# On transfer inequalities in Diophantine approximation, II

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**Abstract.** Let $\Theta$ be a point in $\mathbf{R}^n$. We are concerned with the approximation to $\Theta$ by rational linear subvarieties of dimension $d$ for $0 \le d \le n-1$. To that purpose, we introduce various convex bodies in the Grassmann algebra $\Lambda(\mathbf{R}^{n+1})$. It turns out that our convex bodies in degree $d$ are the $d$-th compound, in the sense of Mahler, of convex bodies in degree one. A dual formulation is also given. This approach enables us both to split and to refine the classical Khintchine transference principle.

## 1. Introduction

Let $n$ be a positive integer and let $\Theta = (\theta_1, \ldots, \theta_n)$ be a point in $\mathbf{R}^n$. We shall assume in all the forthcoming statements that the real numbers $1, \theta_1, \ldots, \theta_n$ are linearly independent over the field $\mathbf{Q}$ of rational numbers. Khintchine's transference principle relates the sharpness of the rational simultaneous approximation to $\theta_1, \ldots, \theta_n$ with the measure of linear independence over $\mathbf{Q}$ of $1, \theta_1, \ldots, \theta_n$. Let us first quantify these notions by introducing the exponents $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$ (the meaning of the indices 0 and $n-1$ will be explained afterwards).

**Definition 1.** *We denote respectively by $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$ the supremum, possibly infinite, of the real numbers $\omega$ for which there exist infinitely many integer $(n+1)$-tuples $(x_0, \ldots, x_n)$ satisfying respectively the inequation*

$$\max_{1\le i\le n} |x_0\theta_i - x_i| \le \Big(\max_{0\le i\le n} |x_i|\Big)^{-\omega} \quad \textit{or} \quad |x_0 + x_1\theta_1 + \cdots + x_n\theta_n| \le \Big(\max_{0\le i\le n} |x_i|\Big)^{-\omega}.$$

Now we can state Khintchine's transference principle [12] (see [15] for an alternative proof, and the monographs [6, 18, 8]) as follows:

**Theorem K.** *The inequalities*

$$\frac{\omega_{n-1}(\Theta)}{(n-1)\omega_{n-1}(\Theta) + n} \le \omega_0(\Theta) \le \frac{\omega_{n-1}(\Theta) - n + 1}{n} \tag{1.1}$$

*hold for any point $\Theta = (\theta_1, \ldots, \theta_n)$ in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$.*

Moreover, Jarník [10, 11] established that both inequalities in (1.1) are optimal, and, consequently, that Theorem K is best possible. It is the main purpose of the present paper to show that, however, Theorem K can be refined if we introduce two further quantities associated with $\Theta$.

Following the general "hat" notations of [3], let us introduce the uniform analogues of the exponents $\omega_0(\Theta)$ and $\omega_{n-1}(\Theta)$.

**Definition 2.** *We denote respectively by $\hat{\omega}_0(\Theta)$ and $\hat{\omega}_{n-1}(\Theta)$ the supremum of the real numbers $\omega$ such that for all sufficiently large real number $X$, there exists a non-zero integer $(n+1)$-tuples $(x_0, \ldots, x_n)$ with supremum norm*

$$\max_{0 \leq i \leq n} |x_i| \leq X,$$

*satisfying respectively the inequation*

$$\max_{1 \leq i \leq n} |x_0\theta_i - x_i| \leq X^{-\omega} \quad \textit{or} \quad |x_0 + x_1\theta_1 + \cdots + x_n\theta_n| \leq X^{-\omega}.$$

We establish the following refinement of Khintchine's theorem, which involves the uniform exponents associated with $\Theta$.

**Theorem 1.** *Suppose $n \geq 2$. The inequalities*

$$\frac{(\hat{\omega}_{n-1}(\Theta) - 1)\omega_{n-1}(\Theta)}{((n-2)\hat{\omega}_{n-1}(\Theta) + 1)\omega_{n-1}(\Theta) + (n-1)\hat{\omega}_{n-1}(\Theta)} \leq \omega_0(\Theta)$$

*and*

$$\omega_0(\Theta) \leq \frac{(1 - \hat{\omega}_0(\Theta))\omega_{n-1}(\Theta) - n + 2 - \hat{\omega}_0(\Theta)}{n-1}$$

*hold for any point $\Theta = (\theta_1, \ldots, \theta_n)$ in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$.*

The above inequalities are stronger than (1.1), since

$$\hat{\omega}_{n-1}(\Theta) \geq n \quad \text{and} \quad \hat{\omega}_0(\Theta) \geq \frac{1}{n},$$

by the Dirichlet Box Principle. Theorem 1 was first established when $n = 2$ in [13] and its statement was announced in [5] and in [14]. It follows from the description given in [13] of the set of all possible quadruples

$$\big(\omega_1(\Theta), \omega_0(\Theta), \hat{\omega}_1(\Theta), \hat{\omega}_0(\Theta)\big),$$

where $\Theta$ ranges over $\mathbf{R}^2$, that Theorem 1 is optimal in dimension two.

Theorem K was extended by Dyson [7] to transfer inequalities between approximation to a system of linear forms and approximation of the tranpose system. It would be interesting to establish a suitable extension of Theorem 1.

The present paper is organized as follows. In Section 2, we define further exponents $\omega_d(\Theta)$ for $d = 1, \dots, n-2$, measuring the accuracy with which $\Theta$ can be approximated by rational linear subvarieties of dimension $d$. We state in Theorems 2 and 3 transference inequalities linking $\omega_d(\Theta)$ and $\omega_{d+1}(\Theta)$, the composition of which gives Theorem K. This was already known [17, 14], but our proof, based on the second theorem of Minkowski, is new. Furthermore, our method allows us to refine inequalities between $\omega_0(\Theta)$ and $\omega_1(\Theta)$ (resp. between $\omega_{n-1}(\Theta)$ and $\omega_{n-2}(\Theta)$), by taking also $\hat{\omega}_0(\Theta)$ (resp. $\hat{\omega}_{n-1}(\Theta)$) into account. Using this, we get Theorem 1, as is explained in Section 7. Section 3 is devoted to some preliminaries of multilinear algebra. In Section 4 and at the beginning of Section 6, we give alternative definitions of the exponents $\omega_d$. Theorems 2 and 3 are established in Sections 5 and 6, respectively.

