Multiplicative Diophantine approximation

Yann Bugeaud (Strasbourg)

1. Introduction

One of the first important results in Diophantine approximation goes back to 1842 and the Dirichlet Theorem [23]. It asserts that if α_{ij}, $1 \le i \le n$, $1 \le j \le m$, are mn real numbers and $Q > 1$ is an integer, then there exist integers $q_1, \ldots, q_m, p_1, \ldots, p_n$ with

$$1 \le \max\{|q_1|, \ldots, |q_m|\} \le Q \tag{1.1}$$

and

$$\max_{1 \le i \le n} |\alpha_{i1} q_1 + \ldots + \alpha_{im} q_m - p_i| \le Q^{-m/n}. \tag{1.2}$$

In his paper, Dirichlet gives a complete proof for $n = 1$ and observes that this proof can be easily extended to arbitrary values of n. Good references on this topic are Chapter II of [52] and Cassels' book [17].

There are in the literature many papers on various generalisations of the Dirichlet Theorem and on closely related problems. A typical question asks whether for a given set of mn real numbers α_{ij}, $1 \le i \le n$, $1 \le j \le m$, the above statement continues to hold with an exponent of Q in (1.2) smaller than $-m/n$. In most of the works, the sup norm is used, exactly as in (1.1) and (1.2). However, it follows from (1.1) and (1.2) that

$$1 \le \left(\prod_{j=1}^{m} \max\{1, |q_j|\} \right)^{1/m} \le Q \tag{1.3}$$

and

$$\left(\prod_{1 \le i \le n} |\alpha_{i1} q_1 + \ldots + \alpha_{im} q_m - p_i| \right)^{1/n} \le Q^{-m/n}. \tag{1.4}$$

Extensions of the Dirichlet Theorem and its relatives with the *geometric mean* like in (1.3) and (1.4), rather than the sup norm, have been much less studied, in particular because this is a much more difficult problem. We call this area *multiplicative Diophantine approximation*, in opposition to *standard* Diophantine approximation (following the terminology from [10]). An emblematic open question in multiplicative Diophantine approximation is

2000 *Mathematics Subject Classification :* 11J13.

the Littlewood conjecture which remains open despite some recent spectacular progress [28]. As for the typical problem mentioned above, it is easily seen that a positive answer to it in the standard setting implies a positive answer in the multiplicative setting, but that the converse does not hold.

Many recent progress, starting with the proof by Kleinbock and Margulis [41] of a conjecture of A. Baker, show that we are now able to attack many questions in multiplicative Diophantine approximation that seemed to be out of reach a decade ago. Kleinbock and Margulis used for the first time in this context the fruitful interplay between Diophantine approximation and dynamical and ergodic properties of actions on homogeneous spaces of Lie groups. Also, Beresnevich, Dickinson and Velani [6, 7] have very recently obtained beautiful results in the multiplicative setting by developing the standard theory. All this motivates the present work, which is essentially a survey, but contains also several new results. We aim at putting forward several, hopefully interesting, open questions. Part of them are certainly within reach, while others would probably need some breakthrough.

Our paper is organized as follows. Section 2 is devoted to the definitions of the multiplicative exponents of Diophantine approximation of matrices and include basic results like transference theorems and Hausdorff dimension of exceptional sets. We focus in Section 3 on the particular case of row and column matrices made up with the n successive powers of a real number. Thus, we define and study the functions Ω_n and Λ_n that complement the exponent w_n introduced by Mahler [44] to classify the set of real numbers. The short Section 4 briefly surveys some of the results obtained by using dynamical and ergodic properties of actions on homogeneous spaces of Lie groups, in the same spirit as in the groundbreaking work [41]. Finally, we discuss in Section 5 the Littlewood conjecture and introduce its inhomogeneous analogue.

2. Exponents of multiplicative Diophantine approximation

2.1. Definitions

For a real number x, let us denote by $\|x\|$ its distance to the ring of integers. More generally, for any (column) vector $\underline{y}$ in $\mathbf{R}^n$, we denote by $|\underline{y}|$ the maximum of the absolute values of its coordinates and by

$$\|\underline{y}\| = \min_{\underline{z} \in \mathbf{Z}^n} |\underline{y} - \underline{z}|$$

the maximum of the distances of its coordinates to the rational integers. There should not be any confusion with the absolute value $| \cdot |$.

We begin by recalling the standard framework in Diophantine approximation. Let n and m be two positive integers and let A be a real $n \times m$ matrix, that is, a matrix with n rows and m columns.

For an n-tuple $\underline{\theta}$ of real numbers, we denote by $w_{n,m}(A, \underline{\theta})$ the supremum of the real numbers w for which, *for arbitrarily large real numbers* X, the inequalities

$$\|A\underline{x} + \underline{\theta}\| \leq X^{-w} \quad \text{and} \quad 0 < |\underline{x}| \leq X \tag{2.1}$$

have a solution $\underline{x}$ in $\mathbf{Z}^m$. Keeping the notation from [15], let $\hat{w}_{n,m}(A, \underline{\theta})$ be the supremum of the real numbers w for which, *for all sufficiently large positive real numbers* X, the inequalities (2.1) have an integer solution $\underline{x}$ in $\mathbf{Z}^m$. We define furthermore two homogeneous exponents $w_{n,m}(A)$ and $\hat{w}_{n,m}(A)$ as in (2.1) with $\underline{\theta} = {}^t(0, \ldots, 0)$.

Although some results on the function $w_{n,m}$ will be recalled below, the reader is directed to [16] for more references.

The functions $w_{n,m}$ and $\hat{w}_{n,m}$ are defined by using the sup norm $|\cdot|$. But the choice of the distance function is arbitrary, and we may replace the sup norm of a vector by, for example, the geometric mean of the absolute values of its non-zero coordinates, or, alternatively, by the square root of the sum of the squares of its coordinates. This would enable us to define further exponents of approximation.

We feel that the former possibility is the most interesting one. Thus, we define the functions Π and Ξ, respectively, for $\underline{x} = (x_1, \ldots, x_m)$ in $\mathbf{Z}^m$ by

$$\Pi(\underline{x}) = \Big(\prod_{1 \leq i \leq m} \max\{1, |x_i|\} \Big)^{1/m},$$

and for $\underline{y} = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ by

$$\Xi(\underline{y}) = \Big({\prod_{1 \leq j \leq n}}' \, \|y_j\| \Big)^{1/n},$$

where $\prod'$ means that the product is taken over the indices j with $\|y_j\| \neq 0$. An empty product is understood to be equal to 0. Obviously, we have

$$|\underline{x}|^{1/m} \leq \Pi(\underline{x}) \leq |\underline{x}|$$

for any $\underline{x}$ in $\mathbf{Z}^m$, and

$$0 < \Xi(\underline{y}) \leq ||\underline{y}||$$

for any $\underline{y}$ in $\mathbf{R}^n \setminus \mathbf{Z}^n$.

