively that

$$q_j \leq 2^j p^{u_j}, \quad j \geq 1. \tag{7}$$

Indeed, $q_1 = 1$, $q_2 = p^{u_2} + 1$, and, assuming that (7) holds for $j = n-1$ and $j = n-2$ for an integer $n \geq 3$, we have

$$q_n \leq 2^{n-2} p^{u_n} + 2^{n-1} p^{u_{n-1}} \leq 2^n p^{u_n},$$

by (4), showing that (7) holds for $j = n$. We conclude that (7) holds for $j \geq 1$.

Let n_0 be such that

$$p^{w^n - w^{n-1}} \geq 2^{2n} p, \quad n \geq n_0,$$

and set $C_1 = 2^{n_0} + 1$. Since $u_n \geq w^n$, we have

$$p^{u_n(1-1/w)} \geq 2^n C_1 p, \quad n \geq n_0 + 1. \tag{8}$$

Furthermore, by (7), we have

$$q_n \leq (C_1 - 1) p^{u_n}, \quad 1 \leq n \leq n_0. \tag{9}$$

We prove by induction on n that

$$q_n \leq (C_1 - 1/n) p^{u_n}, \quad n \geq 1. \tag{10}$$

By (9), inequality (10) holds for $n \leq n_0$. Let $n \geq n_0 + 1$ be an integer such that (10) holds for $n-1$ and for $n-2$. Observe that, by (6) and (8),

$$2^n C_1 p^{u_{n-1}} \leq 2^n C_1 p p^{u_n/w} \leq p^{u_n},$$

thus

$$\begin{aligned} q_n = p^{v_n} q_{n-2} + q_{n-1} &\leq (C_1 - 1/(n-2)) p^{u_n} + C_1 p^{u_{n-1}} \\ &\leq (C_1 - 1/(n-2) + 2^{-n}) p^{u_n} \\ &\leq (C_1 - 1/n) p^{u_n}. \end{aligned}$$

This proves the lemma. □

Lemma 2. *With the above notation, there are positive real numbers C_2 and C_3, depending only on p and w, such that*

$$C_2 q_j^w \leq q_{j+1} \leq C_3 q_j^w, \quad j \geq 1.$$

Proof. Let j be a positive integer. By Lemma 1 and (5), we have

$$q_{j+1} \leq C_1 p^{u_{j+1}} \leq C_1 p^{wu_j+w+1} \leq (C_1 p^{w+1}) q_j^w,$$

while, by Lemma 1 and (6),

$$q_{j+1} \geq p^{u_{j+1}} \geq p^{wu_j-w} \geq (pC)^{-w} q_j^w.$$

This proves the lemma. □

We end these preliminaries with a lemma, which follows from an immediate induction.

Lemma 3. *For $j \geq 0$, we have*

$$p_j \leq (p^{v_1}+1) q_j.$$

For $j \geq 2$, it follows from (3) that

$$\left| \xi - \frac{p_j}{q_j} \right|_p = p^{-v_{j+1}-v_j-\cdots-v_1} = p^{-u_{j+1}-u_j},$$

thus, by Lemma 1, we get

$$q_j^{-1} q_{j+1}^{-1} \leq \left| \xi - \frac{p_j}{q_j} \right|_p \leq C_1^2 q_j^{-1} q_{j+1}^{-1},$$

and, by Lemma 2,

$$\frac{C_3^{-1}}{q_j^{w+1}} \leq \left| \xi - \frac{p_j}{q_j} \right|_p \leq \frac{C_1^2 C_2^{-1}}{q_j^{w+1}}. \tag{11}$$

Consequently, we get

$$w \leq w_1(\xi) \leq \cdots \leq w_d(\xi) \tag{12}$$

for every positive integer d (note that the unknown x_n occurring in the definition of w_n can be equal to 0).

