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Nonarchimedean quadratic Lagrange spectra and continued fractions in power series fields

by

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Abstract. Let $\mathbb{F}_q$ be a finite field of order a positive power q of a prime number. We study the nonarchimedean quadratic Lagrange spectrum defined by Parkkonen and Paulin by considering the approximation by elements of the orbit of a given quadratic power series in $\mathbb{F}_q((Y^{-1}))$, for the action by homographies and anti-homographies of $\mathrm{PGL}_2(\mathbb{F}_q[Y])$ on $\mathbb{F}_q((Y^{-1})) \cup \{\infty\}$. While Parkkonen and Paulin's approach used geometric methods of group actions on Bruhat–Tits trees, ours is based on the theory of continued fractions in power series fields.

1. Introduction. For an irrational real number ξ, define $\lambda(\xi)$ in $(0, +\infty]$ by

$$\lambda(\xi)^{-1} = \liminf_{p,q\in\mathbb{Z},\, q\to+\infty} |q(q\xi - p)|.$$

The *Lagrange spectrum* $\mathcal{L}$ is the set of values taken by the function λ at irrational real numbers. It is included in $[\sqrt{5}, +\infty]$ and has a rather complicated structure, which is not completely understood, despite some recent progress [9, 19]. The first values of $\mathcal{L}$ are $\sqrt{5}$ (sometimes called Hurwitz' constant) and $2\sqrt{2}$, and its first accumulation point is 3. In 1947 Hall [11] established that every real number in the interval $[\sqrt{2} - 1, 4\sqrt{2} - 4]$ can be represented as a sum of two continued fractions with partial quotients at most 4. As an easy consequence, Vinogradov, Delone, and Fuks [27] proved that the whole interval $[5 + \sqrt{2}, +\infty]$ is contained in $\mathcal{L}$ (in his review of [27] for the *Mathematical Reviews*, Cassels observed that 'he was told by Marshall Hall of this application of his result some years ago, and, indeed, this application provided the motivation for Hall's paper'). Subsequently,

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in 1975, Freiman [8] proved that the biggest half-line contained in $\mathcal{L}$ is

$$\left[\frac{2221564096 + 283748\sqrt{462}}{491993569}, +\infty\right].$$

This half-line is called *Hall's ray*. The reader is directed to [6] for additional references (note also that, sometimes, the authors choose to study the set of values taken by the function $1/\lambda$).

Analogous spectra have been defined and studied in various contexts, including inhomogeneous Diophantine approximation (see e.g. Cusick, Moran, and Pollington [7] and Pinner and Wolczuk [25]), Diophantine approximation in imaginary quadratic fields (Maucourant [18]), and in the setting of interval exchange transformations and the Teichmüller flow on moduli spaces of translation surfaces (Hubert, Marchese, and Ulcigrai [14]).

In 2011 Parkkonen and Paulin [20] defined and studied quadratic Lagrange spectra by considering approximation by elements of the orbit of a given real quadratic irrational number for the action by homographies and anti-homographies of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$. These spectra were further investigated by Bugeaud [4], Pejković [23], and Lin [16]. Among other results, the existence of Hall's ray for every quadratic Lagrange spectrum has been established in [16]. Subsequently, Parkkonen and Paulin [21] defined and studied quadratic Lagrange spectra in completions of function fields over finite fields with respect to the absolute values defined by discrete valuations. This setting includes the special case of the field of rational fractions and its valuation at infinity, which was given special attention in [21] and is studied in the present paper.

Let q be a positive power of a prime number and $\mathbf{k} = \mathbb{F}_q$ denote the finite field of order q. Let $R = \mathbb{F}_q[Y]$, $K = \mathbb{F}_q(Y)$, and $\widehat{K} = \mathbb{F}_q((Y^{-1}))$ be, respectively, the ring of polynomials in one variable Y over $\mathbb{F}_q$, the field of rational functions in Y over $\mathbb{F}_q$, and the field of formal power series in Y^{-1} over $\mathbb{F}_q$. Then $\widehat{K}$ is a nonarchimedean local field, the completion of K with respect to the absolute value defined by $|P/Q| = q^{\deg P - \deg Q}$ for all P, Q in $R \setminus \{0\}$. Also, sometimes it is convenient to use the associated valuation v defined for a nonzero element f in $\widehat{K}$ by

$$\mathrm{v}(f) = -\frac{\log |f|}{\log q}.$$

We stress that $\widehat{K}$ is not algebraically closed (for instance, the polynomial $X^2 - Y$ has no roots in $\widehat{K}$). Let

$$K^{(2)} = \{f \in \widehat{K} : [K(f) : K] = 2\}$$

be the set of power series in $\widehat{K}$ which are quadratic over K. Given f in $\widehat{K} \setminus K$, it is well known that f is in $K^{(2)}$ if and only if its continued fraction expansion is eventually periodic. The projective action of $\mathrm{PGL}_2(R)$ on $\widehat{K}\cup\{\infty\}$ preserves $K^{(2)}$, keeping the periodic part of the continued fraction expansions unchanged, up to cyclic permutation and invertible elements; see Lemma 2.5 below. We refer for instance to [15, 26, 22] for background on the above notions.

In odd characteristic, the quadratic polynomials which are irreducible over R are separable. This is not the case in characteristic 2, where there exist quadratic polynomials in $R[X]$ which are irreducible over R and not separable. These polynomials are precisely the polynomials of the form AX^2+C, with A, C in R and A nonzero, whose (double) roots are in $\mathbb{F}_q((Y^{-1/2}))\setminus\widehat{K}$. Consequently, every quadratic power series in $\widehat{K}$ is a root of an irreducible polynomial of the form AX^2+BX+C, where A, B, C are in R and B is nonzero. This short discussion explains that, in characteristic 2, the Galois conjugate α^σ of a quadratic power series α in $\widehat{K}$ is well defined and that $\alpha-\alpha^\sigma=\alpha+\alpha^\sigma$ is nonzero.

