On sequences $(a_n\xi)_{n\geq 1}$ converging modulo 1

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Abstract. *We prove that, for any sequence of positive real numbers $(g_n)_{n\geq 1}$ satisfying $g_n \geq 1$ for $n \geq 1$ and $\lim_{n\to+\infty} g_n = +\infty$, for any real number θ in $[0,1]$ and any irrational real number ξ, there exists an increasing sequence of positive integers $(a_n)_{n\geq 1}$ satisfying $a_n \leq n g_n$ for $n \geq 1$ and such that the sequence of fractional parts $(\{a_n\xi\})_{n\geq 1}$ tends to θ as n tends to infinity. This result is best possible in the sense that the condition $\lim_{n\to+\infty} g_n = +\infty$ cannot be weakened, as recently proved by Dubickas.*

For an increasing sequence $\mathbf{a} = (a_n)_{n\geq 1}$ of positive integers, let $E_{\mathbf{a}}$ denote the set of irrational real numbers ξ such that the sequence $(\{a_n\xi\})_{n\geq 1}$ is not everywhere dense in $[0,1)$. Here, and throughout the present note, $\{x\}$ stands for the fractional part of the real number x. Weyl [4] established in 1916 that $E_{\mathbf{a}}$ has Lebesgue measure zero. No refined general metrical result can be proved since, on the one hand, $E_{\mathbf{a}}$ is empty when $\mathbf{a}$ is the sequence of all positive integers or of all integers of the form $2^k 3^\ell$ (with $k, \ell \geq 0$), and, on the other hand, $E_{\mathbf{a}}$ has full Hausdorff dimension if there exists some τ greater than 1 for which $a_{n+1} \geq \tau a_n$ for $n \geq 1$. We refer to [1,3] for references and further results.

In a recent paper, Dubickas [1] investigated how slowly such a sequence $\mathbf{a}$ can increase for which the set $E_{\mathbf{a}}$ is not empty. More precisely, for any real quadratic number α, he constructed a very slowly increasing sequence $\mathbf{a}$ such that the sequence of fractional parts $(\{a_n\alpha\})_{n\geq 1}$ tends to 0. His proof is quite intricate and makes use of recurrence sequences related to some algebraic integer in the quadratic number field generated by α. In his note Dubickas asked whether, a transcendental real number (or a real algebraic number of degree at least 3) ξ being given, there exists a slowly increasing sequence of positive integers $(a_n)_{n\geq 1}$ such that $\lim_{n\to+\infty} \{a_n\xi\} = 0$.

In the present note, we give a positive answer to (a strong form of) his question.

Theorem. *Let ξ be an irrational real number. Let S be a finite, non-empty set of distinct real numbers in $[0,1]$. Let $(g_n)_{n\geq 1}$ be a sequence of real numbers such that $g_n \geq 1$ for $n \geq 1$ and $\lim_{n\to+\infty} g_n = +\infty$. Then there exists an increasing sequence of positive integers $(a_n)_{n\geq 1}$ satisfying $a_n \leq n g_n$ for $n \geq 1$ and such that the set of limit points of the sequence of fractional parts $(\{a_n\xi\})_{n\geq 1}$ is equal to S.*

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The Theorem extends Theorems 1 and 5 of [1]. Our proof is much simpler; it uses only basic results from the theory of continued fractions and the fact the sequence $(t\xi)_{t\geq 1}$ is dense modulo 1 when ξ is irrational.

The Theorem is best possible in the sense that its conclusion fails if $(g_n)_{n\geq 1}$ does not tend to infinity. Namely, Theorem 2 of Dubickas [1] asserts that, for any irrational real number ξ and any increasing sequence $(a_n)_{n\geq 1}$ satisfying $\liminf_{n\to+\infty} a_n/n < +\infty$, the sequence of fractional parts $(\{a_n\xi\})_{n\geq 1}$ has infinitely many limit points.

Proof of the Theorem. Let $(p_k/q_k)_{k\geq 1}$ be the sequence of convergents to ξ and set

$$\varepsilon_k := \{q_k\xi\}, \quad k \geq 1.$$

Classical results on continued fraction expansions (see e.g., [2]) imply that

$$0 < \varepsilon_{2k+2} < \varepsilon_{2k} < 1/3, \quad k \geq 1.$$

Since ξ is irrational, the sequence $(t\xi)_{t\geq 1}$ is dense modulo 1. This fact (see e.g. [4]) will be implicitly used at several places below.

As explained in [1], we can assume that $g_1, g_2, \ldots$ are integers and that $(g_n)_{n\geq 1}$ is non-decreasing. Set $n_1 = q_2$. For $k \geq 2$, let n_k be the smallest index ℓ such that $\ell > n_{k-1}$ and $g_\ell \geq q_{2k} + 1$. Note that the sequence $(n_k)_{k\geq 1}$ may increase very rapidly.

We proceed now to construct inductively an auxiliary integer sequence $(m_k)_{k\geq 2}$ and a sequence $(a_n)_{n\geq 1}$ with the required property.