Let d be a positive integer with $d < w$. Let $P(X)$ be an integer polynomial of degree at most d and of large height $H(P)$ (recall that the height of an integer polynomial is the maximum of the absolute values of its coefficients). Assume first that $P(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j\geq 1}$. Let j be defined by $q_j \leq H(P) < q_{j+1}$. Observe that, by Lemma 3, the numerator of the rational number $P(p_j/q_j)$ is at most equal to $(d+1)(p^{v_1}+1)^d H(P) q_j^d$, thus

$$|P(p_j/q_j)|_p \geq (d+1)^{-1}(p^{v_1}+1)^{-d} H(P)^{-1} q_j^{-d}.$$

To shorten the formulæ, set

$$C_4 = (d+1)^{-1}(p^{v_1}+1)^{-d}.$$

Since ξ and p_j/q_j are p–adic integers, the mean value theorem (see e.g. [14], Section 5.3) gives

$$|P(p_j/q_j) - P(\xi)|_p \le |\xi - p_j/q_j|_p \le p^{-u_{j+1}-u_j},$$

by (3). Consequently, since

$$|P(p_j/q_j)|_p \ge C_4 H(P)^{-1} q_j^{-d}$$

we get

$$|P(\xi)|_p = |P(p_j/q_j)|_p \ge C_4 H(P)^{-1-d}$$

as soon as $p^{-u_{j+1}-u_j} < C_4 H(P)^{-1} q_j^{-d}$, that is, whenever

$$H(P) < C_4 q_j^{-d} p^{u_{j+1}+u_j}. \tag{13}$$

Similarly, we observe that

$$|P(p_{j+1}/q_{j+1})|_p \ge C_4 H(P)^{-1} q_{j+1}^{-d}$$

and

$$|P(p_{j+1}/q_{j+1}) - P(\xi)|_p \le p^{-u_{j+2}-u_{j+1}} \le C_1^2 C_2^{-1} q_{j+1}^{-1-w}.$$

Since $w > d$ and $H(P) < q_{j+1}$, this implies that, if j (that is, if $H(P)$) is large enough, we have $|P(\xi)|_p \ge C_4 H(P)^{-1} q_{j+1}^{-d}$. In other words, for any positive real number $C_5 < C_4$, we have $|P(\xi)|_p \ge C_5 H(P)^{-1-w}$ if $H(P)^{-w} \le C_5^{-1} C_4 q_{j+1}^{-d}$, that is, if

$$H(P) \ge C_4^{-1/w} C_5^{1/w} q_{j+1}^{d/w}. \tag{14}$$

By Lemma 1, inequality (13) holds if

$$H(P) < C_4 q_j^{-d} C_1^{-2} q_j q_{j+1} = C_1^{-2} C_4 q_{j+1} q_j^{1-d}. \tag{15}$$

Using Lemma 2, we see that (14) certainly holds for

$$H(P) \ge C_4^{-1/w} C_5^{1/w} q_{j+1} (C_3 q_j^w)^{-1+d/w}. \tag{16}$$

Selecting C_5 such that

$$C_4^{-1/w} C_5^{1/w} C_3^{-1+d/w} < C_4 C_1^{-2},$$

we get that, if $1 - d \ge -w + d$, then for every polynomial $P(X)$ whose height is in the interval $[q_j, q_{j+1})$ at least one of the inequalities (15) and (16) is satisfied. This means that the whole range of values $q_j \le H(P) < q_{j+1}$ is covered as soon as

$$w \ge 2d - 1. \tag{17}$$

To summarize, we have proved that, if j is sufficiently large, then, for $w \ge 2d-1$ and for any polynomial $P(X)$ of degree at most d that does not vanish at p_j/q_j and whose height satisfies $q_j \le H(P) < q_{j+1}$, we have

$$|P(\xi)|_p \ge C_5 H(P)^{-w-1}.$$

In particular, if the polynomial $P(X)$ of degree at most d does not vanish at any element of the sequence $(p_j/q_j)_{j\ge1}$ and has sufficiently large height, then it satisfies

$$|P(\xi)|_p \ge C_5 H(P)^{-w-1}. \tag{18}$$

Assume now that there are positive integers $a_1, \ldots, a_h$, distinct positive integers $n_1, \ldots, n_h$ and an integer polynomial $R(X)$ such that the polynomial $P(X)$ of degree at most d can be written as

$$P(X) = (q_{n_1}X - p_{n_1})^{a_1} \ldots (q_{n_h}X - p_{n_h})^{a_h} R(X),$$

where $R(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j\geq 1}$. It follows from (11), (18), Lemma 3, and the so–called Gelfond inequality (see, e.g., [5], Lemma A.3)

$$H(P) \asymp_{d,w} q_{n_1}^{a_1} \ldots q_{n_h}^{a_h} H(R)$$

that

$$\begin{aligned} |P(\xi)|_p &\gg_{d,w} q_{n_1}^{-a_1(w+1)} \ldots q_{n_h}^{-a_h(w+1)} |R(\xi)|_p \\ &\gg_{d,w} q_{n_1}^{-a_1(w+1)} \ldots q_{n_h}^{-a_h(w+1)} H(R)^{-w-1} \\ &\gg_{d,w} \big(q_{n_1}^{a_1} \ldots q_{n_h}^{a_h} H(R)\big)^{-w-1} \gg_{d,w} H(P)^{-w-1}. \end{aligned}$$