Let α be in $K^{(2)}$ and α^σ in $K^{(2)}$ denote its Galois conjugate over K. The *complexity* $h(\alpha)=1/|\alpha-\alpha^\sigma|$ of α was introduced in [13] and studied in [1, Section 17.2]. It measures the size of α, in the same way as $\max\{|p|,|q|\}$ measures the size of the rational number p/q, where p, q are nonzero coprime integers. The complexity $h(\alpha)$ can be expressed in terms of the continued fraction expansion of α (see Lemma 2.2). Let

$$\varTheta_\alpha=\mathrm{PGL}_2(R)\cdot\{\alpha,\alpha^\sigma\}$$

be the union of the orbits of α and α^σ under the projective action of $\mathrm{PGL}_2(R)$. Given f in $\widehat{K}\setminus(K\cup\varTheta_\alpha)$, Parkkonen and Paulin [21] introduced the *quadratic approximation constant* of f, defined by

$$c_\alpha(f)=\liminf_{\beta\in\varTheta_\alpha,\,|\beta-\beta^\sigma|\to 0}\frac{|f-\beta|}{|\beta-\beta^\sigma|},$$

and they studied the *quadratic Lagrange spectrum* of α, defined by

$$\mathrm{Sp}(\alpha)=\{c_\alpha(f):f\in\widehat{K}\setminus(K\cup\varTheta_\alpha)\}.$$

Note that the quadratic Lagrange spectrum of α is contained in $q^{\mathbb{Z}}\cup\{0,+\infty\}$, thus, it is countable. It follows from [13, Theorem 1.6] that if $m_{\widehat{K}}$ is a Haar measure on the locally compact additive group of $\widehat{K}$, then for $m_{\widehat{K}}$-almost every f in $\widehat{K}$, we have $c_\alpha(f)=0$. Hence in particular 0 is in $\mathrm{Sp}(\alpha)$, and therefore the quadratic Lagrange spectrum is closed. Parkkonen and Paulin [21] proved that it is bounded, and defined the (*quadratic*) *Hurwitz constant* of α, denoted by $\mathrm{Hw}(\alpha)$, by

$$\mathrm{Hw}(\alpha)=\max\mathrm{Sp}(\alpha)\in q^{\mathbb{Z}}.$$

They obtained several results on $\mathrm{Hw}(\alpha)$ and $\mathrm{Sp}(\alpha)$, including the existence of a Hall ray, for every α. Parkkonen and Paulin [21] established nonarchimedean analogues of the results obtained in [4, 23, 16] by using geometric methods of group actions on Bruhat–Tits trees. In the present paper, we re-prove many of their results by applying the theory of continued fractions in power series fields, and in addition we establish several new results. In particular, we give alternative proofs of the following two theorems highlighted in [21].

THEOREM 1.1 (Parkkonen and Paulin [21]). *Let α be a quadratic power series in $\widehat{K}$.*

(1) (Upper bound) $\mathrm{Hw}(\alpha) \le q^{-2}$.
(2) (Hall's ray) *There exists an integer m_α such that, for every integer $n > m_\alpha$, the real number q^{-n} belongs to* $\mathrm{Sp}(\alpha)$.

A suitable value for m_α in Theorem 1.1(2) can be given explicitly (see Theorem 4.3).

THEOREM 1.2 (Parkkonen and Paulin [21]). *The Hurwitz constant of any quadratic power series in $\widehat{K}$, whose continued fraction expansion is eventually periodic with period of length at most $q-1$, is equal to q^{-2}.*

Examples of quadratic power series for which the quadratic Lagrange spectrum coincides with its Hall ray are given in [21] and in Theorem 5.1. In particular, we have

$$\mathrm{Sp}([0; Y, Y, \ldots]) = \{0\} \cup \{q^{-n-2} : n \in \mathbb{Z}_{\ge 0}\}. \tag{1.1}$$

Our approach shows that, in order to determine the quadratic spectrum of a quadratic power series α, it is sufficient to compute the quadratic approximation constants of the quadratic power series not in Θ_α.

THEOREM 1.3. *Let α be a quadratic power series in $\widehat{K}$ and q^{-m} a nonzero element of its spectrum. Then there exists a quadratic power series f not in $K \cup \Theta_\alpha$ such that $c_\alpha(f) = q^{-m}$.*

Theorem 1.3 can be regarded as an analogue of a result of Cusick [5] (see also [6, Theorem 2, p. 36]) asserting that the Lagrange spectrum is the closure of the set of the Lagrange constants of quadratic irrationalities.

The proof of Theorem 1.3 shows that if d denotes the maximal degree of a partial quotient in the periodic part of the continued fraction expansion of α, then one can in addition impose that all the partial quotients of f have degree at most $d+1$. Moreover, one can also impose that the length of the period of the continued fraction expansion of α is at most $2 + m/2$. Consequently, an integer m being given, it is sufficient to compute $c_\alpha(f)$ for f being in an explicitly given finite set in order to determine whether q^{-m} is or not in the spectrum of α. Since a suitable value for the integer m_α in Theorem 1.1(2) can be given explicitly, all this shows that a finite amount of computation is sufficient to determine the set $\mathrm{Sp}(\alpha)$ exactly.

Proposition 4.9 of [21] asserts that the function Hw takes arbitrarily small positive values. We establish that it can take every admissible value.

THEOREM 1.4. *For every $m \geq 2$, there exists α in $K^{(2)}$ such that* $\mathrm{Hw}(\alpha) = q^{-m}$.

We feel, however, that there is no simple formula for the Hurwitz constant of a quadratic power series.

Parkkonen and Paulin [21] gave explicit examples of classes of quadratic power series whose quadratic Lagrange spectrum does not coincide with its Hall ray, in other words, which have at least one *gap* in their spectrum. We go slightly further and establish that the number of gaps can be prescribed.

THEOREM 1.5. *For any positive integer ℓ, there exist quadratic power series in $\widehat{K}$ whose Lagrange spectrum has exactly ℓ gaps.*

An interesting question is then to prescribe the number and the lengths of the gaps. This seems to be rather difficult.

Throughout, for a_0 in R and $a_1, \ldots, a_{r+s}$ nonconstant polynomials in R, we use the notation

$$[a_0; a_1, a_2, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

to indicate that the block of partial quotients $a_{r+1}, \ldots, a_{r+s}$ is repeated infinitely many times.

We recall that an irrational power series α is quadratic if and only if its continued fraction expansion is ultimately periodic, that is, of the form

$$\alpha = [a_0; a_1, \ldots, a_r, \overline{b_1, \ldots, b_s}]. \tag{1.2}$$

When we express α as in (1.2) we tacitly assume that s is minimal and $a_r \neq b_s$. We call $b_1, \ldots, b_s$ the *shortest periodic part* in the continued fraction expansion of α.

The present paper is organized as follows. In Section 2, we gather several results on continued fractions in power series fields and apply them in Section 3 to prove Theorem 1.3. Sections 4 and 5 are devoted to the proofs of our further results.

2. Auxiliary lemmas on continued fractions in power series fields. We assume that the reader is familiar with the classical theory of continued fractions of real numbers. Good references include [24, 12] and [15, 26] for the case of power series.

Our first lemma is an analogue for quadratic power series of a theorem of Galois.

Lemma 2.1. *Let $s \geq 1$ be an integer and $b_1, \ldots, b_s$ nonconstant polynomials in R. The Galois conjugate of the quadratic power series*

$$\tau = [b_1; \overline{b_2, \ldots, b_s, b_1}]$$

is the power series

$$\tau^\sigma = -[0; \overline{b_s, \ldots, b_2, b_1}].$$

Although the proof of Lemma 2.1 is contained in that of [1, Proposition 17.7], we give it below for the sake of completeness.