With this notation, we define two exponents of *multiplicative* Diophantine approximation. We denote by $\Omega_{n,m}(A,\underline{\theta})$ the supremum of the real numbers w for which, *for arbitrarily large real numbers* X, the inequalities

$$\Xi(A\underline{x}+\underline{\theta}) \le X^{-w} \quad \text{and} \quad \Pi(\underline{x}) \le X \tag{2.2}$$

have a solution $\underline{x} \ne 0$ in $\mathbf{Z}^m$. Similarly, let $\hat{\Omega}_{n,m}(A,\underline{\theta})$ be the supremum of the real numbers w for which, *for all sufficiently large positive real numbers* X, the inequalities (2.2) have a solution $\underline{x} \ne 0$ in $\mathbf{Z}^m$. We define furthermore the homogeneous exponents $\Omega_{n,m}(A)$ and $\hat{\Omega}_{n,m}(A)$ as in (2.2) with $\underline{\theta} = {}^t(0,\ldots,0)$.

When there is no ambiguity, we simply write

$$w(A,\underline{\theta}), \hat{w}(A,\underline{\theta}), \Omega(A,\underline{\theta}), \hat{\Omega}(A,\underline{\theta}), w(A), \hat{w}(A), \Omega(A), \hat{\Omega}(A),$$

instead of

$$w_{n,m}(A,\underline{\theta}), \hat{w}_{n,m}(A,\underline{\theta}), \Omega_{n,m}(A,\underline{\theta}), \hat{\Omega}_{n,m}(A,\underline{\theta}), w_{n,m}(A), \hat{w}_{n,m}(A), \Omega_{n,m}(A), \hat{\Omega}_{n,m}(A),$$

respectively.

It is easy to check that

$$w(A) \le \Omega(A) \le \begin{cases} mw(A), & \text{if } n=1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

For any real $n \times m$ matrix A, Dirichlet's Theorem implies that

$$\Omega(A) \ge \hat{\Omega}(A) \ge \frac{m}{n}, \tag{2.4}$$

as we have seen in Section 1. Furthermore, Wang and Yu [59] established that we have both equalities in (2.4) for almost all matrices A, with respect to the Lebesgue measure on $\mathbf{R}^{mn}$.

2.2. Badly approximable matrices

By definition, an $n \times m$ real matrix A is *badly approximable* if

$$\inf_{\underline{x} \in \mathbf{Z}^m \setminus \{0\}} \|A\underline{x}\| \cdot |\underline{x}|^{m/n} > 0.$$

Schmidt [49] proved that the set of matrices A with this property has full Hausdorff dimension (see [52], pages 52–53, for additional references, in particular to earlier works of

Cassels and Davenport.) It is still an open question to decide whether, for any n-tuple $\underline{\theta}$ of real numbers, the set of $n \times m$ real matrix A such that

$$\inf_{\underline{x} \in \mathbf{Z}^m \setminus \{0\}} \| A\underline{x} + \underline{\theta} \| \cdot |\underline{x}|^{m/n} > 0$$

has full Hausdorff dimension (see Section 5 of [35]).

Similarly, the matrix A is *multiplicatively badly approximable* if

$$\inf_{\underline{x} \in \mathbf{Z}^m \setminus \{0\}} \Xi(A\underline{x}) \cdot \Pi(\underline{x})^{m/n} > 0.$$

For $m = n = 1$, the matrix $A = (\alpha)$ is (multiplicatively) badly approximable precisely if the irrational real number α is badly approximable, that is, has bounded partial quotients in its continued fraction expansion. For $mn \geq 2$, we do not know whether there exist multiplicatively badly approximable matrices, see Section 5 for some remarks on this celebrated problem, namely the Littlewood conjecture.

Furthermore, the matrix A is badly approximable if, and only if, the transpose matrix ${}^tA$ is badly approximable (see [17], Theorem VIII, page 84).

Most likely, the matrix A is multiplicatively badly approximable if, and only if, the transpose matrix ${}^tA$ is multiplicatively badly approximable. This was established by Cassels and Swinnerton-Dyer [18] for 1×2 matrices.

2.3. Transference theorems

It is well-known that $w(A)$ and $w({}^tA)$ are linked by a transference principle. Dyson [27] established the lower bound

$$w(A) \geq \frac{m\, w({}^tA) + m - 1}{(n-1)w({}^tA) + n},$$

thus extending earlier results of Khintchine [33, 34] who delt with the case $\min\{m, n\} = 1$.

Similarly, the exponents $\Omega(A)$ and $\Omega({}^tA)$ are linked by a transference principle. Extending results by Wang, Yu, and Zhu [60], Schmidt and Wang [53] (see [58]) established that

$$\Omega(A) > m/n \quad \text{if, and only if,} \quad \Omega({}^tA) > n/m, \tag{2.5}$$

a result reproved by Dodson and Kristensen [25] when $\min\{m, n\} = 1$. It is not difficult to deduce from the proof in [53] that, *under some additional assumptions on A*, we have

$$\Omega(A) \geq \frac{m\, \Omega({}^tA) + m - 1}{(n-1)\Omega({}^tA) + n}.$$

Furthermore, one gets from the proofs of [60, 53, 25] that, for any $1 \times m$ matrix A, we have exactly the same inequalities as in the standard transfer, namely

$$\Omega(A) \geq m\,\Omega({}^tA) + m - 1 \tag{2.6}$$

and

$$\Omega({}^tA) \geq \frac{\Omega(A)}{(m-1)\,\Omega(A) + m}.$$

It has been pointed out in [15] that the standard (resp. uniform) inhomogeneous exponents are related to the uniform (resp. standard) homogeneous exponents. In particular, for any n-tuple $\underline{\theta}$ of real numbers, we have the lower bounds

$$w(A, \underline{\theta}) \geq \frac{1}{\hat{w}({}^tA)} \quad \text{and} \quad \hat{w}(A, \underline{\theta}) \geq \frac{1}{w({}^tA)}, \tag{2.7}$$

with equality in (2.7) for almost all $\underline{\theta}$ with respect to the Lebesgue measure on $\mathbf{R}^n$.

It is not at all clear whether this transference principle has some analogue for multiplicative Diophantine approximation. We postpone to Section 5 a discussion on this topic.

2.4. Hausdorff dimension of exceptional sets, spectrum

The generic value of any of the four functions $w_{n,m}$, $\hat{w}_{n,m}$, $\Omega_{n,m}$ and $\hat{\Omega}_{n,m}$ is m/n. It is then desirable to determine the set of values they can take.

Definition 1. *By spectrum of an exponent of Diophantine approximation, we mean the set of values taken by this exponent on the set of $n \times m$ real matrices A of maximal rank, i.e. of rank* $\min\{m, n\}$.

The condition on the rank of A in Definition 1 means that we wish to avoid trivial constructions coming from smaller dimensions.

A useful tool, or, maybe, the most useful tool to determine the spectrum of an exponent of Diophantine approximation is the calculation of Hausdorff dimension and Hausdorff measures of exceptional sets.