We conclude that, if (17) is satisfied, then

$$|P(\xi)|_p \gg_{d,w} H(P)^{-w-1}$$

holds for every polynomial $P(X)$ of degree at most d and sufficiently large height, hence $w_d(\xi) \leq w$. Combined with (12), this completes the proof of Theorem 1, since our construction is flexible enough to yield uncountably many p–adic integers with the required property. □

5. Proof of Theorem 2

Let ξ be an irrational p–adic number. Clearly, we have

$$\lambda_1(\xi) = w_1(\xi) \geq 1$$

and

$$\lambda_1(\xi) \geq \lambda_2(\xi) \geq \ldots \tag{19}$$

Our first lemma establishes a relation between the exponents λ_n and λ_m when m divides n.

Lemma 4. *For any positive integers k and n, and any transcendental p–adic number ξ we have*

$$\lambda_{kn}(\xi) \geq \frac{\lambda_k(\xi) - n + 1}{n}.$$

Proof. Let v be a positive real number and q be a positive integer such that

$$\max_{1\leq j\leq k} |q\xi^j - p_j|_p \leq q^{-v-1},$$

for suitable integers $p_1, \ldots, p_k$. Let h be an integer with $1 \leq h \leq kn$. Write $h = j_1 + \cdots + j_m$ with $m \leq n$ and $1 \leq j_1, \ldots, j_m \leq k$. Then, there are p–adic numbers $\varepsilon_1, \ldots, \varepsilon_m$ such that

$$|\varepsilon_i|_p \leq q^{-v-1}, \quad q\xi^{j_i} = p_{j_i} + \varepsilon_i, \quad (i = 1, \ldots, m).$$

Consequently, we have

$$q^m \xi^h = \prod_{i=1}^m q\xi^{j_i} = \prod_{i=1}^m (p_{j_i} + \varepsilon_i) = \varepsilon' + \prod_{i=1}^m p_{j_i},$$

for a p–adic number ε' satisfying $|\varepsilon'|_p \le q^{-v-1}$. This shows that

$$|q^m \xi^h - p_{j_1} \dots p_{j_m}|_p \le q^{-v-1}$$

and

$$|q^n \xi^h - p_{j_1} \dots p_{j_m} q^{n-m}|_p \le q^{-v-1} = (q^n)^{-1-(v-n+1)/n},$$

independently of h. This proves the lemma. □

We display an immediate consequence of Lemma 4.

Corollary 1. *Let ξ be a p–adic irrational number. Then, $\lambda_n(\xi) = \infty$ holds for every positive n if, and only if, $\lambda_1(\xi) = \infty$.*

We recall two relations between the exponents w_n and λ_n deduced from the p–adic analogue of Khintchine's transference principle due to Mahler [12].

Proposition 1. *For any positive integer n and any p–adic number ξ which is not algebraic of degree at most n, we have*

$$\frac{w_n(\xi)}{(n-1)w_n(\xi) + n} \le \lambda_n(\xi) \le \frac{w_n(\xi) - n + 1}{n}.$$

Proof. See [12]. Note that the value of $w_n(\xi)$ does not change, if, in Definition 1, we only consider tuples $(x_0, x_1, \dots, x_n)$ such that there exists at least one index i for which p does not divide x_i. Similarly, the value of $\lambda_n(\xi)$ does not change, if, in Definition 2, we only consider tuples $(x_0, x_1, \dots, x_n)$ such that p does not divide x_0. □

We are now able to complete the proof of Theorem 2.