Proof of Lemma 2.1. Define

$$\frac{p_s}{q_s} = [0; b_1, \ldots, b_s], \qquad \frac{p_{s-1}}{q_{s-1}} = [0; b_1, \ldots, b_{s-1}]$$

and

$$\frac{p'_s}{q'_s} = [0; b_s, b_{s-1}, \ldots, b_1], \quad \frac{p'_{s-1}}{q'_{s-1}} = [0; b_s, b_{s-1}, \ldots, b_2].$$

Then τ satisfies

$$\frac{1}{\tau} = \frac{p_s\tau + p_{s-1}}{q_s\tau + q_{s-1}},$$

hence,

$$p_s\tau^2 + (p_{s-1} - q_s)\tau - q_{s-1} = 0. \tag{2.1}$$

Likewise, $\tau' = [0; \overline{b_s, \ldots, b_1}]$ satisfies

$$\tau' = \frac{(p'_s/\tau') + p'_{s-1}}{(q'_s/\tau') + q'_{s-1}} = \frac{p'_{s-1}\tau' + p'_s}{q'_{s-1}\tau' + q'_s},$$

hence,

$$q'_{s-1}(\tau')^2 + (q'_s - p'_{s-1})\tau' - p'_s = 0. \tag{2.2}$$

The mirror formula (see e.g. [24, p. 32]) gives

$$p'_s = q_{s-1}, \quad q'_s = q_s, \quad p'_{s-1} = p_{s-1}, \quad q'_{s-1} = p_s.$$

Combining this with (2.2), we obtain

$$p_s(\tau')^2 + (q_s - p_{s-1})\tau' - q_{s-1} = 0. \tag{2.3}$$

Equalities (2.1) and (2.3) show that τ and $-\tau'$ are roots of the same quadratic polynomial. Since they are distinct, they are Galois conjugate. ∎

Our second lemma establishes that the quantity $h(\alpha) = |\alpha - \alpha^\sigma|^{-1}$ can be expressed in a simple way in terms of the continued fraction expansion of the quadratic power series α.

LEMMA 2.2. *Let α be a quadratic power series with ultimately periodic continued fraction expansion given by*

$$\alpha = [a_0; a_1, \dots, a_r, \overline{b_1, \dots, b_s}],$$

where $s \geq 1$ and $a_r \neq b_s$. Denote by α^σ its Galois conjugate. Then

$$h(\alpha) = |\alpha - \alpha^\sigma|^{-1} = q^{2(\sum_{i=1}^{r} \deg a_i) - \deg a_r - \deg b_s + \deg(a_r - b_s)}.$$

Observe that, since a_r is not equal to b_s in Lemma 2.2, the degree of the polynomial $a_r - b_s$ is nonnegative. Lemma 2.2 is the analogue of [4, Lemma 2.1] (see also [3, Lemma 6.1]).

Proof of Lemma 2.2. By Lemma 2.1, the Galois conjugate of

$$\tau = [b_1; \overline{b_2, \dots, b_s, b_1}]$$

is the quadratic number

$$\tau^\sigma = -[0; \overline{b_s, \dots, b_2, b_1}].$$

Let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to α. Since

$$\alpha = \frac{p_r \tau + p_{r-1}}{q_r \tau + q_{r-1}} \quad \text{and} \quad \alpha^\sigma = \frac{p_r \tau^\sigma + p_{r-1}}{q_r \tau^\sigma + q_{r-1}},$$

we get

$$\begin{aligned} (2.4) \qquad |\alpha - \alpha^\sigma| &= \frac{|\tau - \tau^\sigma|}{|q_r \tau + q_{r-1}| \cdot |q_r \tau^\sigma + q_{r-1}|} \\ &= q^{-2 \deg q_r} |\tau - \tau^\sigma| \left| \tau + \frac{q_{r-1}}{q_r} \right|^{-1} \left| \tau^\sigma + \frac{q_{r-1}}{q_r} \right|^{-1}. \end{aligned}$$

Observe that

$$(2.5) \qquad |\tau - \tau^\sigma| = \left| \tau + \frac{q_{r-1}}{q_r} \right| = q^{\deg b_1}$$

and

$$\begin{aligned} (2.6) \qquad \left| \tau^\sigma + \frac{q_{r-1}}{q_r} \right| &= \big| [0; a_r, a_{r-1}, \dots] - [0; \overline{b_s, \dots, b_2, b_1}] \big| \\ &= \left| \frac{[a_r; a_{r-1}, \dots] - [b_s; \overline{b_{s-1}, \dots, b_1, b_s}]}{[a_r; a_{r-1}, \dots] \cdot [b_s; \overline{b_{s-1}, \dots, b_1, b_s}]} \right| \\ &= q^{\deg(a_r - b_s) - \deg a_r - \deg b_s}. \end{aligned}$$

Since $\deg q_r = \sum_{i=1}^{r} \deg a_i$, the lemma follows from (2.4)–(2.6). ∎

Our third auxiliary lemma is the analogue of [4, Lemma 2.2].

LEMMA 2.3. *Let $\alpha = [0; a_1, a_2, \dots]$ and $\beta = [0; b_1, b_2, \dots]$ be power series in $\widehat{K}$. Assume that there exists a nonnegative integer n such that $a_i = b_i$ for any $i = 1, \dots, n$ and $a_{n+1} \neq b_{n+1}$. Then*

$$|\alpha - \beta| = q^{-2(\sum_{i=1}^{n} \deg a_i) - \deg a_{n+1} - \deg b_{n+1} + \deg(a_{n+1} - b_{n+1})}.$$

Proof. Set $\alpha' = [a_{n+1}; a_{n+2}, \ldots]$ and $\beta' = [b_{n+1}; b_{n+2}, \ldots]$. Let $(p_\ell/q_\ell)_{\ell \ge 1}$ denote the sequence of convergents to β. Since $a_{n+1} \ne b_{n+1}$ and the first n partial quotients of α and β are assumed to be the same, we get

$$\alpha = \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} \quad \text{and} \quad \beta = \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}},$$

thus,

$$|\alpha - \beta| = \left| \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} - \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}} \right| = \left| \frac{\alpha' - \beta'}{(q_n \alpha' + q_{n-1})(q_n \beta' + q_{n-1})} \right|.$$

Since $\deg q_n = \deg a_1 + \cdots + \deg a_n$ and

$$|q_n \alpha' + q_{n-1}| = q^{\deg q_n + \deg a_{n+1}}, \quad |q_n \beta' + q_{n-1}| = q^{\deg q_n + \deg b_{n+1}},$$

and

$$|\alpha' - \beta'| = q^{\deg(a_{n+1} - b_{n+1})},$$

this proves the theorem. ∎

We display an easy consequence of Lemma 2.3. Below and in the next sections, it is convenient to take the point of view of combinatorics on words. For an integer $k \ge 1$, let $\mathcal{A}_{\le k}$ (resp., $\mathcal{A}_{=k}$) denote the set of all nonconstant polynomials in R of degree at most equal to k (resp., equal to k). Set

$$\mathcal{A} = \bigcup_{k \ge 1} \mathcal{A}_{\le k} = \bigcup_{k \ge 1} \mathcal{A}_{=k}.$$

If $a_1 \ldots a_r$ is a finite word over $\mathcal{A}$, then $(a_1 \ldots a_r)^\infty$ denotes the infinite word obtained by concatenating on the right infinitely many copies of $a_1 \ldots a_r$.

