On simultaneous uniform approximation to a p-adic number and its square

Yann Bugeaud

Abstract. *Let p be a prime number. We show that a result of Teulié is nearly best possible by constructing a p-adic number ξ such that ξ and ξ^2 are uniformly simultaneously very well approximable by rational numbers with the same denominator. The same conclusion was previously reached by Zelo in his PhD thesis, but our approach using p-adic continued fractions is more direct and simpler.*

1. Introduction

Throughout this paper we set $\lambda = (\sqrt{5} - 1)/2$. In 1969, Davenport and Schmidt [2] established the following statement.

Theorem DS. *Let ξ be a real number that is neither rational nor quadratic. Then, there exists a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\le cX^{-\lambda}, \\ |x_0\xi^2 - x_2| &\le cX^{-\lambda}, \\ |x_0| &\le X \end{aligned}$$

has no non-zero integer solution (x_0, x_1, x_2) for arbitrarily large real numbers X.

It was rather unexpected when, in 2003, Roy [5, 7] proved that Theorem DS cannot be improved.

Theorem R. *There exist a real number ξ which is neither rational nor quadratic and a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1| &\le cX^{-\lambda}, \\ |x_0\xi^2 - x_2| &\le cX^{-\lambda}, \\ |x_0| &\le X \end{aligned} \tag{1.1}$$

has a non-zero integer solution (x_0, x_1, x_2) for every real number $X > 1$.

2010 *Mathematics Subject Classification :* 11J13, 11J61.

Theorem R is quite surprising, since the volume of the convex bodies defined by (1.1) tends rapidly to zero as X grows to infinity. Any real number ξ satisfying a Diophantine condition as in Theorem R was termed by Roy an *extremal number.* He proved [7] that the set of extremal (real) numbers is countable and gave some explicit examples of extremal (real) numbers [5].

Throughout the present paper, p always denotes a prime number. The absolute value $|\cdot|_p$ is normalised in such a way that $|p|_p = p^{-1}$. In 2002, Teulié [8] established the p-adic analogue of Theorem DS.

Theorem T. *Let ξ be a p-adic number that is neither rational nor quadratic. Then, there exists a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1|_p &\le cX^{-1-\lambda}, \\ |x_0\xi^2 - x_2|_p &\le cX^{-1-\lambda}, \\ \max\{|x_0|, |x_1|, |x_2|\} &\le X \end{aligned} \tag{1.2}$$

has no non-zero integer solution (x_0, x_1, x_2) for arbitrarily large real numbers X.

In analogy with the real case, we define an *extremal p-adic number* to be a p-adic number ξ with the property that there is a positive constant c such that, for every real number $X > 1$, the system (1.2) has a non-zero integer solution (x_0, x_1, x_2).

Very recently, in his PhD thesis, Zelo [9] adapted the method initiated by Roy [7] to show that Teulié's result is nearly best possible. Next result follows from his Corollary 2.5.9.

Theorem Z. *Let ε be a positive real number. There exist a p-adic number ξ which is neither rational nor quadratic and a positive real number c such that the system of inequalities*

$$\begin{aligned} |x_0\xi - x_1|_p &\le cX^{-1-\lambda+\varepsilon}, \\ |x_0\xi^2 - x_2|_p &\le cX^{-1-\lambda+\varepsilon}, \\ \max\{|x_0|, |x_1|, |x_2|\} &\le X \end{aligned}$$

has a non-zero integer solution (x_0, x_1, x_2) for every real number $X > 1$.

The purpose of the present note is to give an alternative, simpler proof of Zelo's result. Our approach is inspired by Roy's construction [5] of an extremal number using continued fractions and properties of the infinite Fibonacci word.

2. Result

Let a and b be two symbols. Set $f_1 = b$, $f_2 = a$ and let $f_n = f_{n-1}f_{n-2}$ be the concatenation of the words f_{n-1} and f_{n-2}, for $n \ge 3$. Then,

$$f_\infty = \lim_{n\to+\infty} f_n = abaababaabaab\ldots$$

is the *Fibonacci word* on the alphabet $\{a, b\}$. Roy [5] proved that the real number

$$\xi = [0; 1, 2, 1, 1, 2, 1, 2, 1, \ldots],$$

whose sequence of partial quotients is given by the Fibonacci word on $\{1, 2\}$, is an extremal real number.