Proof of Theorem 2. Let $n \ge 2$ be an integer and ξ be a transcendental p–adic number. Lemma 4 with $k = 1$ implies the lower bound

$$\lambda_n(\xi) \ge \frac{w_1(\xi) - n + 1}{n}. \tag{20}$$

On the other hand, Proposition 1 gives the upper bound

$$\lambda_n(\xi) \le \frac{w_n(\xi) - n + 1}{n}.$$

Now, Theorem 1 asserts that for any given real number $w \ge 2n-1$, there exist uncountably many p–adic integers ξ_w such that

$$w_1(\xi_w) = \dots = w_n(\xi_w) = w.$$

Then,

$$\lambda_k(\xi_w) = \frac{w}{k} - 1 + \frac{1}{k}, \quad k = 1, \dots, n.$$

In particular,

$$\lambda_n(\xi_w) = \frac{w}{n} - 1 + \frac{1}{n},$$

and this gives the required result. □

6. Proof of Theorem 3

As $\mathbb{Q}_p$ can be covered by a countable collection of balls of radius 1 we will only prove the theorem for one such ball, namely $\mathbb{Z}_p$. The arguments are the same for any other ball but some of the constants will change. The proof follows that of [4]. Fix an integer $n \geq 2$. Define the curve $\Gamma \subset \mathbb{Z}_p^n$ as $\Gamma = \{(\xi, \xi^2, \ldots, \xi^n) : \xi \in \mathbb{Z}_p\}$. We will use the notation $|a, b, c|$ to denote the maximum of $|a|$, $|b|$ and $|c|$. If $\mathbf{a}$ is a vector then $|\mathbf{a}|$ is the maximum of the vector entries. The set of points $(\xi, \xi^2, \ldots, \xi^n) \in \Gamma$ which satisfy the inequalities $|q\xi - r|_p \leq |q, r, \mathbf{t}|^{-\tau}$ and $|q\xi^i - t_i|_p \leq |q, r, \mathbf{t}|^{-\tau}$ for infinitely many $q, r \in \mathbb{Z}$ and $\mathbf{t} \in \mathbb{Z}^{n-1}$ will be denoted by $W_\tau(\Gamma)$. The set $W_\tau(\Gamma)$ is closely related to the set of exact order in the statement of Theorem 3 and in order to prove the theorem we will first obtain the Hausdorff dimension and measure of $W_\tau(\Gamma)$ for sufficiently large τ. The proof relies on the following lemma which shows that if $(\xi, \xi^2, \ldots, \xi^n) \in W_\tau(\Gamma)$ then the rational points $(r/q, \mathbf{t}/q)$ also lie on Γ for τ sufficiently large.

Lemma 5. *Let $(\xi, \xi^2, \ldots, \xi^n) \in W_\tau(\Gamma)$ be such that there exist infinitely many $D, r \in \mathbb{Z}$, $\mathbf{t} \in \mathbb{Z}^{n-1}$ such that $|D\xi - r|_p < |D, r, \mathbf{t}|^{-\tau}$ and $|D\xi^i - t_i|_p < |D, r, \mathbf{t}|^{-\tau}$. If $\tau > n$, then $(r/D, \mathbf{t}/D) \in \Gamma$.*

Proof. Let $(\xi, \xi^2, \ldots, \xi^n) \in W_\tau(\Gamma)$. Hence there exist integers r, t_i and D such that $|D\xi - r|_p < |D, r, \mathbf{t}|^{-\tau}$ and $|D\xi^i - t_i|_p < |D, r, \mathbf{t}|^{-\tau}$. Therefore, $|\xi - r/D|_p < |D, r, \mathbf{t}|^{-\tau}|D|_p^{-1}$ and $|\xi^i - \mathbf{t}/D|_p < |D, r, \mathbf{t}|^{-\tau}|D|_p^{-1}$ and there exist $\varepsilon_1, \ldots, \varepsilon_n$, such that $\xi - r/D = \varepsilon_1$ and $\xi^i - t_i/D = \varepsilon_i$ for $i = 2, \ldots, n$ with $|\varepsilon_i|_p < |D, r, \mathbf{t}|^{-\tau}|D|_p^{-1}$. Then,

$$\xi^i = t_i/D + \varepsilon_i = (r/D + \varepsilon_1)^i = (r/D)^i + R(\varepsilon_1)$$

where $R(X)$ is a rational polynomial divisible by X. Hence, $t_i/D - (r/D)^i = R(\varepsilon_1) - \varepsilon_i$ so that