Corollary 2.4. *Let*

$$\tau = [b_1; \overline{b_2, \ldots, b_s, b_1}]$$

be a quadratic power series. Let

$$f = [a_0; a_1, a_2, \ldots]$$

be an irrational power series not in Θ_τ. For a positive integer r such that $a_r \ne b_s$, set

$$\alpha_r = [a_0; a_1, \ldots, a_r, \overline{b_1, \ldots, b_{s-1}, b_s}].$$

If $a_{r+1} \ne b_1$, then put $t = 0$ and $s' = 1$. If $a_{r+1} = b_1$, then let t be the largest integer such that the word $a_{r+1} \ldots a_{r+t}$ coincides with the prefix of length t of the infinite word $(b_1 \ldots b_s)^\infty$. Let s_0 be the integer in $\{1, \ldots, s\}$ such that $t \equiv s_0 \bmod s$ and put $s' = s_0 + 1$ if $s_0 < s$ and $s' = 1$ otherwise. Then

$$\text{(2.7)} \quad \mathrm{v}\left(\frac{|f - \alpha_r|}{|\alpha_r - \alpha_r^\sigma|} \right) = 2\Big(\sum_{j=1}^{t} \deg a_{r+j} \Big) + \deg a_r + \deg b_s - \deg(a_r - b_s) + \deg a_{r+t+1} + \deg b_{s'} - \deg(a_{r+t+1} - b_{s'}).$$

In particular,

$$\frac{|f-\alpha_r|}{|\alpha_r-\alpha_r^\sigma|}\le q^{-2}. \tag{2.8}$$

Furthermore, if $\deg a_r \neq \deg b_s$ *and* $\deg a_{r+t+1} \neq \deg b_{s'}$*, then*

$$\begin{aligned}\mathrm{v}\Big(\frac{|f-\alpha_r|}{|\alpha_r-\alpha_r^\sigma|}\Big) = 2\Big(\sum_{j=1}^{t}\deg a_{r+j}\Big) &+ \min\{\deg a_r, \deg b_s\}\\ &+ \min\{\deg a_{r+t+1}, \deg b_{s'}\}.\end{aligned} \tag{2.9}$$

Proof. This follows directly from Lemmas 2.2 and 2.3. ∎

It remains to describe the orbit of a quadratic power series under the action of $\mathrm{PGL}_2(R)$. For the real analogue, that is, when characterising the orbit of an irrational number under the action of $\mathrm{SL}_2(\mathbb{Z})$, we used in [4] a classical theorem of Serret (see [24, p. 65]), which asserts that the tails of the continued fraction expansions of two irrational real numbers α,β coincide if and only if there exist integers a,b,c,d with $ad-bc=\pm 1$ such that

$$\alpha = \frac{a\beta+b}{c\beta+d}.$$

In the present context, we make use of the version of Serret's theorem established by Schmidt [26, Theorem 1]. Before stating it, let us observe that, for any irrational power series $\alpha=[a_0;a_1,a_2,\dots]$ in $\widehat{K}$ and any a in $\mathbf{k}^\times$, we have

$$a\alpha = [aa_0; a^{-1}a_1, aa_2, a^{-1}a_3,\dots].$$

Furthermore, two power series α,β in $\widehat{K}$ are called *equivalent* if there exist a,b,c,d in R with $ad-bc$ in $\mathbf{k}^\times$ and such that

$$\alpha = \frac{a\beta+b}{c\beta+d}.$$

LEMMA 2.5. *Two irrational power series* $\alpha=[a_0;a_1,a_2,\dots]$ *and* $\beta=[b_0;b_1,b_2,\dots]$ *in* $\widehat{K}$ *are equivalent if and only if there exist nonnegative integers* m,n *and an element* a *in* $\mathbf{k}^\times$ *such that*

$$\begin{aligned}\alpha &= [a_0;a_1,\dots,a_{n-1},a_n,a_{n+1},a_{n+2},\dots],\\ \beta &= [b_0;b_1,\dots,b_{m-1},aa_n,a^{-1}a_{n+1},aa_{n+2},\dots].\end{aligned}$$

Proof. This is [26, Theorem 1]. It is also proved in [17, Section IV.3]. ∎

3. An equivalent formulation for $\mathrm{Sp}(\alpha)$**.** Throughout this section, we fix a quadratic power series α. Let $b_1,\dots,b_s$ be the (shortest) periodic part in its continued fraction expansion and set

$$\tau = [b_1;\overline{b_2,\dots,b_s,b_1}].$$

For $j = 1, \ldots, s$ and a in $\mathbf{k}^\times$, set

$$\tau_{j,a} = [ab_j; \overline{a^{-1}b_{j+1}, \ldots, a^{(-1)^{s-1}}b_{j-1}, a^{(-1)^s}b_j, a^{(-1)^{s+1}}b_{j+1}, \ldots, a^{(-1)^{2s-1}}b_{j-1}, ab_j}]$$

and

$$\tau'_{j,a} = [a^{-1}b_{j-1}; \overline{ab_{j-2}, a^{-1}b_{j-3}, \ldots, a^{(-1)^s}b_j, a^{(-1)^{s+1}}b_{j-1}, a^{(-1)^{s+2}}b_{j-2}, \ldots, a^{(-1)^{2s}}b_j, a^{-1}b_{j-1}}].$$

Here and below, the indices are always understood modulo s. For $n = 1, \ldots, s$, the nth partial quotient of $\tau_{j,a}$ (resp., of $\tau'_{j,a}$) is equal to $a^{(-1)^n}b_{j+n}$ (resp., $a^{(-1)^{n+1}}b_{j-n-1}$) and its $(n+s)$-th partial quotient is equal to $a^{(-1)^{n+s}}b_{j+n}$ (resp., $a^{(-1)^{n+s+1}}b_{j-n-1}$). The overline part in the definition of $\tau_{j,a}$ (resp., of $\tau'_{j,a}$) has length $2s$ and, when s is even, it is the concatenation of two copies of the same string of s partial quotients $a^{-1}b_{j+1}, ab_{j+2}, \ldots, a^{-1}b_{j-1}, ab_j$ (resp., $ab_{j-2}, a^{-1}b_{j-3}, \ldots, ab_j, a^{-1}b_{j-1}$).