For $n = m = 1$, Jarník [30] and, independently, Besicovitch [11] established that, for any $w \geq 1$, we have

$$\dim\{\xi \in \mathbf{R} : w((\xi)) \geq w\} = \frac{2}{1+w},$$

where dim stands for the Hausdorff dimension. A more precise statement, namely

$$\dim\{\xi \in \mathbf{R} : w((\xi)) = w\} = \frac{2}{1+w},$$

follows from a subsequent work of Jarník [31], where he calculated not only Hausdorff dimensions, but also Hausdorff measures. The latter result and the existence of Liouville numbers imply that the spectrum of the exponent w on the 1×1 matrices is equal to the whole interval $[1, +\infty]$.

The spectrum of $w_{n,m}$ has been determined by Dodson [24] (see also [22]). For positive integers m and n, let us denote by $\mathcal{M}_{n,m}(\mathbf{R})$ the set of $n \times m$ real matrices.

Theorem D. *For every real number $w \geq m/n$, we have*

$$\dim\{A \in \mathcal{M}_{n,m}(\mathbf{R}) : w(A) = w\} = (m-1)n + \frac{m+n}{1+w}.$$

In particular, the spectrum of the exponent w on the set of $n \times m$ real matrices is equal to the whole interval $[m/n, +\infty]$.

For a given positive integer n and a real number w with $w \geq 1/n$, Bovey and Dodson [12] proved that the set of $n \times 1$ matrices A for which $\Omega(A) \geq w$ has Hausdorff dimension equal to $n - 1 + 2/(1 + nw)$. This extension of the result of Jarník and Besicovitch has been further generalized to $n \times m$ matrices by Yu [62]. He proved that, for given integers m and n and for a real number $w \geq m/n$, the set of $n \times m$ matrices A for which $\Omega(A) \geq w$ has Hausdorff dimension equal to $mn - 1 + 2/(1 + nw/m)$.

Both results are not sufficient to ensure the existence of matrices A with prescribed values of $\Omega(A)$. However, it is possible, and quite easy, to adapt Bovey and Dodson's and Yu's proofs in order to establish the following assertion.

Theorem 1. *For every real number $w \geq m/n$, we have*

$$\dim\{A \in \mathcal{M}_{n,m}(\mathbf{R}) : \Omega(A) = w\} = mn - 1 + \frac{2}{1 + nw/m}.$$

In particular, the spectrum of the exponent Ω on the set of $n \times m$ real matrices is equal to the whole interval $[m/n, +\infty]$.

Proof. We briefly explain how one should modify Bovey and Dodson's proof to deal with the case $m = 1$. The basic idea is that we have to take dimension functions outside the family $f_s : x \mapsto x^s$. The easiest way is then to work with the two-parameters family $f_{s,t} : x \mapsto x^s (\log 1/x)^t$. With the notation from [12], we choose $s = (k-1) + 2/(\alpha - k + 1)$ and a suitable negative real number t and show that the Hausdorff $f_{s,t}$-measure of the set denoted by $E_\alpha^{(k)}$ in [12] is zero. We leave the details to the reader. $\square$

3. Approximation of dependent quantities

3.1. Exponents of multiplicative approximation for real numbers

Let ξ be a real number and n be a positive integer. In this section, we restrict our attention to the matrices

$$A_n = (\xi, \ldots, \xi^n) \quad \text{and} \quad {}^tA_n = {}^t(\xi, \ldots, \xi^n),$$

made up with the successive powers of ξ.

Mahler [44] defined in 1932 the first classification of real numbers (actually, of complex numbers, but we restrict our attention to real numbers) in terms of their properties of Diophantine approximation. Keeping his notation, for every integer $n \geq 1$, let us denote by $w_n(\xi)$ the supremum of the real numbers w such that the inequality

$$0 < |P(\xi)| \leq H(P)^{-w}$$

holds for infinitely many polynomials $P(X)$ with integer coefficients and degree at most n (the height $H(P)$ of a polynomial $P(X)$ is the maximum of the moduli of its coefficients). With the notation of Section 2, we have $w_n(\xi) = w_{1,n}(A_n)$ when ξ is transcendental or algebraic of degree greater than n.

Rather than considering small values of the linear form whose coefficients are the successive powers of ξ, we may also study the simultaneous rational approximation of successive powers of ξ. Following [14], we denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that the inequality

$$\max_{1 \leq m \leq n} |x_0 \xi^m - x_m| \leq |x_0|^{-\lambda}$$

has infinitely many solutions in integers $x_0, \ldots, x_n$ with $x_0 \neq 0$. With the notation of Section 2, we have $\lambda_n(\xi) = w_{n,1}({}^tA_n)$ when ξ is transcendental or algebraic of degree greater than n.

The reader is directed to [13, 16] for results on the exponents w_n and λ_n. We just recall a theorem of Sprindžuk [54].

Theorem S. *For almost all real numbers ξ, we have $w_n(\xi) = n$ and $\lambda_n(\xi) = 1/n$.*

Motivated by several recent works [41, 7, 3], we introduce two new exponents, which correspond to the *multiplicative* versions of the exponents w_n and λ_n, respectively.

Instead of the height $H(P)$ of an integer, non-constant polynomial $P(X) = x_n X^n + \ldots + x_1 X + x_0$, we consider the function Π defined by

$$\Pi(P) := \left(\prod_{1 \leq i \leq \deg P} \max\{1, |x_i|\} \right)^{1/\deg P} = \Pi((x_1, \ldots, x_{\deg P})).$$

Observe that we have

$$\Pi(P) \leq \max\{|x_1|, \ldots, |x_{\deg P}|\} \leq \Pi(P)^{\deg P}.$$

Consequently, there exist positive constants $c_1(\xi, \deg P)$ and $c_2(\xi, \deg P)$, depending only on $\deg P$ and on ξ, such that

$$c_1(\xi, \deg P)\, \Pi(P) \le H(P) \le c_2(\xi, \deg P)\, \Pi(P)^{\deg P}, \quad \text{if } |P(\xi)| \le 1.$$

This fact will be used repeatedly throughout the rest of this section.

Definition 2. *Let $n \ge 1$ be an integer and let ξ be a real number. We denote by $\Omega_n(\xi)$ the supremum of the real numbers w such that the inequality*

$$0 < |P(\xi)| \le \Pi(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most n.

The exponent of approximation w_n^+ introduced in [13], page 112, differs from the exponent Ω_n by a factor n. However, we feel that Ω_n is the 'right' function to study.

Definition 3. *Let $n \ge 1$ be an integer and let ξ be a real number. We denote by $\Lambda_n(\xi)$ the supremum of the real numbers λ such that the inequality*

$$0 < \left(\prod_{1 \le m \le n} |x_0 \xi^m - x_m| \right)^{1/n} \le |x_0|^{-\lambda} \tag{3.1}$$

has infinitely many solutions in integers $x_0, \ldots, x_n$ with $x_0 \neq 0$.

Note that (3.1) can be rewritten as $0 < \Xi(x_0{}^t A_n) \le |x_0|^{-\lambda}$ when ξ is irrational.

Throughout the rest of this section, we present various results on the functions Ω_n and Λ_n. We begin with stating immediate consequences of (2.3) and of the transference principles from Section 2.3.