$$D^{i-1}t_i - r^i = D^i(R(\varepsilon_1) - \varepsilon_i).$$

Clearly, $D^{i-1}R(X) \in \mathbb{Z}[X]$, so that $|D^iR(\varepsilon_1)|_p \leq |D|_p|\varepsilon_1|_p < |D, r, \mathbf{t}|^{-\tau}$. Thus,

$$|D^{i-1}t_i - r^i|_p \leq |D, r, \mathbf{t}|^{-\tau}.$$

Since $D^{i-1}t_i - r^i$ is an integer, its p–adic absolute value is either 0, or at least equal to $|D^{i-1}t_i - r^i|^{-1}$. Combined with our assumption that τ exceeds n, the above inequality shows that $D^{i-1}t_i = r^i$ for $i = 2, \ldots, n$. This implies that $(r/D, \mathbf{t}/D)$ lies on Γ, as asserted. □

Define the point P_{rq} as

$$P_{rq} = \left(\frac{r}{q}, \ldots, \frac{r^n}{q^n}\right) = \left(\frac{rq^{n-1}}{q^n}, \ldots, \frac{r^n}{q^n}\right).$$

If the highest common factor of r and q is 1 then the lowest common denominator of the coordinates of P_{rq} is q^n. On the other hand, if $(r,q) = h > 1$ then we can write $r = r_1 h$ and $q = q_1 h$ so that

$$P_{rq} = \left(\frac{r_1 q_1^{n-1}}{q_1^n}, \ldots, \frac{r_1^n}{q_1^n}\right) = P_{r_1 q_1}.$$

We may therefore assume without loss of generality that $(r,q) = 1$. If $\Xi = (\xi, \xi^2, \ldots, \xi^n) \in W_\tau(\Gamma)$ and $\tau > n$, then Lemma 5 asserts that Ξ must be approximated by infinitely many points P_{rq} with $(r,q) = 1$ and must satisfy the inequalities $|q^n\xi - rq^{n-1}|_p < |q^n, r^n|^{-\tau}$, $|q^n\xi^2 - r^2q^{n-2}|_p < |q^n, r^n|^{-\tau}$, $\ldots$, $|q^n\xi^n - r^n|_p < |q^n, r^n|^{-\tau}$.

The proof of the theorem now follows that in [4]. First, we move from the set $W_\tau(\Gamma)$ to the set

$$V_\tau(\Gamma) = \{\xi \in \mathbb{Z}_p : (\xi, \xi^2, \ldots, \xi^n) \in W_\tau(\Gamma)\}.$$

It is not difficult to show that for all ξ_1, ξ_2 in $\mathbb{Z}_p$, we have

$$|\xi_1 - \xi_2|_p = \max_{i=1,\ldots,n} |\xi_1^i - \xi_2^i|_p.$$

Thus, there is a bi–Lipschitz transformation between any ball $B(\xi, r) \subset \mathbb{Z}_p$ and the image of that ball on Γ. To determine the Hausdorff dimension of $W_\tau(\Gamma)$ it is therefore enough to find the Hausdorff dimension of $V_\tau(\Gamma)$. It can be readily verified that the following inclusions hold for $V_\tau(\Gamma)$

$$\bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B(r/q, |r^n, q^n|^{-\tau}) \subset V_\tau(\Gamma) \subset \bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B(r/q, |r^n, q^n|^{-\tau}|q^n|_p^{-1}). \tag{21}$$

To prove the exact order result it is necessary to obtain dimension and measure results for $W_\tau(\Gamma)$. The fact that $\dim W_\tau(\Gamma) = \dim V_\tau(\Gamma) \geq \frac{2}{n\tau}$ and the fact that the Hausdorff $2/n\tau$ measure is infinite follows directly from Theorem 16 in [2] by using the LHS of (21) and putting $\psi(r) = r^{-n\tau}$ and $f(r) = r^s$. It is therefore only necessary to prove the upper bound for the Hausdorff dimension.