Recall that Θ_τ is equal to $\mathrm{PGL}_2(R) \cdot \{\alpha, \alpha^\sigma\}$. Observe that $\tau = \tau_{1,1}$ and, for a in $\mathbf{k}^\times$, we have

$$\Theta_\tau = \Theta_{\tau_{1,a}} = \cdots = \Theta_{\tau_{s,a}} = \Theta_{\tau'_{1,a}} = \cdots = \Theta_{\tau'_{s,a}} = \Theta_\alpha.$$

Furthermore, by Lemma 2.1, we have

$$\tau^\sigma_{j,a} = -[0; \overline{a^{-1}b_{j-1}, ab_{j-2}, \ldots, a^{-1}b_{j+1}, ab_j}] = -1/\tau'_{j,a},$$

for $j = 1, \ldots, s$.

Let

$$f = [a_0; a_1, a_2, \ldots]$$

be an irrational power series not in Θ_τ, which we wish to approximate by power series from Θ_τ. A trivial way to do this consists in keeping the first r partial quotients of f and putting then the sequence of partial quotients of one of the power series $\tau_{j,a}$ or $\tau'_{j,a}$, with $1 \le j \le s$ and a in $\mathbf{k}^\times$. For instance, for $r \ge 1$, a in $\mathbf{k}^\times$, and $j = 1, \ldots, s$, the quadratic power series

$$\begin{aligned} \alpha_{r,j,a} &= [a_0; a_1, \ldots, a_r, \overline{ab_j, a^{-1}b_{j+1}, \ldots, a^{-1}b_{j-1}}] \\ &= [a_0; a_1, \ldots, a_r, \tau_{j,a}] \end{aligned}$$

and

$$\begin{aligned} \alpha'_{r,j,a} &= [a_0; a_1, \ldots, a_r, \overline{a^{-1}b_{j-1}, ab_{j-2}, \ldots, ab_j}] \\ &= [a_0; a_1, \ldots, a_r, \tau'_{j,a}] \end{aligned}$$

are quite good approximations to f in Θ_τ and

$$\ell_{j,a}(f) = \liminf_{r \to +\infty} |f - \alpha_{r,j,a}| \cdot h(\alpha_{r,j,a}), \quad \ell'_{j,a}(f) = \liminf_{r \to +\infty} |f - \alpha'_{r,j,a}| \cdot h(\alpha'_{r,j,a})$$

are greater than or equal to $c_\tau(f)$, thus

$$c_\tau(f) \le \min_{1\le j\le s} \min_{a\in\mathbf{k}^\times} \min\{\ell_{j,a}(f), \ell'_{j,a}(f)\}. \tag{3.1}$$

Unlike the real case, we do have equality in (3.1).

LEMMA 3.1. *Under the above notation, we have*

$$c_\tau(f) = \min_{1\le j\le s} \min_{a\in\mathbf{k}^\times} \min\{\ell_{j,a}(f), \ell'_{j,a}(f)\}.$$

The analogue of Lemma 3.1 does not hold in the real case: see e.g. [4, Section 3.6]. Lemma 3.1 shows that the power series case is simpler than its real analogue.

Proof of Lemma 3.1. We have to estimate the quantities $|f-\beta|\cdot h(\beta)$ for quadratic power series β of the form

$$\zeta_{j,a} = [a_0; a_1, \dots, a_r, c_1, \dots, c_t, \tau_{j,a}]$$

and

$$\zeta'_{j,a} = [a_0; a_1, \dots, a_r, c'_1, \dots, c'_t, \tau'_{j,a}],$$

where $1\le j\le s$, $t\ge 1$, a in $\mathbf{k}^\times$, and $c_1,\dots,c_t,c'_1,\dots,c'_t$ are nonconstant polynomials in R, with $c_1 \ne a_{r+1}$, $c_t \ne a^{-1}b_{j-1}$, $c'_1 \ne a_{r+1}$, and $c'_t \ne ab_j$. For simplicity, we only treat the case of $\tau_{j,1}$, the other cases being completely analogous.

It follows from Lemmas 2.2 and 2.3 that

$$\begin{aligned} \mathrm{v}\left(\frac{|f-\zeta_{j,1}|}{|\zeta_{j,1}-\zeta_{j,1}^{(\sigma)}|}\right) &= \deg a_{r+1} + \deg c_1 - \deg(a_{r+1}-c_1) \\ &\quad - 2\sum_{j=1}^{t} \deg c_j + \deg c_t + \deg b_{j-1} - \deg(c_t - b_{j-1}), \end{aligned} \tag{3.2}$$

where $j-1$ is understood modulo s. If $t\ge 2$, then the right hand side of (3.2) is

$$\le -\deg(a_{r+1}-c_1)+\deg a_{r+1}-\deg c_1-\deg c_t-\deg(c_t-b_{j-1})+\deg b_{j-1}\le 0.$$

If $t=1$, then the right hand side of (3.2) is equal to

$$-\deg(a_{r+1}-c_1) + \deg a_{r+1} - \deg(c_1 - b_{j-1}) + \deg b_{j-1}.$$

Since $\deg(P_1+P_2) \le \max\{\deg P_1, \deg P_2\}$ for all polynomials P_1, P_2, the last displayed quantity is

$$\le -\max\{\deg(a_{r+1}-b_{j-1}), 0\} + \deg a_{r+1} + \deg b_{j-1}.$$

Recalling that

$$\alpha_{r,j-1,1} = [a_0; a_1, \dots, a_r, \tau_{j-1,1}],$$

it follows from (2.6) (resp., (2.7)) that if $a_{r+1} \neq b_{j-1}$ (resp., i $a_{r+1} = b_{j-1}$) then

$$\frac{|f-\alpha_{r,j-1,1}|}{|\alpha_{r,j-1,1}-\alpha^{(\sigma)}_{r,j-1,1}|} \le q^{-\deg a_{r+1}-\deg b_{j-1}+\max\{\deg(a_{r+1}-b_{j-1}),0\}} \le \frac{|f-\zeta_{j,1}|}{|\zeta_{j,1}-\zeta^{(\sigma)}_{j,1}|}.$$

Consequently, it is sufficient to restrict our attention to the approximants of the form $\alpha_{r,j,a}$ and $\alpha'_{r,j,a}$ in order to compute $c_\tau(f)$. This proves the lemma. ∎

Notation. Let $W = w_1 \dots w_h$ with $h \ge 1$ denote a finite word over the alphabet $\mathcal{A}$. Then, f_W denotes the quadratic power series with purely periodic continued fraction expansion of period W, that is,

$$f_W = [0; \overline{w_1, \dots, w_h}].$$

An important consequence of Lemma 3.1 is that the spectrum of α is determined by the set of values taken by the function c_α at quadratic power series. This is precisely the content of Theorem 1.3.