Proposition 1. *For every positive integer n and every real number ξ we have*

$$w_n(\xi) \le \Omega_n(\xi) \le n w_n(\xi) \tag{3.2}$$

and

$$\lambda_n(\xi) \le \Lambda_n(\xi) \le +\infty.$$

Furthermore, we have

$$\Omega_n(\xi) \ge n\Lambda_n(\xi) + n - 1, \quad \Lambda_n(\xi) \ge \frac{\Omega_n(\xi)}{(n-1)\Omega_n(\xi) + n}.$$

A. Baker conjectured that, for almost all real numbers ξ and for any positive real number ε, the equation $|P(\xi)| \le \Pi(P)^{-n-\varepsilon}$ has only finitely many solutions in integer polynomials $P(X)$ of degree at most n. This has been established by Sprindžuk [55] for $n = 2$ (see also Theorem 2 of Yu [63]), by Bernik and Borbat [9] for $n = 3, 4$, and by Kleinbock and Margulis [41] for arbitrary n .

Theorem KM. *For almost all real numbers ξ, we have $\Omega_n(\xi) = n$ and $\Lambda_n(\xi) = 1/n$ for every positive integer n.*

Proposition 1 shows that Theorem KM refines Theorem S.

Theorem 2. *Let n be a positive integer. For every real algebraic number ξ, we have $\Omega_n(\xi) = n$ and $\Lambda_n(\xi) = 1/n$.*

Proof. We adapt suitably the proof of Theorem 1D, page 152, of [52], that rests on the Schmidt Subspace Theorem [51]. We prove by induction on n that, for every non-zero real algebraic numbers $\xi_1, \ldots, \xi_n$ and every positive ε, the inequality

$$0 < |x_0 + \xi_1 x_1 + \ldots + \xi_n x_n| \leq \Pi((x_1, \ldots, x_n))^{-n(1+\varepsilon)}, \tag{3.3}$$

has only finitely many solutions in integer $(n+1)$-tuples $(x_0, x_1, \ldots, x_n)$. For $n = 1$, this is nothing but Roth's Theorem. If $n \geq 2$, then the subspace theorem implies that all the integer solutions to (3.3) are contained in finitely many rational subspaces, say in the union of the T subspaces $A_0^{(i)} z_0 + \ldots + A_n^{(i)} z_n = 0$, $1 \leq i \leq T$. Let then $(A_0, \ldots, A_n)$ be an $(n+1)$-tuple and $(x_0, \ldots, x_n)$ a solution to (3.3) such that $A_0 x_0 + \ldots + A_n x_n = 0$ and $A_n \neq 0$. Setting $\zeta_i = \xi_i - \xi_n A_i / A_n$, we get

$$\begin{aligned} 0 < |x_0 + \xi_1 x_1 + \ldots + \xi_n x_n| &= |x_0 + \zeta_1 x_1 + \ldots + \zeta_{n-1} x_{n-1}| \\ &\leq \Pi((x_1, \ldots, x_n))^{-n(1+\varepsilon)} \\ &\leq \Pi((x_1, \ldots, x_{n-1}))^{-(n-1)(1+\varepsilon)}. \end{aligned}$$

The induction hypothesis implies that there are finitely many possibilities for the n-tuple $(x_0, \ldots, x_{n-1})$. Since $\xi_n \neq 0$ and the number of subspaces is finite, this shows that $\Omega_n(\xi) \leq n$.

Let d be the degree of ξ. If ξ is rational, then we check easily that $\Omega_n(\xi) \geq n$. Otherwise, we have $d \geq 2$ and, by the Dirichlet *Schubfachprinzip*, there are integers $x_n, x_{d-2}, x_{d-3}, \ldots, x_0$, not all zero, with $X := \max\{|x_n|, |x_{d-2}|, \ldots, |x_1|\}$ arbitrarily large, such that

$$|x_n \xi^n + x_{d-2} \xi^{d-2} + \ldots + x_1 \xi + x_0| \leq \Pi((x_n, x_{d-2}, \ldots, x_1))^{-d+1}.$$

Since ξ has degree d, the left-hand side of the inequality above does not vanish. Furthermore, x_n is non-zero when X is sufficiently large, since $\Omega_{d-2}(\xi) \leq d-2$. Thus, there exist integer polynomials $P(X)$ of degree exactly n such that $0 < |P(\xi)| \leq \Pi(P)^{-n}$. This gives $\Omega_n(\xi) \geq n$, as wanted. □

Mahler [44] used the functions w_n to define a classification of real numbers. We could proceed similarly with the functions Ω_n, however, we are not completely convinced that this classification would be relevant. In particular, it does not seem to be at all clear whether two algebraically dependent real numbers are in the same class. We stress that, for any integer $n \geq 2$, there do not exist a constant $c(n)$, depending only on n, such that $\Pi(P) \cdot \Pi(Q) \leq c(n)\, \Pi(PQ)$ for every integer polynomials $P(X)$, $Q(X)$ of degree at most n.

This shows that the situation is very different from the standard setting, where the naïve height H is used instead of Π.

Bernik [8] proved that the spectrum of w_n is equal to $[n, +\infty]$ and, even, that

$$\dim\{\xi \in \mathbf{R} : w_n(\xi) = w\} = \frac{n+1}{1+w}, \quad (w \geq n).$$

At present no method is known that avoids the theory of Hausdorff dimension and yields that the spectrum of w_n is equal to the interval $[n, +\infty]$. In particular, we do not have any constructive proof of Bernik's result.

We propose to consider three problems about spectra.

Question 1. *Let $\lambda > 1/n$ be real. Do there exist real numbers ξ with $\lambda_n(\xi) = \lambda$?*

A partial answer to this question can be found in Theorem 10 from [16], where it is proved (with explicit examples) that the spectrum of λ_n includes the interval $((1 + \sqrt{4n^2+1})/(2n), +\infty]$.

For $\lambda > 1/n$, it does not seem to be easy to compute

$$\dim\{\xi \in \mathbf{R} : \lambda_n(\xi) = \lambda\}.$$

This problem has been solved for $n = 2$ and $\lambda \in [1/2, 1]$ by Beresnevich, Dickinson, Vaughan and Velani [6, 57], who proved that

$$\dim\{\xi \in \mathbf{R} : \lambda_2(\xi) = \lambda\} = \frac{2-\lambda}{1+\lambda}.$$

As kindly pointed out to me by Michel Laurent, for $\lambda > 1$, the method from [57] yields that

$$\dim\{\xi \in \mathbf{R} : \lambda_2(\xi) = \lambda\} = \frac{1}{1+\lambda}. \tag{3.4}$$

According to the referee, the proof of Theorem 19 from [5] can be adapted to the case of the parabola to get (3.4). This completes the resolution of Question 1 for $n = 2$.