## 2. Going-up and going-down transfers

It is convenient to view $\mathbf{R}^n$ as a subset of $\mathbf{P}^n(\mathbf{R})$ via the usual embedding $(x_1, \dots, x_n) \mapsto (1, x_1, \dots, x_n)$. We shall identify $\Theta = (\theta_1, \dots, \theta_n)$ with its image in $\mathbf{P}^n(\mathbf{R})$.

Following [14], let us introduce for each integer $d$ with $0 \leq d \leq n-1$ an exponent $\omega_d(\Theta)$ which measures the approximation to the point $\Theta \in \mathbf{P}^n(\mathbf{R})$ by rational linear projective subvarieties of dimension $d$, in terms of their height. Denote by $\mathrm{d}$ the projective distance on $\mathbf{P}^n(\mathbf{R})$ (it will be defined in §4 below ; notice however that the normalization used there does not matter for our purpose). For any real linear subvariety $L$ of $\mathbf{P}^n(\mathbf{R})$, we denote by

$$\mathrm{d}(\Theta, L) = \min_{P \in L} \mathrm{d}(\Theta, P)$$

the minimal distance between $\Theta$ and the real points $P$ of $L$. When $L$ is rational over $\mathbf{Q}$, we indicate moreover by $H(L)$ its height, that is the Weil height of any system of Plücker coordinates of $L$. It is convenient to normalize the height by using the Euclidean norm at the Archimedean place of $\mathbf{Q}$. We refer to §1 of [17] for more information on the notion of height of a linear subspace.

**Definition 3.** *Let $d$ be an integer with $0 \leq d \leq n-1$. We denote by $\omega_d(\Theta)$ the supremum of the real numbers $\omega$ for which there exist infinitely many rational linear subvarieties $L \subset \mathbf{P}^n(\mathbf{R})$ such that*

$$\dim(L) = d \quad \textit{and} \quad \mathrm{d}(\Theta, L) \leq H(L)^{-1-\omega}.$$

Definitions 1 and 3 are consistent, since $\mathrm{d}(\Theta, L)$ compares respectively with

$$\max_{1 \leq i \leq n} \left| \theta_i - \frac{x_i}{x_0} \right| \quad \text{and} \quad \frac{|y_0 + y_1\theta_1 + \cdots + y_n\theta_n|}{\max_{0 \leq i \leq n} |y_i|}$$

when $L$ is either the rational point (case $d = 0$) with homogeneous coordinates $(1, x_1/x_0, \ldots, x_n/x_0)$, or the hyperplane (when $d = n-1$) with homogeneous equation $y_0X_0 + \cdots + y_nX_n = 0$.

Theorem 1 is a consequence of the following two statements.

**Theorem 2 (Going-up transfer).** *Let $\Theta = (\theta_1, \ldots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integer $d$ with $0 \le d \le n-2$, we have the lower bound*

$$\omega_{d+1}(\Theta) \ge \frac{(n-d)\omega_d(\Theta) + 1}{n-d-1}. \tag{2.1}$$

*Furthermore,*

$$\omega_1(\Theta) \ge \frac{\omega_0(\Theta) + \hat{\omega}_0(\Theta)}{1 - \hat{\omega}_0(\Theta)}. \tag{2.2}$$

**Theorem 3 (Going-down transfer).** *Let $\Theta = (\theta_1, \ldots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integer $d$ with $1 \le d \le n-1$, we have the lower bound*

$$\omega_{d-1}(\Theta) \ge \frac{d\,\omega_d(\Theta)}{\omega_d(\Theta) + d + 1}. \tag{2.3}$$

*Furthermore,*

$$\omega_{n-2}(\Theta) \ge \frac{(\hat{\omega}_{n-1}(\Theta) - 1)\omega_{n-1}(\Theta)}{\omega_{n-1}(\Theta) + \hat{\omega}_{n-1}(\Theta)}. \tag{2.4}$$

The lower bounds (2.1) and (2.3) are implicit in [17] and are stated in [14]. It is shown in [14] that their composition produces Khintchine's theorem. The same splitting principle is used here. We prove Theorem 1 in §7 by iterating successively the finer Going-up estimates (2.2) and (2.1), and in the other direction the Going-down inequalities (2.4) and (2.3). Note that $\hat{\omega}_0(\Theta) \le 1$ if at least one of the $\theta_k$'s is irrational [12]. The right hand side of (2.2) is then understood to be $+\infty$ when $\hat{\omega}_0(\Theta) = 1$.

In contrast with the previous works [13, 14, 17], our approch is based here on the use of the second theorem of Minkowski on the successive minima of a convex body, combined with Mahler's theory of compound convex bodies [16].

We conclude this section by formulating the transfer inequalities between $\omega_d(\Theta)$ and $\omega_{d'}(\Theta)$ that easily follow from repeated applications of (2.1) and (2.3).

**Corollary 1.** *Let $\Theta = (\theta_1, \ldots, \theta_n)$ be in $\mathbf{R}^n$ with $1, \theta_1, \ldots, \theta_n$ linearly independent over $\mathbf{Q}$. For any integers $d, d'$ with $0 \le d < d' \le n-1$, we have*

$$\frac{(d+1)\omega_{d'}(\Theta)}{(d'-d)\omega_{d'}(\Theta) + d' + 1} \le \omega_d(\Theta) \le \frac{(n-d')\omega_{d'}(\Theta) - d' + d}{n-d}.$$

## 3. Multilinear algebra

We collect in this section some classical results of multilinear algebra and their geometrical interpretation in terms of join and intersection of linear varieties in the space $\mathbf{R}^{n+1}$. For more details, we refer to [1].

First, we equip the real vector space $\mathbf{R}^{n+1}$ with the usual scalar product

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_{n+1}y_{n+1}, \quad \mathbf{x} = (x_1, \ldots, x_{n+1}), \quad \mathbf{y} = (y_1, \ldots, y_{n+1}),$$

and extend it naturally to the Grassmann algebra $\Lambda(\mathbf{R}^{n+1})$, by requiring that for any orthonormal basis $\{\mathbf{e}_i\}_{1\le i\le n+1}$ of $\mathbf{R}^{n+1}$, the family of wedge products

$$\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_r}\,; \quad 1\le i_1<\cdots<i_r\le n+1,\ 0\le r\le n+1,$$

is an orthonormal basis of $\Lambda(\mathbf{R}^{n+1})$. Then, the Cauchy-Binet formula shows that

$$\mathbf{X}\cdot\mathbf{Y} = \det\Big(\mathbf{x}_i\cdot\mathbf{y}_j\Big)_{1\le i,j\le r} \tag{3.1}$$

for any pair of decomposable $r$-vectors $\mathbf{X} = \mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_r$ and $\mathbf{Y} = \mathbf{y}_1\wedge\ldots\wedge\mathbf{y}_r$. The scalar product $\cdot$ enables us to identify the dual of the real vector space $\Lambda^r(\mathbf{R}^{n+1})$ with itself. For any multivector $\mathbf{X}\in\Lambda(\mathbf{R}^{n+1})$, we denote by $|\mathbf{X}| = \sqrt{\mathbf{X}\cdot\mathbf{X}}$ the Euclidean norm of $\mathbf{X}$.