Proof of Theorem 1.3. Let $f = [a_0; a_1, a_2 \dots]$ be in $\widehat{K} \setminus (K \cup \Theta_\alpha)$ and assume that there exists a positive integer m such that $c_\alpha(f) = q^{-m}$. Then, by Lemma 3.1, there exist j in $\{1, \dots, s\}$ and a in $\mathbf{k}^\times$ such that $\ell_{j,a}(f) = q^{-m}$ or $\ell'_{j,a}(f) = q^{-m}$. Without any loss of generality, we may assume that $j = a = 1$ and $\ell_{1,1}(f) = q^{-m}$. Let d be an upper bound for the degrees of the partial quotients of α. Observe that if, for some $r \ge 1$, the degree of a_r exceeds d and if g denotes the power series whose continued fraction expansion is the same as that of f, except that a_r is replaced by Y^{d+1}, then by (2.9) we have

$$\begin{aligned}|f-[a_0; a_1, \dots, a_r, \tau]| &\cdot h([a_0; a_1, \dots, a_r, \tau]) \\ &= |g-[a_0; a_1, \dots, a_{r-1}, Y^{d+1}, \tau]| \cdot h([a_0; a_1, \dots, a_{r-1}, Y^{d+1}, \tau]),\end{aligned}$$

where $\tau = \tau_{1,1}$. Consequently, letting $\tilde{f} = [\tilde{a}_0; \tilde{a}_1, \tilde{a}_2, \dots]$ be the power series obtained from f by replacing by Y^{d+1} every partial quotient of f of degree at least $d+1$, we get $\ell_{1,1}(\tilde{f}) = q^{-m}$ and there exists an infinite set $\mathcal{R}$ of positive integers such that, for any r in $\mathcal{R}$, the quadratic number

$$\alpha_{r,1,1} = [\tilde{a}_0; \tilde{a}_1, \dots, \tilde{a}_r, \tau]$$

satisfies

$$|\tilde{f} - \alpha_{r,1,1}| \cdot h(\alpha_{r,1,1}) = q^{-m}.$$

Since the partial quotients $\tilde{a}_n$ belong to a finite set and $\mathcal{R}$ is infinite, it follows from Corollary 2.4 that there are polynomials P_1, P_2 of degree at most $d+1$ and a word W_0 (possibly empty) which is a prefix of $(b_1 \dots b_s)^\infty$ such that there exist $t \ge 0$ and arbitrarily large integers r in $\mathcal{R}$ with

$$\tilde{a}_r = P_1, \quad \tilde{a}_{r+1} \dots \tilde{a}_{r+t} = W_0, \quad \tilde{a}_{r+t+1} = P_2,$$

and

$$|\tilde{f} - \alpha_{r,1,1}| \cdot h(\alpha_{r,1,1}) = q^{-m}.$$

Setting $W = P_1 W_0 P_2$ and writing $W = w_1 \dots w_{t+2}$, we see that

$$|f_W - [0; w_1, \dots, w_{h(t+2)}, \tilde{a}_r, \tau]| \cdot h([0; w_1, \dots, w_{h(t+2)}, \tilde{a}_r, \tau]) = q^{-m}$$

for $h \geq 0$. This shows that $c_\alpha(f_W) \leq q^{-m}$. The inequality cannot be strict, since otherwise we would have $c_\alpha(\tilde{f}) < q^{-m}$. Consequently, $c_\alpha(f_W) = q^{-m}$ and the theorem is proved. ∎

4. First results on $\mathrm{Sp}(\alpha)$ **for an arbitrary** α**.** We display several immediate consequences of the preceding lemmas. Our first result is a reformulation of the first assertion of Theorem 1.1.

THEOREM 4.1. *For every quadratic power series* α *in* $\widehat{K}$,

$$\mathrm{Sp}(\alpha) \subset \{0\} \cup \{q^{-n-2} : n \in \mathbb{Z}_{\geq 0}\}.$$

Proof. This follows from (2.8). ∎

Following [21], for every power series f in $\widehat{K} \setminus K$, set

$$M(f) = \limsup_{k \to +\infty} \deg a_k \geq 1,$$

$$M_2(f) = \limsup_{k \to +\infty} (\deg a_k + \deg a_{k+1}) \geq 2,$$

$$m(f) = \liminf_{k \to +\infty} \deg a_k \geq 1.$$

Corollary 2.4 allows us to re-prove Lemma 4.4 and Corollaries 4.6 and 4.7 of [21].

Throughout the end of this section, we keep the notation of Section 3. We denote by $b_1, \dots, b_s$ the (shortest) periodic part in the continued fraction expansion of a power series α in $K^{(2)}$ and we define τ and $\tau_{j,a}$ for $j = 1, \dots, s$ and a in $\mathbf{k}^\times$ as in Section 3.

PROPOSITION 4.2. *Let* α *be in* $K^{(2)}$ *and* f *in* $\widehat{K} \setminus (K \cup \Theta_\alpha)$.

(1) *If* $m(f) > M(\alpha)$, *then* $c_\alpha(f) = q^{-M_2(\alpha)}$. *Consequently,* $\mathrm{Hw}(\alpha) \geq q^{-M_2(\alpha)}$.
(2) *If* $M(f) < m(\alpha)$, *then* $c_\alpha(f) = q^{-M_2(f)}$. *Consequently, the quadratic spectrum of* α *includes* $q^{-2}, q^{-3}, \dots, q^{-2m(\alpha)+2}$.
(3) *If* $M(\alpha) = 1$ *or* $m(\alpha) \geq 2$, *then* $\mathrm{Hw}(\alpha) = q^{-2}$.

Proof. Replacing if necessary α by $\tau_{j,1}$ for a suitable j in $\{1, \dots, s\}$, we may assume that $\deg b_1 + \deg b_s = M_2(\alpha)$. If $m(f) > M(\alpha)$, then, for r large enough, the right hand side of (2.9) is equal to $M_2(\alpha)$. Combined with Lemma 3.1, this proves the first assertion. For the second assertion, by considering the infinite sequence of integers r such that $\deg a_r + \deg a_{r+1} = M_2(f)$ and using the quadratic power series $[a_0; a_1, \dots, a_r, \overline{b_1, \dots, b_{s-1}, b_s}]$ to approximate f, we conclude that $c_\alpha(f) \leq q^{-M_2(f)}$. Actually, equality holds by Lemma 3.1. Then, choosing $f = [0; \overline{Y^d, Y^{d'}}]$ for integers d, d' with $1 \le d, d' < m(\alpha)$, we get the second part of (2). The assertion (3) is an immediate consequence of the first two. ∎

We now confirm the existence of Hall's ray in the quadratic Lagrange spectrum of an arbitrary quadratic power series. This establishes Theorem 1.1(2).