In this note we show that a similar construction works in the p-adic setting. Before stating our main result, it is convenient to define an exponent of approximation.

Definition. *Let $n \geq 1$ be an integer and let ξ be a p-adic number. We denote by $\hat{\lambda}_n(\xi)$ the supremum of the real numbers $\hat{\lambda}$ such that, for every sufficiently large real number X, the system of inequalities*

$$\max_{1 \leq m \leq n} |x_0 \xi^m - x_m|_p \leq X^{-1-\hat{\lambda}},$$
$$0 < \max\{|x_0|, |x_1|, \ldots, |x_n|\} \leq X$$

has a solution in integers $x_0, \ldots, x_n$.

It follows from the Dirichlet *Schubfachprinzip* that $\hat{\lambda}_n(\xi) \geq 1/n$ for every positive integer n and every irrational number ξ. Teulié [8] derived upper bounds for $\hat{\lambda}_n(\xi)$ when ξ is not algebraic of degree at most n. His Theorem T implies that $\hat{\lambda}_2(\xi) \leq \lambda$ for every p-adic number ξ which is neither rational nor quadratic, while Theorem Z asserts that

$$\sup\{\hat{\lambda}_2(\xi) : \xi \in \mathbf{Q}_p, \ \xi \text{ is neither rational nor quadratic}\} = \lambda. \tag{2.1}$$

As in the real case, it remains unknown whether there are transcendental p-adic numbers ξ and integers $n \geq 3$ such that $\hat{\lambda}_n(\xi) > 1/n$.

Our Theorem gives a constructive proof of (2.1).

Theorem. *Let v be a positive integer and let $(v_n)_{n \geq 1}$ be the Fibonacci word on $\{v, v+1\}$ starting with v. Let ξ_v denote the p-adic number*

$$\xi_v := 1 + \lim_{n \to +\infty} \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{\ldots + p^{v_n}}}}.$$

Then we have $\hat{\lambda}_2(\xi_v) \geq (1 - 7/v)\lambda$ and

$$\sup\{\hat{\lambda}_2(\xi_v) : v \geq 1\} = \lambda.$$

Remark 1. It does not seem that Zelo's approach allows him to replace X^ε in Theorem Z by a function of X which increases less rapidly, like e.g. $X^{1/\log\log X}$. The same applies for the constructive method described in the present note. In particular, it remains an interesting open problem to decide whether there exist extremal p-adic numbers and even whether there exist p-adic numbers ξ with $\hat{\lambda}_2(\xi) = \lambda$.

Remark 2. It follows from the p-adic version of the Schmidt Subspace Theorem that any p-adic number ξ satisfying $\hat{\lambda}_2(\xi) > 1/2$ is either rational, or quadratic, or transcendental.

Remark 3. Zelo's approach is more complicated than ours, but it gives more information. Indeed, it yields a characterization of extremal p-adic numbers (if such numbers exist) as well as a characterization of p-adic numbers ξ with $\hat{\lambda}_2(\xi)$ sufficiently close to λ. One may hope that, combined with ideas from [6], it could be used to prove the existence of p-adic numbers that are very badly approximable by cubic integers.

3. Proof

Before proceeding with the construction of p-adic numbers enjoying special approximation properties, we make several general remarks which were inspired by [3].

- *Definition of p-adic continued fractions.*

Set

$$p_{-1} = 1, q_{-1} = 0, p_0 = 1, q_0 = 1.$$

Let $\mathbf{v} = (v_n)_{n\geq 1}$ be a sequence of positive integers and set

$$p_n = p^{v_n} p_{n-2} + p_{n-1}, q_n = p^{v_n} q_{n-2} + q_{n-1}, \quad (n \geq 1).$$

Observe that

$$\left| \frac{p_1}{q_1} - \frac{p_0}{q_0} \right|_p = p^{-v_1}$$

and that, for $n \geq 2$, we have

$$\begin{aligned} \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|_p &= \left| \frac{(p^{v_n} p_{n-2} + p_{n-1})q_{n-1} - (p^{v_n} q_{n-2} + q_{n-1})p_{n-1}}{q_n q_{n-1}} \right|_p \\ &= p^{-v_n} \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} \right|_p, \end{aligned}$$

since p does not divide $q_n q_{n-1} q_{n-2}$.