Let $\mathbf{X}\in\Lambda^r(\mathbf{R}^{n+1})$ and $\mathbf{Y}\in\Lambda^s(\mathbf{R}^{n+1})$ be two multivectors of respective degree $r$ and $s$ with $s\le r$. We define the *internal product* (also called *contraction* ) of $\mathbf{X}$ by $\mathbf{Y}$, as the unique multivector

$$\mathbf{Y}\lrcorner\mathbf{X}\in\Lambda^{r-s}(\mathbf{R}^{n+1})$$

for which the equality

$$\mathbf{Z}\cdot(\mathbf{Y}\lrcorner\mathbf{X}) = (\mathbf{Z}\wedge\mathbf{Y})\cdot\mathbf{X} \tag{3.2}$$

holds for any $\mathbf{Z}\in\Lambda^{r-s}(\mathbf{R}^{n+1})$. In other words, the application $\mathbf{X}\mapsto\mathbf{Y}\lrcorner\mathbf{X}$ is the transpose of the linear map $\mathbf{Z}\mapsto\mathbf{Z}\wedge\mathbf{Y}$ with respect to the dot pairing.

Assume now that $\mathbf{X} = \mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_r$ and $\mathbf{Y} = \mathbf{y}_1\wedge\ldots\wedge\mathbf{y}_s$ are decomposable multivectors. When $s = 1$, we deduce from (3.1) and (3.2) the explicit formula

$$\mathbf{y}\lrcorner\mathbf{X} = \sum_{j=1}^{r}(-1)^{r-j}(\mathbf{y}\cdot\mathbf{x}_j)\mathbf{x}_1\wedge\ldots\wedge\hat{\mathbf{x}}_j\wedge\ldots\wedge\mathbf{x}_r \tag{3.3}$$

for any vector $\mathbf{y}\in\Lambda^1(\mathbf{R}^{n+1})$. It formally follows from (3.2) that

$$(\mathbf{Y}\wedge\mathbf{Y}')\lrcorner\mathbf{X} = \mathbf{Y}\lrcorner(\mathbf{Y}'\lrcorner\mathbf{X}) \tag{3.4}$$

for any pair of multivectors $\mathbf{Y}$ and $\mathbf{Y}'$ with respective degree $s$ and $s'$ such that $s+s'\leq r$. Starting with (3.3) and using (3.4), we obtain by induction on $s$ the formula

$$\mathbf{Y} \lrcorner \mathbf{X} = \sum \operatorname{sgn}(\sigma)(\mathbf{y}_1 \cdot \mathbf{x}_{\sigma(r-s+1)}) \cdots (\mathbf{y}_s \cdot \mathbf{x}_{\sigma(r)})\mathbf{x}_{\sigma(1)} \wedge \ldots \wedge \mathbf{x}_{\sigma(r-s)} \tag{3.5}$$

where the sum is taken over all the substitutions $\sigma$ of $\{1,\ldots,r\}$ such that $\sigma(1)<\cdots<\sigma(r-s)$.

Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_{n+1}\}$ be any positively oriented (meaning that $\det(\mathbf{e}_1,\ldots,\mathbf{e}_{n+1})=1$) orthonormal basis of $\mathbf{R}^{n+1}$. Remark that the volume form $\mathbf{e}_1\wedge\ldots\wedge\mathbf{e}_{n+1}$ does not depend upon the choice of such a basis.

**Definition 4.** *For every $\mathbf{X}$ in $\Lambda^r(\mathbf{R}^{n+1})$, we denote by*

$$*\mathbf{X} = \mathbf{X} \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1}) \in \Lambda^{n+1-r}(\mathbf{R}^{n+1})$$

*the Hodge dual of $\mathbf{X}$.*

Expanding

$$\mathbf{X} = \sum_{1\leq i_1<\cdots<i_r\leq n+1} X_{i_1,\ldots,i_r}\mathbf{e}_{i_1}\wedge\ldots\wedge\mathbf{e}_{i_r}$$

in the induced orthonormal basis of $\Lambda^r(\mathbf{R}^{n+1})$, we find

$$*\mathbf{X} = \sum_{1\leq i_1<\cdots<i_r\leq n+1} \varepsilon_{i_1,\ldots,i_r} X_{i_1,\ldots,i_r}\mathbf{e}_{j_1}\wedge\ldots\wedge\mathbf{e}_{j_{n+1-r}},$$

where $\{j_1,\ldots,j_{n+1-r}\} = \{1,\ldots,n+1\}\setminus\{i_1,\ldots,i_r\}$ with $j_1<\cdots<j_{n+1-r}$, and $\varepsilon_{i_1,\ldots,i_r}$ stands for the signature of the shuffle substitution $(1,\ldots,n+1)\mapsto(j_1,\ldots,j_{n+1-r},i_1,\ldots,i_r)$. The Hodge star operator

$$* : \Lambda^r(\mathbf{R}^{n+1}) \xrightarrow{\sim} \Lambda^{n+1-r}(\mathbf{R}^{n+1})$$

is clearly an isometry for the dot scalar product and iterating twice the Hodge star, we get

$$* \circ * = (-1)^{r(n+1-r)}\mathrm{Id}. \tag{3.6}$$

**Lemma 1.** *Let $\mathbf{X}=\mathbf{x}_1\wedge\ldots\wedge\mathbf{x}_r$ be a system of Plücker coordinates ($\natural$) of a $r$-dimensional subspace*

$$V = <\mathbf{x}_1,\ldots,\mathbf{x}_r>$$

($\natural$) The word "coordinates" classically refers to the canonical basis of $\Lambda^r(\mathbf{R}^{n+1})$.