THEOREM 4.3. *For every α in $K^{(2)}$, there exists m_α such that q^{-m} is in $\mathrm{Sp}(\alpha)$ for every integer $m \ge m_\alpha$. If we denote by s the length of the periodic part of the continued fraction expansion of α and by d the maximum of the degrees of its partial quotients, an admissible value for m_α is $2d(s+1)$.*

Proof. Replacing if necessary α by $\tau_{j,1}$ for a suitable j in $\{1, \dots, s\}$, we may assume that $d = \deg b_s$ is the maximum of the degrees of $b_1, \dots, b_s$. Define b_m for $m > s$ by setting $b_m = b_j$, where $1 \le j \le s$ and $j \equiv m \bmod s$. Let u be an integer with $u \ge s$. Let P_1, P_2 be nonconstant polynomials in R such that $P_1 \ne b_s$ and $P_2 \ne b_{u+1}$. It follows from Corollary 2.4 applied with $r = 3$ that

$$
\begin{aligned}
-\frac{\log c_\tau([0; \overline{P_1, b_1, \dots, b_u, P_2, Y^{d+1}}])}{\log q}& \\
&= 2\sum_{i=1}^{u} \deg b_i + \deg b_s + \deg P_1 - \deg(b_s - P_1) \\
&\quad + \deg b_{u+1} + \deg P_2 - \deg(b_{u+1} - P_2),
\end{aligned}
$$

where equality holds since $u \ge s$ and our assumption that $P_1 \ne b_s$ and $P_2 \ne b_{u+1}$ guarantees that neither $P_1 b_1 \dots b_s$ nor $b_2 \dots b_s P_2$ is a factor of the periodic part of some $\tau_{j,a}$. Some condition on u is indeed necessary: it may happen that P_1 is one of $b_1, \dots, b_{s-1}$, say $P_1 = b_\ell$, and the word $b_\ell b_1 b_2$ is a factor of $b_1 \dots b_s$ (the partial quotient Y^{d+1} has been added to guarantee that the condition $u \ge s$ is sufficient).

If we select $P_1 = P_2 = Y^{d+1}$ and recall that $\deg b_s = d$, this shows at once that

$$q^{-2\sum_{i=1}^{u} \deg b_i - d - \deg b_{u+1}}$$

is in $\mathrm{Sp}(\alpha)$.

To establish the theorem, it is sufficient to show that, with suitable choices of P_1 and P_2, the quantity

$$\deg b_s + \deg P_1 - \deg(b_s - P_1) + \deg b_{u+1} + \deg P_2 - \deg(b_{u+1} - P_2)$$

takes every integer value between $d + \deg b_{u+1}$ and $d + 2\deg b_{u+1} + \deg b_{u+2}$. We proceed as follows. For $k = 0, \dots, d$, there exists a polynomial $P_{1,k}$ of degree d such that $\deg(P_{1,k} - b_s) = d - k$. For $h = 0, \dots, \deg b_{u+1}$, there exists a polynomial $P_{2,h}$ of degree $\deg b_{u+1}$ such that $\deg(P_{2,h} - b_{u+1}) = \deg b_{u+1} - h$. Then

$$\deg b_s + \deg P_{1,k} - \deg(b_s - P_{1,k}) + \deg b_{u+1} + \deg P_{2,h} - \deg(b_{u+1} - P_{2,h}) \\ = d + \deg b_{u+1} + k + h,$$

which takes all values between $d + \deg b_{u+1}$ and $d + 2\deg b_{u+1} + \deg b_{u+2}$, since $d = \deg b_s \geq \deg b_{u+2}$. This shows that every rational number of the form q^{-m}, with m an integer at least equal to $2\sum_{i=1}^{s} \deg b_i + d + \deg b_{s+1}$, is in $\mathrm{Sp}(\alpha)$. Consequently, a suitable value for m_α is given by $2\sum_{i=1}^{s} \deg b_i + d + \deg b_{u+1}$, which is at most equal to $2d(s+1)$. ∎

5. Gaps in the spectra and further results. We begin with an alternative proof of [21, Theorem 4.11] and establish (1.1).

THEOREM 5.1 (Parkkonen and Paulin [21]). *For every polynomial P in R of degree* 1, *we have*

$$\mathrm{Sp}([0; \overline{P}]) = \{0\} \cup \{q^{-n} : n \in \mathbb{Z}_{\geq 2}\}.$$

Proof. Set $\alpha = [0; \overline{P}\,]$. Let m be a non-negative integer and set

$$g_m = [0; \overline{Y^2, P, \ldots, P}] \quad \text{and} \quad h_m = [0; \overline{Y^2, P+1, P, \ldots, P}],$$

where P is repeated m times. Then we check that $c_\alpha(g_m) = q^{-2m-2}$ and $c_\alpha(h_m) = q^{-2m-3}$. Indeed, by Lemma 3.1, the best approximations of g_m (resp., h_m) by elements of the orbit of α are obtained by truncating the continued fraction expansion of g_m (resp., h_m) after Y^2 and completing by infinitely many copies of P. We then apply Corollary 2.4. This shows that every q^{-n} with $n \geq 2$ is in the spectrum of α. Since we already observed in the Introduction that 0 is in the spectrum of any element of $K^{(2)}$, this proves the theorem. ∎

We continue with an alternative proof of [21, Proposition 4.8].

PROPOSITION 5.2 (Parkkonen and Paulin [21]). *If α is in $K^{(2)}$ and the period of its continued fraction expansion contains no more than $q-2$ partial quotients of degree* 1, *then* $\mathrm{Hw}(\alpha) = q^{-2}$.

Proof. The argument is the same as in [21, proof of Proposition 4.8]. There exists a polynomial P in R of degree 1 such that, for every partial quotient b of degree 1 of the period of α, the polynomial $P - b$ is nonconstant. It then follows from Corollary 2.4 and Lemma 3.1 that $c_\alpha(f_P) = q^{-2}$. ∎

Proof of Theorem 1.2. Let α be a quadratic power series in $\widehat{K}$ and denote by k the length of the period in its continued fraction expansion. Theorem 1.2 follows from Proposition 5.2 if $k \leq q-2$ or if $k = q-1$ and at least one partial quotient is of degree at least 2. If $k = q-1$ and all the partial quotients are of degree 1, then $M(\alpha) = 1$ and Theorem 1.2 follows from Proposition 4.2(3). ∎

Proof of Theorem 1.4. Let $k \geq 2$ be an integer. Consider a finite word W over the alphabet $\mathcal{A}_{\leq k}$ constructed by concatenating a copy of each different block of length $k-1$ over $\mathcal{A}_{\leq k}$. The order is irrelevant. Let α in $K^{(2)}$ have period given by W.