Question 2. *Let $\Omega > n$ be real. Do there exist real numbers ξ with $\Omega_n(\xi) = \Omega$?*

Question 3. *Let $\Lambda > 1/n$ be real. Do there exist real numbers ξ with $\Lambda_n(\xi) = \Lambda$?*

We give now a partial answer to Questions 2 and 3, and show that the right-hand side inequality of (3.2) is sharp. The key idea lies at heart of a work of Güting [29], where the existence of real numbers ξ with prescribed values for $w_n(\xi)$ for some integers n is established in a constructive way. We adapt Theorem 7.7 from [13] for our purpose.

Theorem 3. *Let $n \geq 1$ be an integer. Let d be a real number with*

$$(d-n)(d-n+1) > n^2(d+1). \tag{3.5}$$

Then, there exist real numbers ξ with

$$w_n(\xi) = d \tag{3.6}$$

and

$$\Omega_n(\xi) = nd. \tag{3.7}$$

In particular, the function Ω_n takes any value Ω with

$$\Omega \geq n^2(n+2).$$

Proof. Let d and n be as in the statement of the theorem. Throughout the proof, the numerical constants implied by $\ll$ and $\gg$ depend, at most, on n and d. Let $(n_i)_{i\geq 1}$ be a strictly increasing sequence of positive integers such that n_{i+1}/n_i tends to $d+1$ and $\gcd(n_i, n) = 1$ for any $i \geq 1$ (take for instance $n_i = n\,[(d+1)^i] + 1$, for $i \geq 1$). Define the positive real number ξ by

$$\xi^n = \sum_{j\geq 1} 2^{-n_j}.$$

Let ε be a real number with $0 < \varepsilon < 1$. Let $i_0 \geq 1$ be such that $d+1-\varepsilon < n_{i+1}/n_i < d+1+\varepsilon$ holds for any integer $i \geq i_0$. For $i \geq i_0$, the polynomial

$$P_i(X) := 2^{n_i}X^n - 2^{n_i}\sum_{j=1}^{i} 2^{-n_j}$$

satisfies $H(P_i) = 2^{n_i}$ and $\Pi(P_i) = 2^{n_i/n}$. Furthermore, since n_i and n are coprime, a result of Dumas [26] asserts that $P_i(X)$ is irreducible. It follows from

$$P_i(\xi) = \sum_{j>i} 2^{n_i-n_j} = \sum_{j>i} H(P_i)^{1-n_j/n_i}, \quad P_i'(\xi) = n\xi^{n-1}H(P_i),$$

and Lemma A.5 from [13] that $P_i(X)$ has a root α_i such that

$$H(P_i)^{-d-1-\varepsilon} \ll |\xi - \alpha_i| \ll H(P_i)^{-d-1+\varepsilon}. \tag{3.8}$$

It is immediate that, for $i \geq i_0$, we have

$$H(P_i)^{d+1-\varepsilon} \leq H(P_{i+1}) \leq H(P_i)^{d+1+\varepsilon} \tag{3.9}$$

and

$$\Pi(P_i)^{-nd-n\varepsilon} \ll |P_i(\xi)| \ll \Pi(P_i)^{-nd+n\varepsilon},$$

thus

$$\Omega_n(\xi) \geq nd. \tag{3.10}$$

Let $P(X)$ be an integer polynomial of degree at most n. Since the polynomials $P_i(X)$, $i \geq i_0$, are irreducible of degree n, we assume without any loss of generality that $P(X)$ is not a constant multiple of one of the polynomials $P_i(X)$ with $i \geq i_0$.

There exists an integer i with

$$\Pi(P_i) \leq \Pi(P) < \Pi(P_{i+1}). \tag{3.11}$$

We may assume that $\Pi(P)$ is sufficiently large in order to ensure that $i \geq i_0$. Thus, (3.8) and (3.9) are satisfied. We distinguish two cases and introduce a real number $u > n$, which will be specified later on.

First, we assume that

$$H(P)^n \leq \Pi(P_i)^u. \tag{3.12}$$

The Liouville inequality (see e.g. Corollary A.2 from [13]) asserts that $|P(\alpha_i)| \gg H(P)^{-n+1}\, H(P_i)^{-n}$, whence, by (3.12), we get

$$|P(\alpha_i)| \gg H(P) \cdot \Pi(P_i)^{-u} \cdot H(P_i)^{-n} \gg H(P) \cdot \Pi(P_i)^{-n^2-u}.$$

We infer from (3.8) and Rolle's Theorem that

$$|P(\alpha_i) - P(\xi)| \ll H(P) \cdot \Pi(P_i)^{-n(d+1)+n\varepsilon}.$$

Consequently, by (3.11),

$$|P(\xi)| \gg H(P) \cdot \Pi(P_i)^{-nd+n\varepsilon} \gg \Pi(P)^{-nd+n\varepsilon} \tag{3.13}$$

holds as soon as

$$n^2 + u < nd - n\varepsilon,$$

which can be rewritten

$$u < n(d - n - \varepsilon). \tag{3.14}$$

We assume now that

$$H(P)^n > \Pi(P_i)^u. \tag{3.15}$$

We then get $|P(\alpha_{i+1})| \gg H(P)^{-n+1}\, H(P_{i+1})^{-n}$ by Corollary A.2 from [13], and we infer from (3.8) and Rolle's Theorem that

$$|P(\xi) - P(\alpha_{i+1})| \ll H(P) \cdot \Pi(P_{i+1})^{-n(d+1)+n\varepsilon}.$$

Observe that

$$H(P) \cdot \Pi(P_{i+1})^{-n(d+1)+n\varepsilon} \ll H(P)^{-n+1}\, H(P_{i+1})^{-n}$$

since $H(P) \leq \Pi(P_{i+1})^n$ and $d \geq 2n + \varepsilon$. Consequently,

$$|P(\xi)| \geq \Pi(P)^{-nd+n\varepsilon} \tag{3.16}$$

holds if $H(P)^{-n+1} H(P_{i+1})^{-n} \gg \Pi(P)^{-nd+n\varepsilon}$. The latter inequality holds as soon as

$$H(P)^{n-1} \cdot H(P_{i+1})^n \ll H(P)^{d-\varepsilon},$$

thus, by (3.15), as soon as,

$$\Pi(P_i)^{u(d-\varepsilon)/n} \cdot \Pi(P_i)^{-u(n-1)/n} \gg H(P_{i+1})^n. \tag{3.17}$$

We deduce from (3.9) that $H(P_{i+1})^n \ll \Pi(P_i)^{n^2(d+1+\varepsilon)}$, thus (3.17) holds if

$$u > \frac{n^3(d+1+\varepsilon)}{d-\varepsilon-n+1}. \tag{3.18}$$

Since the positive real number ε can be taken arbitrarily small, we can select a real number u such that (3.14) and (3.18) hold simultaneously when

$$n(d-n) > \frac{n^3(d+1)}{d-n+1}.$$

Consequently, if (3.5) holds, then we infer from (3.13) and (3.16) that

$$\Omega_n(\xi) \leq nd.$$

Combined with (3.10), this gives (3.7). To conclude the proof, it remains to note that (3.6) follows from Theorem 7.7 of [13]. Note that there is a misprint in the statement of this theorem, namely, one should read $d > (2n-1+\sqrt{4n^2+1})/2$ instead of $d > (2n+1+\sqrt{4n^2+1})/2$. $\square$

Question 2 for $n = 1$ is solved by the Jarník–Besicovitch theorem. Yu [63] (see also [13], Ex. 5.6) established that, for any $w \geq 2$, the Hausdorff dimension of the set of real numbers ξ with $\Omega_2(\xi) \geq w$ is equal to $4/(2+w)$. This has been rediscovered by Beresnevich and Bernik [4]. As observed in [13], Ex. 5.6, a slight modification of their proof yields that

$$\dim\{\xi \in \mathbf{R} : \Omega_2(\xi) = w\} = \frac{4}{2+w}.$$

In particular, the spectrum of the exponent Ω_2 is equal to $[2, +\infty]$.