Lemma 6. *For any $n \geq 2$ and $\tau > n$ we have*

$$\dim V_\tau(\Gamma) \leq \frac{2}{n\tau}.$$

The proof follows that of [4, Lemma 2]. Using the RHS of (21) gives a cover of $V_\tau(\Gamma)$ so that

$$\begin{aligned}
\mathcal{H}^s(V_\tau(\Gamma)) &\ll \sum_{k>N} \sum_{r,q:\max(r,q)=k} |r^n, q^n|^{-\tau s}|q^n|_p^{-s} \\
&\ll \sum_{k>N} \left(\sum_{r,q:\max(r,q)=q=k} |r^n, q^n|^{-\tau s}|q^n|_p^{-s} + \sum_{r,q:\max(r,q)=r=k} |r^n, q^n|^{-\tau s}|q^n|_p^{-s} \right) \\
&\ll \sum_{k>N} \left(k k^{-n\tau s}|k|_p^{-ns} + k^{-\tau ns} \sum_{q=1}^{k} |q|_p^{-ns} \right).
\end{aligned}$$

Consider the second sum first and let α be such that $p^\alpha \le k < p^{\alpha+1}$. Then, as $|k|_p = 1$ if p does not divide k, we have

$$\begin{aligned}
\sum_{k>N} k^{-\tau ns} \sum_{q=1}^{k} |q|_p^{-ns} &= \sum_{k>N} k^{-\tau ns} \left(\sum_{q\le k, p\not| q} 1 + \sum_{q \le k: p|q \text{ and } p^2 \not| q} p^{ns} + \cdots + \sum_{q\le k : p^\alpha | q} p^{\alpha ns} \right) \\
&\ll \sum_{k>N} k^{-\tau ns} \left(k + \frac{k}{p} p^{ns} + \frac{k}{p^2} p^{2ns} + \cdots + \frac{k}{p^\alpha} p^{\alpha ns} \right) \\
&\ll \sum_{k>N} k^{1-\tau ns} \left(\sum_{i=0}^{\alpha} p^{i(ns-1)} \right) \ll \sum_{k>N} k^{ns-\tau ns} < \infty
\end{aligned}$$

for $s > \frac{1}{n\tau - n}$. Clearly, for $\tau > n \ge 2$, $\frac{2}{n\tau} > \frac{1}{n\tau-n}$ so for $s > \frac{2}{n\tau}$ the series converges. Now, using the same arguments consider the first sum to obtain

$$\begin{aligned}
\sum_{k>N} k k^{-n\tau s} |k|_p^{-ns} &\ll \sum_{k>N : p \not| k} k^{1-n\tau s} + \sum_{r>N : p\not| r} (pr)^{1-n\tau s} p^{ns} + \sum_{r>N: p \not| r} (p^2 r)^{1-n\tau s} p^{2ns} + \dots \\
&\ll \sum_{k>N} k^{1-n\tau s} \sum_{i=0}^{\infty} p^{i(1+ns-n\tau s)}
\end{aligned}$$

The last geometric series again converges if $s > \frac{1}{n\tau-n}$. Thus for $s > \frac{2}{n\tau}$ both sums converge which is enough to prove $\dim W_\tau(\Gamma) = \dim V_\tau(\Gamma) \le \frac{2}{n\tau}$ for $\tau > n$. □

It is now possible to obtain the dimension of the set

$$E_\lambda := \{\xi \in \mathbb{Z}_p : \lambda_n(\xi) = \lambda\},$$

when λ exceeds $n-1$. Clearly, $E_\lambda \subset W_{\lambda+1}(\Gamma)$ so that

$$\dim E_\lambda \le \frac{2}{n(1+\lambda)},$$

by Lemma 6. Note that

$$E_\lambda = \lim_{n\to\infty} W_{\lambda+1}(\Gamma) \setminus W_{\lambda+1+1/n}(\Gamma).$$

Also, $\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1}(\Gamma)) = \infty$, ([2, Theorem 16]) and $\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1+1/n}(\Gamma)) = 0$ from the definition of Hausdorff dimension. Thus,

$$\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1}(\Gamma) \setminus W_{\lambda+1+1/n}(\Gamma)) = \infty,$$

which implies that

$$\dim E_\lambda \ge \frac{2}{n(1+\lambda)}.$$

This proves Theorem 3. □

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Yann Bugeaud
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 STRASBOURG
France
bugeaud@math.unistra.fr

Natalia Budarina, Detta Dickinson & Hugh O'Donnell
Department of Mathematics
Logic House
NUI Maynooth
Co. Kildare
Republic of Ireland
ddickinson@maths.nuim.ie