*in $\mathbf{R}^{n+1}$. Then $*\mathbf{X}$ is a system of Plücker coordinates of the orthogonal $V^\perp$ of $V$.*

**Proof.** That is the assertion of Theorem I of Chapter VII §3 in [9]. Using the notion of contraction, we may argue as follows. Take any orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ of $V$ and extend it to an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ of $\mathbf{R}^{n+1}$. Then

$$\mathbf{X} = \rho(\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_r)$$

for some non-zero real number $\rho$. Now, it follows from (3.5) that

$$*\mathbf{X} = \pm\rho(\mathbf{e}_{r+1} \wedge \ldots \wedge \mathbf{e}_{n+1}).$$

□

**Remark.** The same argument shows more generally that if $\mathbf{Y} = \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_s$ is a system of Plücker coordinates of an $s$-dimensional vector space $W = < \mathbf{y}_1, \ldots, \mathbf{y}_s >$ with $s \geq r$, then $\mathbf{X} \lrcorner \mathbf{Y}$ is a system of Plücker coordinates of the intersection $W \cap V^\perp$, provided that this intersection has dimension $s - r$.

**Lemma 2.** *For any $\mathbf{X} \in \Lambda^r(\mathbf{R}^{n+1})$ and $\mathbf{Y} \in \Lambda^s(\mathbf{R}^{n+1})$ with $r + s \leq n+1$, we have the duality formula*

$$*(\mathbf{Y} \wedge \mathbf{X}) = \mathbf{Y} \lrcorner (*\mathbf{X})$$

**Proof.** Using (3.4), we find

$$*(\mathbf{Y} \wedge \mathbf{X}) = (\mathbf{Y} \wedge \mathbf{X}) \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1}) = \mathbf{Y} \lrcorner (\mathbf{X} \lrcorner (\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{n+1})) = \mathbf{Y} \lrcorner (*\mathbf{X}).$$

□

## 4. Alternative definition of the intermediate exponents

Let $P$ and $Q$ be points in $\mathbf{P}^n(\mathbf{R})$ with homogeneous coordinates $\mathbf{x}$ and $\mathbf{y}$. As in [14], we define the projective distance $\mathrm{d}(P, Q)$ between $P$ and $Q$ by

$$\mathrm{d}(P, Q) = \frac{|\mathbf{x} \wedge \mathbf{y}|}{|\mathbf{x}||\mathbf{y}|}.$$

It has been shown in Lemma 1 of [14] that for any point $\Theta$ in $\mathbf{P}^n(\mathbf{R})$ with homogeneous coordinates $\mathbf{y} = (1, \theta_1, \ldots, \theta_n)$ and any linear subvariety $L$ of $\mathbf{P}^n(\mathbf{R})$ with Plücker coordinates $\mathbf{X}$, the minimal distance $\mathrm{d}(\Theta, L)$ between $\Theta$ and the set of real points of $L$ is equal to

$$\mathrm{d}(\Theta, L) = \frac{|\mathbf{y} \wedge \mathbf{X}|}{|\mathbf{y}||\mathbf{X}|}. \quad (4.1)$$

We can now reformulate Definition 3 in terms of integer solutions of the following system of linear inequations.

**Proposition.** *For any integer $d$ with $0 \le d \le n-1$, the exponent $\omega_d(\Theta)$ is the supremum of the real numbers $\omega$ for which there exist infinitely many integer multivectors $\mathbf{X} \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that*

$$|\mathbf{y} \wedge \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

In relation with Definition 4 of [14], we do not assume here that the multivectors $\mathbf{X}$ occurring in the Proposition are decomposable. To suppress this additional condition, we expand the remark given on page 312 of [14]. The following lemma will be as well our main ingredient to prove Theorem 2.

**Lemma 3.** *Let $\mathbf{y} = (1, \theta_1, \dots, \theta_n) \in \mathbf{R}^{n+1}$ and let $U, V$ be positive real numbers with $V \le U$. The convex body $\mathcal{C}$ of $\Lambda^{d+1}(\mathbf{R}^{n+1})$ consisting of the $\mathbf{Z}$ such that*

$$|\mathbf{Z}| \le UV^d \quad \text{and} \quad |\mathbf{y} \wedge \mathbf{Z}| \le V^{d+1} \tag{4.2}$$

*is comparable (†) to the $(d+1)$-th compound of the convex body $\mathcal{C}'$ consisting of the $\mathbf{z} \in \mathbf{R}^{n+1}$ such that*

$$|\mathbf{z}| \le U \quad \text{and} \quad |\mathbf{y} \wedge \mathbf{z}| \le V. \tag{4.3}$$

**Proof.** The convex body $\mathcal{C}'$ is comparable to the parallelepiped $\mathcal{P}$ defined by

$$|x_0| \le U, \quad |x_0\theta_i - x_i| \le V, \quad 1 \le i \le n.$$

However, $\mathcal{P}$ is comparable to the convex hull of the points

$$\pm U\mathbf{y}, \pm V\mathbf{e}_1, \dots, \pm V\mathbf{e}_n,$$

where

$$\mathbf{e}_1 = (0,1,0,\dots,0), \dots, \mathbf{e}_n = (0,\dots,0,1).$$

The convex compound $\mathcal{C}'^{d+1}$ is then comparable to the convex hull in $\Lambda^{d+1}(\mathbf{R}^{n+1})$ of the exterior products of $d+1$ of these points, that is, of

$$\pm V^{d+1}\mathbf{e}_{i_0} \wedge \ldots \wedge \mathbf{e}_{i_d}, \quad 1 \le i_0 < \cdots < i_d \le n,$$

(†) We say that two families $\mathcal{C}_1$ and $\mathcal{C}_2$ of symmetrical convex bodies, parametrized by (say) $U$ and $V$, are *comparable* if there exists a real number $\kappa > 1$, such that the inclusions $\kappa^{-1}\mathcal{C}_1(U,V) \subseteq \mathcal{C}_2(U,V) \subseteq \kappa\mathcal{C}_1(U,V)$ hold for any parameters $U, V$. Accordingly, the constants implied in the forthcoming symbols $\ll$, $\gg$ and $\asymp$ may depend on $n$ and $\Theta$, but not on $U$ and $V$. The relation $f \asymp g$ means that we have both $f \ll g$ and $f \gg g$.

and

$$\pm UV^d \mathbf{y} \wedge \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_d}, \quad 1 \le i_1 < \cdots < i_d \le n.$$