Yu further conjectured that, for $n \geq 3$ and $w \geq n$, the set $\{\xi \in \mathbf{R} : \Omega_n(\xi) \geq w\}$ has Hausdorff dimension equal to $(2n)/(n+w)$. The fact that this is a lower bound for the dimension is contained in Theorem 7 from [7]. To prove the reverse inequality seems to be quite difficult. Yu's conjecture is consistent with the results from Subsection 2.4.

Beresnevich and Velani ([7], Theorem 5) proved that, for $n \geq 2$ and $\lambda > 1/n$, the Hausdorff dimension of the set $\{\xi \in \mathbf{R} : \Lambda_n(\xi) = \lambda\}$ is at least $2/(1+n\lambda)$. In view of Theorem 1, it is likely that we have equality. This guess is supported by a recent result of Badziahin and Levesley [3], who established that

$$\dim\{\xi \in \mathbf{R} : \Lambda_2(\xi) \geq \lambda\} \leq \frac{2}{1+2\lambda}.$$

Consequently, the spectrum of Λ_2 is equal to the whole interval $[1/2, +\infty]$.

3.2. On uniform Diophantine approximation

The exponents w_n and λ_n have in common to be defined by the occurrence of infinitely many solutions to some Diophantine inequalities. In [14], we attached to them the functions $\hat{w}_n$ and $\hat{\lambda}_n$ defined by a condition of *uniform* existence of solutions. Similarly, we introduce two exponents of uniform, multiplicative Diophantine approximation of real numbers.

Definition 4. *Let $n \geq 1$ be an integer and let ξ be a real number. We denote by $\hat{\Omega}_n(\xi)$ the supremum of the real numbers w such that, for any sufficiently large real number X, the inequality*

$$0 < |P(\xi)| \leq X^{-w}$$

is satisfied by an integer polynomial P of degree at most n and with $\Pi(P) \leq X$. We denote by $\hat{\Lambda}_n(\xi)$ the supremum of the real numbers λ such that, for any sufficiently large real number X, the inequality

$$0 < \Big(\prod_{1 \leq m \leq n} |x_0 \xi^m - x_m| \Big)^{1/n} \leq X^{-\lambda}$$

has a solution in integers $x_0, \ldots, x_n$ with $1 \leq |x_0| \leq X$.

Recent important works of Roy (see [48] and his earlier papers quoted therein) show that real numbers ξ with $\hat{w}_2(\xi) > 2$ (and, even, with $\hat{w}_2(\xi) = (3+\sqrt{5})/2$) do exist. In particular, these numbers satisfy $\hat{\Omega}_2(\xi) > 2$.

There is no result on the exponents $\hat{\Omega}_n$ and $\hat{\Lambda}_n$ in the literature. Among the many interesting open problems, let us display the following.

Question 4. *To find the suprema of the functions $\hat{\Omega}_n$ and $\hat{\Lambda}_n$.*

Davenport and Schmidt [20] proved that the supremum of $\hat{w}_n$ is at most $2n-1$, consequently, $\hat{\Omega}_n$ cannot exceed $n(2n-1)$. For $n=2$, Arbour and Roy [2] established that the supremum of $\hat{w}_2$ is $(3+\sqrt{5})/2$, thus $\hat{\Omega}_2$ cannot exceed $3+\sqrt{5}$. It is likely that the above bounds for $\hat{\Omega}_n$ can be refined.

4. Approximation on submanifolds and friendly measures

According to the terminology of Sprindžuk and of Kleinbock and Margulis, a submanifold $\mathcal{M} \subset \mathbf{R}^n$ is *extremal* (resp. *strongly extremal*) if almost every point $\underline{y}$ on $\mathcal{M}$ satisfies $w_{1,n}(\underline{y}) = n$ (resp. $\Omega_{1,n}(\underline{y}) = n$). It follows from Theorem A from [41] that the manifold $\{(x, x^2, \ldots, x^n) : x \in \mathbf{R}\}$ is strongly extremal, and this gives the above Theorem KM. Its proof is based on the correspondence between approximation properties of real n-tuples and behaviour of certain orbits in the space of unimodular lattices in $\mathbf{R}^{n+1}$. This correspondence dates back to the works of Davenport and Schmidt [21], Schmidt [50] and Dani [19]. The reader is directed to the survey papers [36, 39, 45] and to the book of Starkov [56] for a detailed exposition of the main principle behind a reduction of Theorem KM to a dynamical statement.

Theorem KM is the first general result concerning multiplicative Diophantine approximation in arbitrary dimension. It has been subsequently extended and generalized in many directions, see e.g. [10, 37, 38, 39, 40, 42]. We briefly state a recent result from [40].

Theorem KLW. *Any friendly measure μ on $\mathbf{R}^n$ is strongly extremal.*

Friendly measures are defined in [40] using purely geometric conditions. They include volume measures on smooth manifolds considered in [41] and the Hausdorff measure on the Cantor set. In particular, Theorem KLW extends a previous result of Weiss [61] that establishes the extremality of the natural measure supported by the middle-third Cantor set.

5. The Littlewood conjecture

It follows from the theory of continued fractions that, for any real number α, there exist infinitely many positive integers q such that $q \cdot \|q\alpha\| < 1$. In particular, for any given pair (α, β) of real numbers, there exist infinitely many positive integers q such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| < 1.$$

A famous open problem in simultaneous Diophantine approximation, called the Littlewood conjecture [43], claims that in fact, for any given pair (α, β) of real numbers, a stronger result should hold, namely

$$\inf_{q\geq 1} q\cdot \|q\alpha\|\cdot\|q\beta\| = 0. \tag{5.1}$$

With the definition from Section 2.2, the Littlewood conjecture predicts that the matrix (α, β) is never multiplicatively badly approximable. Recently, Pollington and Velani [46] proved that for any badly approximable number α, the set of badly approximable numbers β such that (5.1) holds has maximal Hausdorff dimension (actually, their result is stronger). This was improved by Einsiedler, Katok and Lindenstrauss [28], who established that the set of real pairs (α, β) for which (5.1) does not hold has Hausdorff dimension zero. Despite this beautiful result, the Littlewood conjecture remains unsolved.