Let j be the integer such that $q_{2j} \geq n_2 > q_{2j-2}$. Observe that $j \geq 2$ and set $m_2 = q_{2j}$. Define

$$a_n = n, \quad n = 1, \ldots, m_2,$$

and observe that

$$\{m_2\xi\} = \{a_{m_2}\xi\} \leq \varepsilon_2, \quad a_{m_2} \leq m_2 q_4 \leq m_2(g_{m_2} - 1), \quad g_{m_2} \geq g_{n_2} \geq q_4 + 1.$$

Let us proceed with the induction step. Set $\varepsilon_0 = 1$. Let $k \geq 2$ be an integer and assume that m_k and a_{m_k} have been constructed such that

$$\{a_{m_k}\xi\} \leq \varepsilon_{2k-2}, \quad a_{m_k} \leq m_k(g_{m_k} - 1), \quad g_{m_k} \geq g_{n_k} \geq q_{2k} + 1.$$

Set $b_0 = a_{m_k}$ and let $b_1 < b_2 < \ldots$ be the (infinite) increasing sequence of all integers t satisfying $t > a_{m_k}$ and $\{t\xi\} \leq \varepsilon_{2k-2}$. Observe that if the integer t satisfies $\{t\xi\} \leq \varepsilon_{2k-2}$, then

$$\{(t + q_{2k})\xi\} = \{t\xi\} + \varepsilon_{2k} < 2\varepsilon_{2k-2}$$

and we have either

$$\{(t + q_{2k})\xi\} \leq \varepsilon_{2k-2}$$

or

$$\{(t + q_{2k} - q_{2k-2})\xi\} \leq \varepsilon_{2k-2}.$$

From this, we deduce that

$$b_{j+1} \le b_j + q_{2k}, \quad \text{for } j \ge 0,$$

and

$$b_j \le m_k(g_{m_k} - 1) + jq_{2k} \le (m_k + j)(g_{m_k+j} - 1), \quad \text{for } j \ge 0.$$

Let m_{k+1} be the smallest integer ℓ satisfying $\ell \ge \max\{m_k + 1, n_{k+1}\}$ and

$$\{b_{\ell - m_k}\xi\} \le \varepsilon_{2k}.$$

This integer is well defined since the sequence $(t\xi)_{t\ge1}$ is dense modulo 1. Setting

$$a_{m_k+j} = b_j, \quad j = 1, \ldots, m_{k+1} - m_k,$$

we thus have

$$\{a_n\xi\} \le \varepsilon_{2k-2}, \;\; a_n \le n(g_n - 1), \quad n = m_k + 1, \ldots, m_{k+1},$$

and

$$\{a_{m_{k+1}}\xi\} \le \varepsilon_{2k}, \quad g_{m_{k+1}} \ge g_{n_{k+1}} \ge q_{2k+2} + 1.$$

This completes the inductive set.

To summarize, we have constructed inductively an increasing sequence $(a_n)_{n\ge1}$ of positive integers satisfying

$$a_n = n, \quad \text{for } n = 1, \ldots, m_2 - 1,$$

$$a_n \le n(g_n - 1), \quad \text{for } n \ge m_2,$$

and

$$\lim_{n\to+\infty} \{a_n\xi\} = 0.$$

This proves the Theorem when $S = \{0\}$.

Assume now that $S \ne \{0\}$. For θ in $(0, 1]$, let $(d_n^{(\theta)})_{n\ge1}$ be an increasing sequence of non-negative integers such that $d_1^{(\theta)} = 0$, $\lim_{n\to+\infty} \{d_n^{(\theta)}\xi\} = \theta$ and $\{d_n^{(\theta)}\xi\} < \theta$, for $n \ge 1$. Let also $(d_n^{(0)})_{n\ge1}$ be an increasing sequence of positive integers such that $\lim_{n\to+\infty} \{d_n^{(0)}\xi\} = 0$. Assume that $S = \{\theta_1, \ldots, \theta_r\}$ for some positive integer r, and denote by $(d_n)_{n\ge1}$ the increasing sequence of integers obtained by taking the union of the r sequences $(d_n^{(\theta_1)})_{n\ge1}, \ldots, (d_n^{(\theta_r)})_{n\ge1}$. For every d in $(d_n)_{n\ge1}$, let $f(d)$ denote an integer i such that d belongs to the sequence $(d_n^{(\theta_i)})_{n\ge1}$. Note that this integer is uniquely determined when d is sufficiently large.

Let n_0 be an integer such that $n_0 \ge m_2$ and $\{a_n\xi\} < \theta$ for every non-zero θ in S and for every $n \ge n_0$. Let $(c_n)_{n\ge n_0}$ be a non-decreasing sequence of integers from $\{d_1, d_2, d_3, \ldots\}$ such that $\lim_{n\to+\infty} c_n = +\infty$,

$$c_n \le n, \quad |\theta_{f(c_n)} - \{c_n\xi\}| > \{a_n\xi\}, \quad \text{for } n \ge n_0,$$

and, for every $i = 1, \ldots, r$, the set $\mathcal{N}_i := \{n \ge n_0 : f(c_n) = i\}$ is infinite.

Setting $b_n = a_n$ for $n = 1, \ldots, n_0 - 1$ and $b_n = a_n + c_n$ for $n \ge n_0$, we check that

$$b_n \le n g_n, \quad \text{for } n \ge 1,$$

and that, for every $i = 1, \ldots, r$, we have

$$\lim_{\mathcal{N}_i \ni n \to +\infty} \{b_n \xi\} = \theta_i.$$

In particular, the set of limit points of $(\{b_n \xi\})_{n \ge 1}$ is equal to the set S. This ends the proof of the Theorem.

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