The points $\mathbf{Z}$ of this form satisfy

$$|\mathbf{Z}| \ll UV^d, \quad |\mathbf{y} \wedge \mathbf{Z}| \ll V^{d+1}.$$

Conversely, let $\mathbf{Z}$ be in $\Lambda^{d+1}(\mathbf{R}^{n+1})$ for which (4.2) holds and express it in the base composed of the $d+1$ exterior products of the base $(\mathbf{y}, \mathbf{e}_1, \ldots, \mathbf{e}_n)$, that is,

$$\mathbf{Z} = \sum a_{i_0, i_1, \ldots, i_d}\, \mathbf{e}_{i_0} \wedge \ldots \wedge \mathbf{e}_{i_d} + \sum b_{i_1, i_2, \ldots, i_d}\, \mathbf{y} \wedge \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_d}.$$

Then, we have the estimates

$$\sum |a_{i_0, i_1, \ldots, i_d}| + \sum |b_{i_1, i_2, \ldots, i_d}| \asymp |\mathbf{Z}| \le UV^d \quad \text{and} \quad \sum |a_{i_0, i_1, \ldots, i_d}| \asymp |\mathbf{y} \wedge \mathbf{Z}| \le V^{d+1}.$$

This completes the proof of the lemma. □

With this lemma, we are able to establish our Proposition.

**Proof of the Proposition.** Let $\omega$ be a real number with $\omega \ge -1$ and let $\mathbf{X}$ be a non-zero point in $\Lambda^{d+1}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{y} \wedge \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

The first minimum of the convex body $\mathcal{C}$ composed of the $\mathbf{Z} \in \Lambda^{d+1}(\mathbf{R}^{n+1})$ such that

$$|\mathbf{Z}| \le |\mathbf{X}| \quad \text{and} \quad |\mathbf{y} \wedge \mathbf{Z}| \le |\mathbf{X}|^{-\omega}$$

is therefore at most equal to 1 since $\mathbf{X}$ belongs to $\mathcal{C}$. Setting

$$U = |\mathbf{X}|^{(d\omega+d+1)/(d+1)}, \quad V = |\mathbf{X}|^{-\omega/(d+1)}, \tag{4.4}$$

we observe that $V \le U$ and that

$$|\mathbf{X}| = UV^d, \quad |\mathbf{X}|^{-\omega} = V^{d+1}.$$

By Lemma 3, the convex $\mathcal{C}$ is comparable to the $(d+1)$-th compound of the convex body $\mathcal{C}' \subset \mathbf{R}^{n+1}$ defined by the inequalities (4.3). Now, Mahler's theory on compound convex bodies tells us that the integer point where $\mathcal{C}$ reaches its first minimum is essentially obtained as the wedge product $\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_{d+1}$ of the integer points $\mathbf{x}_i, 1 \le i \le d+1$, where $\mathcal{C}'$ reaches its $i$-th minimum. We may therefore assume that $\mathbf{X} = \mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_{d+1}$.

Let $L \subset \mathbf{P}^n(\mathbf{R})$ be the $d$-dimensional rational linear subvariety $L = \mathbf{P}(V)$ where $V = \langle \mathbf{x}_1, \ldots, \mathbf{x}_{d+1} \rangle$. By (4.1), we obtain

$$\mathrm{d}(\Theta, L) = \frac{|\mathbf{y} \wedge \mathbf{X}|}{|\mathbf{y}||\mathbf{X}|} \leq |\mathbf{y}|^{-1}|\mathbf{X}|^{-1-\omega} \ll H(L)^{-1-\omega},$$

so that $\omega_d(\Theta) \geq \omega$.

Conversely, if $L$ satisfies $\mathrm{d}(\Theta, L) \leq H(L)^{-1-\omega}$, choose a system of coprime integer Plücker coordinates $\mathbf{X}$ of $L$, so that $H(L) = |\mathbf{X}|$. Then (4.1) shows that the upper bound $|\mathbf{y} \wedge \mathbf{X}| \ll |\mathbf{X}|^{-\omega}$ holds true. □

Our Proposition enables us to recover the following corollary, which was already obtained in [14] and earlier in [17], using different arguments.

**Corollary 2.** *For any integer $d$ with $0 \leq d \leq n-1$, we have the lower bound*

$$\omega_d(\Theta) \geq \frac{d+1}{n-d}.$$

**Proof.** The linear map $\Lambda^{d+1}(\mathbf{R}^{n+1}) \longrightarrow \Lambda^{d+2}(\mathbf{R}^{n+1})$ which sends $\mathbf{X} \mapsto \mathbf{y} \wedge \mathbf{X}$ has rank $M = \binom{n+1}{d+1} - \binom{n}{d}$. Now, view the coordinates of $\mathbf{y} \wedge \mathbf{X}$ in the canonical basis of $\Lambda^{d+2}(\mathbf{R}^{n+1})$ as linear forms in the $N = \binom{n+1}{d+1}$ coordinates of $\mathbf{X}$, and select among them $M$ linearly independent forms $L_1(\mathbf{X}), \ldots, L_M(\mathbf{X})$. By Minkowski's first theorem on convex bodies, the system of linear inequalities

$$|\mathbf{X}| \leq H, \quad |L_1(\mathbf{X})| \leq cH^{-(N-M)/M}, \ldots, |L_M(\mathbf{X})| \leq cH^{-(N-M)/M}$$

has a non-zero integer solution $\mathbf{X}$ for any $H \gg 1$ and for some positive coefficient $c$ independent of $H$. It follows that

$$\omega_d(\Theta) \geq \frac{N-M}{M} = \frac{\binom{n}{d}}{\binom{n+1}{d+1} - \binom{n}{d}} = \frac{d+1}{n-d},$$

as claimed. □

## 5. Proof of Theorem 2

We use the Proposition as a more convenient characterization of the exponents $\omega_d(\Theta)$ and take again the notations of Section 4. Let $\omega$ be a real number with $-1 \leq \omega < \omega_d(\Theta)$ and let $\mathbf{X} \in \Lambda^{d+1}(\mathbf{Z}^{n+1})$ be such that

$$|\mathbf{y} \wedge \mathbf{X}| \leq |\mathbf{X}|^{-\omega},$$

where $\mathbf{y}$ denotes the homogeneous coordinates of $\Theta$. Recall that $U$ and $V$ are given by (4.4) and that the convex bodies $\mathcal{C}$ and $\mathcal{C}'$ are defined by (4.2) and (4.3), respectively. The first minimum $\lambda_1$ of the convex body $\mathcal{C}$ is at most equal to 1 since $\mathbf{X}$ belongs to $\mathcal{C}$. Replacing, if necessary, $\mathbf{X}$ by an integer point where this first minimum is reached, and suitably increasing $\omega$, we may assume that $\lambda_1 = 1$.