Cassels and Swinnerton-Dyer [18] proved that (5.1), which can be rewritten as

$$\inf_{q\geq 1} q\cdot \Xi((q\alpha, q\beta))^2 = 0,$$

is equivalent to the equality

$$\inf_{(x,y)\in \mathbf{Z}\times\mathbf{Z}\setminus\{(0,0)\}} \max\{|x|, 1\}\cdot\max\{|y|,1\}\cdot\|x\alpha + y\beta\| = 0,$$

that is, with the present notation, to

$$\inf_{(x,y)\in \mathbf{Z}\times\mathbf{Z}\setminus\{(0,0)\}} \|x\alpha + y\beta\|\cdot \Pi((x,y))^2 = 0.$$

This corresponds to the result mentioned at the end of Section 2.2. They used this to show that (5.1) holds if α and β belong to the same cubic number field. The reader is directed to [1, 47] and to Chapter 10 of [13] for additional bibliographic references.

Transference principles that link homogeneous approximation with inhomogeneous uniform approximation are well-known [17, 15], and one of them is recalled at the end of Section 2.3. These suggest us to introduce the inhomogeneous version of the Littlewood conjecture.

Question 5. *Let α and β be real numbers with $1, \alpha, \beta$ being linearly independent over the rationals. Let α_0, β_0 and γ be real numbers. To prove or to disprove that*

$$\inf_{q\neq 0} |q|\cdot\|q\alpha - \alpha_0\|\cdot\|q\beta - \beta_0\| = 0$$

and/or that

$$\inf_{(x,y)\neq(0,0)} \|x\alpha + y\beta - \gamma\|\cdot\max\{|x|,1\}\cdot\max\{|y|,1\} = 0.$$

Recall that Minkowski's Theorem (see [17], page 48) asserts that if α is irrational and if α_0 is not of the form $m\alpha + n$ for integers m, n, then there are infinitely many integers q such that

$$|q|\cdot\|q\alpha - \alpha_0\| < 1/4.$$

Consequently, for α, β, α_0, β_0 and γ as in Question 5, we have

$$\liminf_{|q|\to+\infty} |q| \cdot \|q\alpha - \alpha_0\| \cdot \|q\beta - \beta_0\| < 1$$

and

$$\liminf_{\max\{|x|,|y|\}\to+\infty} \|x\alpha + y\beta - \gamma\| \cdot \max\{|x|, 1\} \cdot \max\{|y|, 1\} < 1.$$

As far as we are aware, Question 5 has not been studied up to now.

Good candidates for disproving the conclusion of Question 5 could be the pairs of real numbers (α, β) such that $\hat{w}((\alpha, \beta)) = +\infty$. The existence of such pairs has been proved by Khintchine [34]. In view of the transference principle enounced at the end of Section 2.3, we have $w({}^t(\alpha, \beta), \underline{\theta}) = 0$ for almost all $\underline{\theta}$. This means that for almost all pairs (α_0, β_0) and for every positive real number ε, we have

$$\max\{\|q\alpha - \alpha_0\|, \|q\beta - \beta_0\|\} > q^{-\varepsilon},$$

for every sufficiently large positive integer q. By a result of Jarník [32], these pairs (α, β) also satisfy $\hat{w}({}^t(\alpha, \beta)) = 1$, and the same transference principle yields that, for almost all real numbers γ, we have

$$\|x\alpha + y\beta - \gamma\| > \max\{|x|, |y|\}^{-1-\varepsilon},$$

when $\max\{|x|, |y|\}$ is sufficiently large.

Acknowledgements: I am pleased to thank Dmitry Kleinbock, Michel Laurent and Maurice Dodson for useful comments and suggestions. I am also very grateful to the referee for his very careful reading and his numerous remarks.

References

[1] B. Adamczewski and Y. Bugeaud, *On the Littlewood conjecture in simultaneous Diophantine approximation*, J. London Math. Soc. 73 (2006), 355–366.

[2] B. Arbour and D. Roy, *A Gel'fond type criterion in degree two*, Acta Arith. 111 (2004), 97–103.

[3] D. Badziahin and J. Levesley, *Convergence Theorems for Simultaneous and Multiplicative Diophantine Approximation on Planar Curves*, Glasgow Math. J. 49 (2007), 367–375.

[4] V. Beresnevich and V. Bernik, *A. Baker's conjecture and Hausdorff dimension*, Publ. Math. Debrecen 56 (2000), 263–269.

[5] V. Beresnevich, H. Dickinson and S. L. Velani, Measure theoretic laws for lim sup sets. Mem. Amer. Math. Soc. 179 (2006), no. 846, x+91 pp.

[6] V. Beresnevich, H. Dickinson and S. L. Velani, *Diophantine approximation on planer curves and the distribution of rational points*, with an appendix by R.C. Vaughan: 'Sums of two squares near perfect squares', Ann. of Math. 166 (2007), 367–426.

[7] V. Beresnevich and S. L. Velani, *A note on simultaneous Diophantine approximation on planar curve*, Math. Ann. 337 (2007), 769–796.

[8] V. I. Bernik, *On the best approximation of zero by values of integral polynomials*, Acta Arith. 53 (1989), 17–28 (in Russian).

[9] V. I. Bernik and V. N. Borbat, *Polynomials with overfalls in values of coefficients and a conjecture of A. Baker*, Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.–Mat. Navuk (1997), no. 3, 5–8 (in Russian).

[10] V. Bernik, D. Kleinbock, and G. A. Margulis, *Khintchine–type theorems on manifolds: the convergence case for standard and multiplicative versions*, Internat. Math. Res. Notices (2001), no. 9, 453–486.

[11] A. S. Besicovitch, *Sets of fractional dimension (IV): on rational approximation to real numbers*, J. London Math. Soc. 9 (1934), 126–131.

[12] J. D. Bovey and M. M. Dodson, *The fractional dimension of sets whose simultaneous rational approximations have errors with a small product*, Bull. London Math. Soc. 10 (1978), 213–218.

[13] Y. Bugeaud, Approximation by algebraic numbers, Cambridge Tracts in Mathematics, Cambridge, 2004.

[14] Y. Bugeaud and M. Laurent, *Exponents of Diophantine Approximation and Sturmian Continued Fractions*, Ann. Inst. Fourier (Grenoble) 55 (2005), 773–804.

[15] Y. Bugeaud and M. Laurent, *Exponents of homogeneous and inhomogeneous Diophantine Approximation*, Moscow Math. J. 5 (2005), 747–766.

[16] Y. Bugeaud and M. Laurent, *Exponents of Diophantine approximation*, In: Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa, Ser. CRM, vol. 4, 2007, 101–121.

[17] J. W. S. Cassels, An introduction to Diophantine Approximation, Cambridge Tracts in Math. and Math. Phys., vol. 99, Cambridge University Press, 1957.

[18] J. W. S. Cassels and H. P. F. Swinnerton-Dyer, *On the product of three homogeneous linear forms and indefinite ternary quadratic forms*, Philos. Trans. Roy. Soc. London, Ser. A, 248 (1955), 73–96.

[19] S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and Diophantine approximation*, J. reine angew. Math. 359 (1985), 55–89.