Consequently, for $n \geq 0$ and $k \geq 1$, we have

$$\left| \frac{p_{n+k}}{q_{n+k}} - \frac{p_n}{q_n} \right|_p = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p = p^{-v_{n+1} - v_n - \ldots - v_1}. \tag{3.1}$$

This shows that the sequence $(p_n/q_n)_{n\geq 1}$ converges p-adically. Let $\xi_{\mathbf{v}}$ denote its limit. It follows from (3.1) that

$$\left| \xi_{\mathbf{v}} - \frac{p_n}{q_n} \right|_p = p^{-v_{n+1} - v_n - \ldots - v_1}, \quad n \geq 1, \tag{3.2}$$

and we can write

$$\xi_{\mathbf{v}} := 1 + \lim_{n \to +\infty} \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{\ldots + p^{v_n}}}}.$$

- *Palindromes.*

Let n be a positive integer. We have

$$\frac{p_n}{q_n} = 1 + \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{\ldots + p^{v_n}}}}.$$

Furthermore, the classical *mirror formula* (see [4], page 12) asserts that

$$\frac{p_n}{p_{n-1}} = 1 + \cfrac{p^{v_n}}{1 + \cfrac{p^{v_{n-1}}}{1 + \cfrac{p^{v_{n-2}}}{\ldots + p^{v_1}}}}.$$

Consequently, if the word $v_1 \ldots v_n$ is a *palindrome*, that is, if $v_j = v_{n+1-j}$ for $j = 1, \ldots, n$, then

$$\frac{p_n}{q_n} = \frac{p_n}{p_{n-1}},$$

hence,

$$q_n = p_{n-1}.$$

This implies that

$$\begin{aligned} \left| \xi_{\mathbf{v}}^2 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_n}{q_n} \right|_p &= \left| \left(\xi_{\mathbf{v}} - \frac{p_{n-1}}{q_{n-1}} \right) \cdot \left(\xi_{\mathbf{v}} + \frac{p_n}{q_n} \right) + \xi_{\mathbf{v}} \left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right) \right|_p \\ &\leq p^{-v_n - v_{n-1} - \ldots - v_1}, \end{aligned}$$

by (3.1) and (3.2). We then derive from (3.2) that

$$\max\{|q_{n-1}\xi_{\mathbf{v}} - p_{n-1}|_p, |q_{n-1}\xi_{\mathbf{v}}^2 - p_n|_p\} \leq p^{-v_n - v_{n-1} - \ldots - v_1}, \tag{3.3}$$

showing that $\xi_{\mathbf{v}}$ and its square are simultaneously well approximable by rational numbers of denominator q_{n-1}.

- *Completion of the proof.*

In the sequel, v denotes a positive integer and we assume that the sequence $\mathbf{v} = (v_n)_{n\geq 1}$ takes its values in the set $\{v, v+1\}$. We assume that $v \geq 8$ since the theorem obviously holds for $v \leq 7$. From the inequalities

$$p^v q_{n-2} \leq q_n \leq q_{n-1} + p^{v+1} q_{n-2}, \quad n \geq 1,$$

we deduce that there exist positive constants c_1 and c_2 such that

$$c_1 p^{nv/2} \leq q_n \leq c_2 p^{n(v+2)/2}, \quad n \geq 1. \tag{3.4}$$

Furthermore, we observe that

$$nv \leq v_1 + \ldots + v_n \leq n(v+1), \quad n \geq 1. \tag{3.5}$$

Take for $(v_n)_{n\geq 1}$ the Fibonacci word on $\{v, v+1\}$ starting with v. For simplicity, let us write ξ_v instead of $\xi_{\mathbf{v}}$. Let $(F_k)_{k\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0, F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$. For $k \geq 4$, set $n_k = F_k - 3$. It is well known (see e.g. [1]) that, for $k \geq 4$, the prefix of length $n_k + 1$ of the word

$$v_1 v_2 v_3 \ldots = v(v+1)vv(v+1)v(v+1)v\ldots$$

is a palindrome.