By Lemma 3, the convex $\mathcal{C}$ is comparable to the $(d+1)$-th compound of the convex body $\mathcal{C}'$ of volume

$$\mathrm{vol}(\mathcal{C}') \asymp UV^n = |\mathbf{X}|^{(-(n-d)\omega+d+1)/(d+1)}.$$

By Minkowski's Theorem, the successive minima $\lambda'_1 \le \ldots \le \lambda'_{n+1}$ of $\mathcal{C}'$ satisfy

$$\lambda'_1 \times \ldots \times \lambda'_{n+1} \asymp \mathrm{vol}(\mathcal{C}')^{-1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(d+1)}.$$

Since $\mathcal{C}$ is comparable to the $(d+1)$-th compound of $\mathcal{C}'$, Mahler's theorem on compound convex bodies asserts that $\lambda_1$, the first minimum of $\mathcal{C}$, is comparable to the product $\lambda'_1 \times \ldots \times \lambda'_{d+1}$. Consequently,

$$\lambda'_1 \times \ldots \times \lambda'_{d+1} \asymp 1 \tag{5.1}$$

and

$$(\lambda'_{d+2})^{n-d} \le \lambda'_{d+2} \times \ldots \times \lambda'_{n+1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(d+1)},$$

whence

$$\lambda'_{d+2} \ll |\mathbf{X}|^{((n-d)\omega-d-1)/((d+1)(n-d))}. \tag{5.2}$$

Now, since the $(d+2)$-th compound of $\mathcal{C}'$ has its first minimum comparable to

$$\lambda'_1 \times \ldots \times \lambda'_{d+2} \asymp \lambda'_{d+2},$$

it follows from Lemma 3 with $d+1$ in place of $d$ that there exists $\tilde{\mathbf{X}} \in \Lambda^{d+2}(\mathbf{Z}^{n+1})$ such that

$$|\tilde{\mathbf{X}}| \ll \lambda'_{d+2} U V^{d+1}, \quad |\mathbf{y} \wedge \tilde{\mathbf{X}}| \ll \lambda'_{d+2} V^{d+2}.$$

A rapid computation using (5.2) yields that

$$\lambda'_{d+2} U V^{d+1} \ll |\mathbf{X}|^{(n-d-1)/(n-d)}$$

and

$$\lambda'_{d+2} V^{d+2} \ll |\mathbf{X}|^{-((n-d)\omega+1)/(n-d)}.$$

This gives

$$|\mathbf{y} \wedge \tilde{\mathbf{X}}| \ll |\tilde{\mathbf{X}}|^{-((n-d)\omega+1)/(n-d-1)},$$

and we get (2.1) since $\omega$ can be taken arbitrarily close to $\omega_d(\Theta)$.

To establish (2.2), let us first observe that (5.2) with $d = 0$ gives

$$\lambda_2' \ll |\mathbf{X}|^{\omega - 1/n}. \tag{5.3}$$

One can get a better upper bound for $\lambda_2'$ when $d = 0$ by taking the uniform exponents into account, as we show now. In that case $\mathcal{C} = \mathcal{C}'$ and $\lambda_1' = \lambda_1 = 1$. The vector $\mathbf{X}$ is necessarily primitive in $\mathbf{Z}^{n+1}$, since the convex body $\mathcal{C}'$ attains its first minimum at that point. Let $\hat{\omega}$ be a real number with $\hat{\omega} < \min(\omega, \hat{\omega}_0(\Theta))$. By Definition 2, there exists a non-zero integer point $\mathbf{x}$ such that

$$|\mathbf{x}| < |\mathbf{X}|, \quad |\mathbf{y} \wedge \mathbf{x}| \leq |\mathbf{X}|^{-\hat{\omega}}.$$

Since $\mathbf{X}$ is primitive, the vectors $\mathbf{x}$ and $\mathbf{X}$ are linearly independent. Furthermore, we have

$$|\mathbf{x}| < |\mathbf{X}| \leq |\mathbf{X}|^{1+\omega-\hat{\omega}} = U|\mathbf{X}|^{\omega-\hat{\omega}},$$

since $\omega \geq \hat{\omega}$, and

$$|\mathbf{y} \wedge \mathbf{x}| \leq |\mathbf{X}|^{-\hat{\omega}} = |\mathbf{X}|^{-\omega+\omega-\hat{\omega}} = V|\mathbf{X}|^{\omega-\hat{\omega}}.$$

This gives

$$\lambda_2' \ll |\mathbf{X}|^{\omega-\hat{\omega}}. \tag{5.4}$$

Note that the upper estimate (5.4) may be sharper than (5.3) since $\hat{\omega}_0(\Theta) \geq 1/n$.

Observing that $U = |\mathbf{X}|$ and $V = |\mathbf{X}|^{-\omega}$ and proceeding as above, we infer from (5.4) that

$$\lambda_2' UV \ll |\mathbf{X}|^{1-\hat{\omega}}$$

and

$$\lambda_2' V^2 \ll |\mathbf{X}|^{-(\omega+\hat{\omega})},$$

whence

$$|\mathbf{y} \wedge \tilde{\mathbf{X}}| \ll |\tilde{\mathbf{X}}|^{-(\omega+\hat{\omega})/(1-\hat{\omega})}.$$

Letting $\omega$ tends to $\omega_0(\Theta)$ and $\hat{\omega}$ tends to $\hat{\omega}_0(\Theta) \leq \omega_0(\Theta)$, this gives

$$\omega_1(\Theta) \geq \frac{\omega_0(\Theta) + \hat{\omega}_0(\Theta)}{1 - \hat{\omega}_0(\Theta)}.$$

We have proved (2.2).

## 6. Proof of Theorem 3

The proof is parallel to that of Theorem 2. We use Hodge duality to reverse the Going-down transfer into a Going-up transfer, noting that the duality permutes the dimension with the codimension.

Let us start with the following dual version of the above Proposition.