[20] H. Davenport and W. M. Schmidt, *Approximation to real numbers by algebraic integers*, Acta Arith. 15 (1969), 393–416.

[21] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on Diophantine approximation. II*, Acta Arith. 16 (1970), 413–423.

[22] H. Dickinson and S. L. Velani, *Hausdorff measure and linear forms*, J. reine angew. Math. 490 (1997), 1–36.

[23] L. G. P. Dirichlet, *Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einige Anwendungen auf die Theorie der Zahlen*, S.-B. Preuss. Akad. Wiss. (1842), 93–95.

[24] M. M. Dodson, *Hausdorff dimension, lower order and Khintchine's theorem in metric Diophantine approximation*, J. reine angew. Math. 432 (1992), 69-76.

[25] M. M. Dodson and S. Kristensen, *Khintchine's theorem and transference principle for star bodies*, Int. J. Number Theory 2 (2006), 431–453.

[26] G. Dumas, *Sur quelques cas d'irréductibilité des polynomes à coefficients rationnels*, J. Math. pures appl. (6) 2 (1906), 191–258.

[27] F. J. Dyson, *On simultaneous Diophantine approximations*, Proc. London Math. Soc. 49 (1947), 409–420.

[28] M. Einsiedler, A. Katok and E. Lindenstrauss, *Invariant measures and the set of exceptions to the Littlewood conjecture*, Ann. of Math. 164 (2006), 513–560.

[29] R. Güting, *Zur Berechnung der Mahlerschen Funktionen* w_n, J. reine angew. Math. 232 (1968), 122–135.

[30] V. Jarník, *Diophantischen Approximationen und Hausdorffsches Mass*, Mat. Sbornik 36 (1929), 371–382.

[31] V. Jarník, *Über die simultanen Diophantische Approximationen*, Math. Z. 33 (1931), 505–543.

[32] V. Jarník, *Zum Khintchineschen "Übertragungssatz"*, Trav. Inst. Math. Tbilissi 3 (1938), 193–212.

[33] A. Ya. Khintchine, *Zwei Bemerkungen zu einer Arbeit des Herrn Perron*, Math. Z. 22 (1925), 274–284.

[34] A. Ya. Khintchine, *Über eine Klasse linearer diophantischer Approximationen*, Rendiconti Circ. Mat. Palermo 50 (1926), 170–195.

[35] D. Kleinbock, *Badly approximable systems of affine forms*, J. Number Theory 79 (1999), 83–102.

[36] D. Kleinbock, *Some applications of homogeneous dynamics to number theory*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proceedings of Symposia in Pure Mathematics Volume 69, Amer. Math. Soc., Providence, 2001.

[37] D. Kleinbock, *Baker–Sprindžuk conjectures for complex analytic manifolds.* Algebraic groups and Arithmetic, 539–553, T.I.F.R., Mumbai, 2004.

[38] D. Kleinbock, *An extension of quantitative nondivergence and applications to Diophantine exponents*, Trans. Amer. Math. Soc. To appear.

[39] D. Kleinbock, *Diophantine properties of measures and homogeneous dynamics.* Preprint.

[40] D. Kleinbock, E. Lindenstrauss, and B. Weiss, *On fractal measures and Diophantine approximation*, Selecta Math. 10 (2004), 479–523.

[41] D. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math. 148 (1998), 339–360.

[42] D. Kleinbock and G. Tomanov, *Flows on S-arithmetic homogeneous spaces and applications to metric Diophantine approximation*, Comm. Math. Helvet. 82 (2007), 519–581.

[43] J. E. Littlewood, Some problems in real and complex analysis. D. C. Heath and Co. Raytheon Education Co., Lexington, Mass., 1968.

[44] K. Mahler, *Zur Approximation der Exponentialfunktionen und des Logarithmus. I, II*, J. reine angew. Math. 166 (1932), 118–150.

[45] G. Margulis, *Diophantine Approximation, Lattices and Flows on Homogeneous Spaces*, in: A Panorama of Number Theory or The View from Baker's Garden. Edited by G. Wüstholz, pp. 280–310, Cambridge University Press, 2002.

[46] A. D. Pollington and S. Velani, *On a problem in simultaneous Diophantine approximation: Littlewood's conjecture*, Acta Math. 185 (2000), 287–306.

[47] M. Queffélec, *An introduction to Littlewood's conjecture.* This volume.

[48] D. Roy, *On two exponents of approximation related to a real number and its square*, Canad. J. Math. 59 (2007), 211–224.

[49] W. M. Schmidt, *Badly approximable systems of linear forms*, J. Number Theory 1 (1969), 139–154.

[50] W. M. Schmidt, *Diophantine approximation and certain sequences of lattices*, Acta Arith. 18 (1971), 165–178.

[51] W. M. Schmidt, *Norm form equations*, Ann. of Math. 96 (1972), 526–551.

[52] W. M. Schmidt, Diophantine Approximation, Lecture Notes in Math. 785, Springer, Berlin, 1980.

[53] W. M. Schmidt and Y. Wang, *A note on a transference theorem of linear forms*, Sci. Sinica 22 (1979), 276–280.

[54] V. G. Sprindžuk, Mahler's problem in metric number theory, Izdat. "Nauka i Tehnika", Minsk, 1967 (in Russian). English translation by B. Volkmann, Translations of Mathematical Monographs, Vol. 25, American Mathematical Society, Providence, R.I., 1969.

[55] V. G. Sprindžuk, Metric theory of Diophantine approximation, Izdat. "Nauka", Moscow, 1977 (in Russian). English translation by R. A. Silverman, Scripta Series in Mathematics, John Wiley & Sons, New-York–Toronto–London, 1979.

[56] A. N. Starkov, Dynamical systems on homogeneous spaces, Translations of Mathematical Monographs, 190, American Mathematical Society, Providence, RI, 2000.

[57] R. C. Vaughan and S. Velani, *Diophantine approximation on planar curves: the convergence theory*, Invent. Math. 166 (2006), 103–124.

[58] Y. Wang, Selected papers of Wang Yuan. Translated from the Chinese original. Edited by the author. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[59] Y. Wang and K. Yu, *A note on some metrical theorems in Diophantine approximation*, Chinese Ann. Math. 2 (1981), 1–12.

[60] Y. Wang, K. Yu, and Y. C. Zhu, *A transfer theorem for linear forms*, Acta Math. Sinica 22 (1979), 237–240 (in Chinese).

[61] B. Weiss, *Almost no points on a Cantor set are very well approximable*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 457 (2001), 949–952.

[62] K. Yu, *Hausdorff dimension and simultaneous rational approximation*, J. London Math. Soc. 24 (1981), 79–84.

[63] K. Yu, *A note on a problem of Baker in metrical number theory*, Math. Proc. Cambridge Philos. Soc. 90 (1981), 215–227.

Yann Bugeaud

Université Louis Pasteur
U. F. R. de mathématiques
7, rue René Descartes
67084 STRASBOURG (FRANCE)

bugeaud@math.u-strasbg.fr