In view of the preceding discussion, for $k \geq 4$, we have

$$\begin{aligned} \max\{|q_{n_k}\xi_v - p_{n_k}|_p, |q_{n_k}\xi_v^2 - p_{n_k+1}|_p\} &\leq p^{-v_{n_k+1} - v_{n_k} - \ldots - v_1} \\ &\leq c_3 q_{n_k}^{-2+4/v}, \end{aligned} \tag{3.6}$$

by (3.3), (3.4) and (3.5). Here and below, $c_3, \ldots, c_7$ denote positive real numbers independent of k.

Let Q be a large positive integer. Let $k \geq 4$ be the integer defined by the inequalities

$$q_{n_k} \leq Q < q_{n_{k+1}}.$$

Since n_k/n_{k+1} tends to λ as k tends to infinity, we may assume that Q is sufficiently large in order to guarantee that

$$\lambda n_{k+1} \leq \frac{v+3}{v+2} n_k.$$

Let u be the largest non-negative integer such that $q_{n_k} p^u \leq Q$, and set

$$q'_{n_k} = p^u q_{n_k}, \;\; p'_{n_k} = p^u p_{n_k}, \;\; p'_{n_k+1} = p^u p_{n_k+1}.$$

We then have

$$Q^\lambda \leq q_{n_{k+1}}^\lambda \leq c_2 p^{\lambda n_{k+1}(v+2)/2} \leq c_2 p^{n_k(v+3)/2} \leq c_4 q_{n_k}^{1+3/v},$$

and it follows from (3.6) that

$$\max\{|q'_{n_k}\xi_v - p'_{n_k}|_p, |q'_{n_k}\xi_v^2 - p'_{n_k+1}|_p\} \le c_3 p^{-u} q_{n_k}^{-2+4/v}$$
$$\le c_5 Q^{-1} q_{n_k}^{-1+4/v} \le c_6 Q^{-1} Q^{-(1-7/v)\lambda}.$$

Since $0 < p'_{n_k}, p'_{n_k+1}, q'_{n_k} \le c_7 Q$, this shows that

$$\hat\lambda_2(\xi_v) \ge (1 - 7/v)\lambda,$$

and the proof of the theorem is complete. □

Acknowledgements. The author is pleased to thank Damien Roy for several useful remarks.

References

[1] J.-P. Allouche, J. L. Davison, M. Queffélec, and L. Q. Zamboni, *Transcendence of Sturmian or morphic continued fractions*, J. Number Theory 91 (2001), 39–66.

[2] H. Davenport and W. M. Schmidt, *Approximation to real numbers by algebraic integers*, Acta Arith. 15 (1969), 393–416.

[3] K. Mahler, *Zur Approximation P-adischer Irrationalzahlen*, Nieuw Arch. Wisk. 18 (1934), 22–34.

[4] O. Perron, Die Lehre von den Kettenbrüchen. Teubner, Leupzig, 1929.

[5] D. Roy, *Approximation simultanée d'un nombre et son carré*, C. R. Acad. Sci. Paris 336 (2003), 1–6.

[6] D. Roy, *Approximation to real numbers by cubic algebraic numbers, II*, Annals of Math. 158 (2003), 1081–1087.

[7] D. Roy, *Approximation to real numbers by cubic algebraic numbers, I*, Proc. London Math. Soc. 88 (2004), 42–62.

[8] O. Teulié, *Approximation d'un nombre p-adique par des nombres algébriques*, Acta Arith. 102 (2002), 137–155.

[9] D. Zelo, Simultaneous approximation to real and p-adic numbers. PhD thesis. Univ. Ottawa. 2009. `arXiv:0903.0086`.

Yann Bugeaud
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 STRASBOURG (FRANCE)

`bugeaud@math.unistra.fr`