**Lemma 4.** *For $d = 0, \dots, n-1$, the exponent $\omega_d(\Theta)$ of a point $\Theta$ in $\mathbf{R}^n$ with homogeneous coordinates $\mathbf{y}$ is the supremum of the real numbers $\omega$ such that there are infinitely many $\mathbf{X} \in \Lambda^{n-d}(\mathbf{Z}^{n+1})$ with*

$$|\mathbf{y} \lrcorner \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

**Proof.** By Lemma 2 and (3.6), we have

$$*(\mathbf{y} \wedge *\mathbf{X}) = (-1)^{(d+1)(n-d)}(\mathbf{y} \lrcorner \mathbf{X}),$$

for every $\mathbf{X}$ in $\Lambda^{n-d}(\mathbf{R}^{n+1})$. Note that $*$ maps $\Lambda^{n-d}(\mathbf{Z}^{n+1})$ isometrically onto $\Lambda^{d+1}(\mathbf{Z}^{n+1})$, so that

$$|*\mathbf{X}| = |\mathbf{X}| \quad \text{and} \quad |\mathbf{y} \wedge *\mathbf{X}| = |\mathbf{y} \lrcorner \mathbf{X}|.$$

Now, replace $\mathbf{X}$ by $*\mathbf{X}$ in the Proposition to conclude the proof. □

Here is now the dual version of Lemma 3.

**Lemma 5.** *Let $d$ be an integer with $0 \le d \le n-1$ and let $U, V$ be positive real numbers with $V \le U$. The convex body $\mathcal{C}$ of $\Lambda^{n-d}(\mathbf{R}^{n+1})$ consisting of the $\mathbf{Z}$ such that*

$$|\mathbf{Z}| \le U^{n-d} \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le U^{n-d-1}V \tag{6.1}$$

*is comparable to the $(n-d)$-th compound of the convex body $\mathcal{C}'$ composed of the $\mathbf{z} \in \mathbf{R}^{n+1}$ such that*

$$|\mathbf{z}| \le U \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{z}| \le V. \tag{6.2}$$

**Proof.** Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of the orthogonal of $\mathbf{y}$ in $\mathbf{R}^{n+1}$. The convex body $\mathcal{C}'$ is comparable to the parallelepiped $\mathcal{P}$ consisting of the points

$$x_0\mathbf{y} + x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \quad \text{where} \quad |x_0| \le V, \quad |x_i| \le U, \quad 1 \le i \le n.$$

Note that $\mathcal{P}$ is comparable to the convex hull of the points

$$\pm V\mathbf{y}, \pm U\mathbf{e}_1, \dots, \pm U\mathbf{e}_n.$$

The compound convex body $\mathcal{C}'^{n-d}$ is then comparable to the convex hull in $\Lambda^{n-d}(\mathbf{R}^{n+1})$ of the exterior products of $n-d$ of these points, that is, of

$$\pm U^{n-d}\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d}}, \quad 1 \le i_1 < \cdots < i_{n-d} \le n, \tag{6.3}$$

and

$$\pm U^{n-d-1}V\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{n-d-1}} \wedge \mathbf{y}, \quad 1 \le i_1 < \cdots < i_{n-d-1} \le n. \tag{6.4}$$

Express now any point $\mathbf{Z}$ in $\Lambda^{n-d}(\mathbf{R}^{n+1})$ in the base composed of the $n-d$ exterior products of the base $(\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{y})$, that is,

$$\mathbf{Z} = \sum a_{i_1,\dots,i_{n-d}}\, \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_{n-d}} + \sum b_{i_1,\dots,i_{n-d-1}}\, \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_{n-d-1}} \wedge \mathbf{y}.$$

Then, formula (3.3) shows that

$$\mathbf{y} \lrcorner \mathbf{Z} = |\mathbf{y}|^2 \left( \sum b_{i_1,\dots,i_{n-d-1}}\, \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_{n-d-1}} \right).$$

Therefore, the points $\mathbf{Z}$ of the form (6.3) or (6.4) satisfy

$$|\mathbf{Z}| \le U^{n-d} \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le |\mathbf{y}|^2 U^{n-d-1} V. \tag{6.5}$$

Conversely, for any point $\mathbf{Z}$ satisfying (6.5), the coefficients $a_{i_1,\dots,i_{n-d}}$ (resp. $b_{i_1,\dots,i_{n-d-1}}$) are bounded in absolute value by $U^{n-d}$ (resp. by $U^{n-d-1}V$). This completes the proof of the lemma. □

With these two lemmata, we are able to establish Theorem 3.

**Proof of Theorem 3.** Let $\omega$ be a positive real number with $\omega < \omega_d(\Theta)$. By Lemma 4, there exist infinitely many points $\mathbf{X} \in \Lambda^{n-d}(\mathbf{Z}^{n+1})$ such that

$$|\mathbf{y} \lrcorner \mathbf{X}| \le |\mathbf{X}|^{-\omega}.$$

Fix such a point $\mathbf{X}$ with large norm $|\mathbf{X}|$ and consider the convex body $\mathcal{C}$ composed of the multivectors $\mathbf{Z} \in \Lambda^{n-d}(\mathbf{R}^{n+1})$ such that

$$|\mathbf{Z}| \le |\mathbf{X}| \quad \text{and} \quad |\mathbf{y} \lrcorner \mathbf{Z}| \le |\mathbf{X}|^{-\omega}.$$

It contains the integer point $\mathbf{X}$. Replacing possibly $\mathbf{X}$ by a smaller point and enlarging suitably $\omega$, one can assume that $\mathbf{X}$ is the smallest non-zero integer point in $\mathcal{C}$. Thus, we may assume that the first minimum of $\mathcal{C}$ is equal to 1. Setting

$$U = |\mathbf{X}|^{1/(n-d)} \quad \text{and} \quad V = |\mathbf{X}|^{-((n-d)\omega + n-d-1)/(n-d)},$$

we observe that $V \le U$ and that

$$|\mathbf{X}| = U^{n-d}, \quad |\mathbf{X}|^{-\omega} = U^{n-d-1} V.$$

By Lemma 5, the convex body $\mathcal{C}$ is therefore comparable to the $(n-d)$-th compound of the convex body $\mathcal{C}'$ consisting of the real $(n+1)$-tuples $\mathbf{z}$ such that

$$|\mathbf{z}| \le U \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{z}| \le V.$$