Let $f = [0; a_1, a_2, \ldots]$ be in $\widehat{K} \setminus (K \cup \Theta_\alpha)$. Let h be the largest integer in $\{0, 1, \ldots, k-1\}$ for which there are arbitrarily large integers n with the property that the h polynomials $a_n, a_{n+1}, \ldots, a_{n+h-1}$ are of degree at most k. If $h = k-1$, then $c_\alpha(f) \leq q^{-2(k-1)-2} = q^{-2k}$. If $h = k-2$ and $k \geq 3$, then there exists a polynomial b of degree k and infinitely many integers n such that $a_n a_{n+1} \ldots a_{n+h-1} b$ is a factor of W and $\deg a_{n+h} > k$. This implies that $c_\alpha(f) \leq q^{-2(k-2)-1-k} = q^{-3k+3}$. If $h < k-2$ or if $k = 2$ and $h = 0$, then there exists a polynomial b of degree k and infinitely many integers n such that $b a_n a_{n+1} \ldots a_{n+h-1} b$ is a factor of W, $\deg a_{n-1} > k$, and $\deg a_{n+h} > k$. This implies that $c_\alpha(f) \leq q^{-2h-2k} \leq q^{-2k}$. Since, by Proposition 4.2(1),

$$c_\alpha([0; \overline{Y^{k+1}}]) = q^{-2k},$$

all this shows that $\mathrm{Hw}(\alpha) = q^{-2k}$ for $k \geq 2$.

It remains to treat the case of q^{-m} with m odd. Consider a finite word W' over the alphabet $\mathcal{A}_{\leq k}$ constructed by concatenating a copy of each different block of length k over $\mathcal{A}_{\leq k}$. The order is irrelevant; however, for technical reasons, we assume that the last letter of W' is Y^k. Let β in $K^{(2)}$ have period given by the word $W' Y^{k+1}$.

Let $f = [0; a_1, a_2, \ldots]$ be in $\widehat{K} \setminus (K \cup \Theta_\alpha)$. Let h be the largest integer in $\{0, 1, \ldots, k\}$ for which there are arbitrarily large integers n such that the h polynomials $a_n, a_{n+1}, \ldots, a_{n+h-1}$ are of degree at most k. If $h = k$, then $c_\alpha(f) \leq q^{-2k-2}$. If $h = k-1$, then there exists a polynomial b of degree k and infinitely many integers n such that $a_n a_{n+1} \ldots a_{n+h-1} b$ is a factor of W and $\deg a_{n+h} > k$. This implies that $c_\alpha(f) \leq q^{-2(k-1)-k-1} = q^{-3k+1} \leq q^{-2k-1}$. If $0 < h \leq k-2$, then there exists a polynomial b of degree k and infinitely many integers n such that $b a_n a_{n+1} \ldots a_{n+h-1} b$ is a factor of W, $\deg a_{n-1} > k$, and $\deg a_{n+h} > k$. This implies that $c_\alpha(f) \leq q^{-2h-2k} \leq q^{-2k-2}$.

So, we are left with the case where all but finitely many a_n's are polynomials of degree at least $k+1$. Since there are infinitely many pairs Y^k, Y^{k+1} in the sequence of partial quotients of β, we get $c_\beta(f) \leq q^{-1-2k}$, with equality, for instance, for $f = [0; \overline{Y^{k+2}}]$. All this implies that $\mathrm{Hw}(\beta) = q^{-2k-1}$.

Consequently, and taking also Proposition 4.2(3) into account, we have shown that the function Hw takes any value q^{-m} with $m = 2$ or $m \geq 4$. To conclude, let W'' be a finite word over $\mathcal{A}_{\leq 2}$ of even length such that every nonconstant polynomial of degree at most 2 occurs in W'' and only them, and any two consecutive polynomials are of different degree. Let γ in $K^{(2)}$ have period given by W''. Then it is easy to check that $\mathrm{Hw}(\gamma) = q^{-3} = c_\gamma([0; \overline{Y^3}])$. This completes the proof of the theorem. ∎

Proof of Theorem 1.5. Let $k \geq 2$ and $\ell \geq 1$ be integers. Consider a cyclic de Bruijn word [2, 10] of order ℓ over $\mathcal{A}_{=k}$, that is, a word W of length $(\mathrm{Card}\,\mathcal{A}_{=k})^\ell$ such that every word of length ℓ occurs exactly once in the prefix of length $(\mathrm{Card}\,\mathcal{A}_{=k})^\ell + \ell - 1$ of W^∞. Let α in $K^{(2)}$ have period given by W.

Let j be a nonnegative integer. Let d, d' be positive integers different from k. Consider a factor $W_j = w_1 \dots w_j$ of W^∞ of length j. It follows from Corollary 2.4 and Lemma 3.1 that

$$c_\alpha([0; \overline{Y^d, w_1, \dots, w_j, Y^{d'}}]) = q^{-2kj - \min\{d,k\} - \min\{d',k\}}.$$

Suitable choices of d and d' show that the function c_α can take every value between q^{-2kj-2} and $q^{-2kj-2k}$.

Let us now study which values of the form q^{-2kj-1} can be taken by the function c_α.

Let $f = [0; a_1, a_2, \dots]$ be in $\widehat{K} \setminus K$. Let h be the largest integer in $\{0, 1, \dots, \ell+1\}$ for which there are arbitrarily large integers n such that the polynomials $a_n, a_{n+1}, \dots, a_{n+h-1}$ are of degree k. If $h \leq \ell$, then

$$q^{-2kh-2k} \leq c_\alpha(f) \leq q^{-2kh-2};$$

otherwise

$$c_\alpha(f) \leq q^{-2\ell k - 2}.$$

This shows that the points $q^{-2k-1}, q^{-4k-1}, \dots, q^{-2\ell k-1}$ are not in the spectrum of α. It remains to establish that if k is sufficiently large, then q^{-2jk-1} is in $\mathrm{Sp}(\alpha)$ for every $j \geq \ell+1$.

Observe that every factor of W^∞ of length ℓ can be prolonged in only one way to a factor of W^∞ of length $\ell+1$. Let Z be a word of length $\ell+1$ over $\mathcal{A}_{=k}$ which is a factor of W^∞. Let P_1 denote its last letter and write $Z = Z'P_1$. Let P_2 be a polynomial of degree k, different from P_1, such that $\deg(P_1 - P_2) \leq k-1$ and with the property that neither $Z'P_2$, nor its mirror image, nor any of their twists by an element of $\mathbf{k}^\times$ as described in Section 3, is a factor of W^∞. The number of polynomials P_2 of degree k which do not have the above property is bounded from above in terms of the cardinality of $\mathbf{k}$ only. Consequently, the existence of Z, P_1, P_2 is guaranteed if k is sufficiently large in terms of q.

Let d be a positive integer not equal to k. If f is a quadratic power series whose period is composed of Y^d followed by the letters of Z' and by P_2, then one gets

$$c_\alpha(f) = q^{-2k\ell - \min\{d,k\} - 2k + \deg(P_1 - P_2)}.$$

Choosing d such that $\min\{d,k\} = \deg(P_1 - P_2) + 1$ shows that there exists a power series g such that $c_\alpha(g)$ is equal to $q^{-2(\ell+1)k-1}$.

A similar argument shows that c_α takes every value of the form q^{-2hk-1} with $h \geq \ell + 2$. We omit the details. ∎

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