Let

$$\lambda'_1 \leq \ldots \leq \lambda'_{n+1}$$

be the successive minima of the convex body $\mathcal{C}'$. Since the Euclidean volume of $\mathcal{C}'$ is $\asymp U^n V$, the second theorem of Minkowski gives

$$\lambda'_1 \times \ldots \times \lambda'_{n+1} \asymp (U^n V)^{-1} = |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}.$$

Since the first minimum of the $(n-d)$-th compound of $\mathcal{C}'$ is comparable to 1, one gets

$$\lambda'_1 \times \ldots \times \lambda'_{n-d} \asymp 1,$$

hence

$$\lambda'_{n-d+1} \times \ldots \times \lambda'_{n+1} \asymp |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}.$$

Consequently,

$$(\lambda'_{n-d+1})^{d+1} \ll |\mathbf{X}|^{((n-d)\omega-d-1)/(n-d)}, \tag{6.6}$$

and

$$\lambda'_{n-d+1} U \ll |\mathbf{X}|^{\omega/(d+1)}.$$

Since the first minimum of the $(n-d+1)$-th compound of $\mathcal{C}'$ is comparable to the product $\lambda'_1 \times \ldots \times \lambda'_{n-d+1}$, hence to $\lambda'_{n-d+1}$, we infer from Lemma 5 that there exists a non-zero integer point $\tilde{\mathbf{X}} \in \Lambda^{n-d+1}(\mathbf{Z}^{n+1})$ such that

$$|\tilde{\mathbf{X}}| \ll \lambda'_{n-d+1} U^{n-d+1} = \lambda'_{n-d+1} U |\mathbf{X}| \ll |\mathbf{X}|^{(\omega+d+1)/(d+1)}$$

and

$$|\mathbf{y} \lrcorner \tilde{\mathbf{X}}| \ll \lambda'_{n-d+1} U^{n-d} V = \lambda'_{n-d+1} U |\mathbf{X}|^{-\omega} \ll |\mathbf{X}|^{-d\omega/(d+1)}.$$

Since $\omega$ can be taken arbitrarily close to $\omega_d(\Theta)$, Lemma 4 gives (2.3).

For $d = n-1$, it is possible to get a sharper result. In that case $\mathcal{C} = \mathcal{C}'$ is a convex body in $\mathbf{R}^{n+1}$ and (6.6) reads

$$\lambda'_2 \ll |\mathbf{X}|^{-1+\omega/n}. \tag{6.7}$$

Enlarging possibly $\omega$, we may assume that

$$|\mathbf{y} \cdot \mathbf{X}| = |\mathbf{X}|^{-\omega}.$$

The vector $\mathbf{X}$ is necessarily primitive in $\mathbf{Z}^{n+1}$, since the convex body $\mathcal{C}'$ attains its first minimum at that point. Let $\hat{\omega}$ be a positive real number with $\hat{\omega} < \min(\omega, \hat{\omega}_{n-1}(\Theta))$. By Definition 2, there exists a non-zero integer point $\mathbf{x} \in \mathbf{Z}^{n+1}$ such that

$$|\mathbf{x}| \leq |\mathbf{X}|^{\omega/\hat{\omega}} \quad \text{and} \quad |\mathbf{y} \cdot \mathbf{x}| < |\mathbf{X}|^{-\omega}.$$

Since $\mathbf{X}$ is primitive, the vectors $\mathbf{x}$ and $\mathbf{X}$ are linearly independent ; otherwise $\mathbf{x}$ should be an integer multiple of $\mathbf{X}$ and $|\mathbf{y}\cdot\mathbf{x}|$ should be greater than or equal to $|\mathbf{y}\cdot\mathbf{X}| = |\mathbf{X}|^{-\omega}$. Thus, we obtain the upper bound

$$\lambda_2' \ll |\mathbf{X}|^{-1+\omega/\hat{\omega}}, \tag{6.8}$$

which may be better than (6.7) since $\hat{\omega}_{n-1}(\Theta) \geq n$. Now, we take again the preceding arguments. Noting that $U = |\mathbf{X}|$ and $V = |\mathbf{X}|^{-\omega}$, we obtain a non-zero point $\tilde{\mathbf{X}} \in \Lambda^2(\mathbf{Z}^{n+1})$ satisfying

$$|\tilde{\mathbf{X}}| \ll \lambda_2' U^2 \ll |\mathbf{X}|^{1+\omega/\hat{\omega}}$$

and

$$|\mathbf{y} \lrcorner \tilde{\mathbf{X}}| \ll \lambda_2' UV \ll |\mathbf{X}|^{-\omega+\omega/\hat{\omega}}.$$

Then, Lemma 4 gives

$$\omega_{n-2}(\Theta) \geq \frac{(\hat{\omega}-1)\omega}{\omega+\hat{\omega}}.$$

Letting $\omega$ and $\hat{\omega}$ tend respectively to $\omega_{n-1}(\Theta)$ and $\hat{\omega}_{n-1}(\Theta)$, we have established (2.4).

## 7. Proof of Theorem 1

It is a formal consequence of the finer estimates (2.1)–(2.4).

Using the second inequality of Corollary 1 with $d = 1$ and $d' = n-1$, we get the estimate

$$\omega_{n-1}(\Theta) \geq (n-1)\omega_1(\Theta) + n - 2,$$

which, combined with (2.2), yields the second claimed estimate of Theorem 1, namely

$$\omega_{n-1}(\Theta) \geq (n-1)\frac{\omega_0(\Theta)+\hat{\omega}_0(\Theta)}{1-\hat{\omega}_0(\Theta)} + n - 2 = \frac{(n-1)\omega_0(\Theta)+\hat{\omega}_0(\Theta)+n-2}{1-\hat{\omega}_0(\Theta)}.$$

Using now the first inequality of Corollary 1 with $d = 0$ and $d' = n-2$, we get

$$\omega_0(\Theta) \geq \frac{\omega_{n-2}(\Theta)}{(n-2)\omega_{n-2}(\Theta)+n-1}$$

which, combined with (2.4), yields the first claimed inequality of Theorem 1, namely

$$\omega_0(\Theta) \geq \frac{(\hat{\omega}_{n-1}(\Theta)-1)\omega_{n-1}(\Theta)}{((n-2)\hat{\omega}_{n-1}(\Theta)+1)\omega_{n-1}(\Theta)+(n-1)\hat{\omega}_{n-1}(\Theta)}.$$

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