Effective irrationality measures for real and p-adic roots of rational numbers close to 1, with an application to parametric families of Thue–Mahler equations

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Abstract. *We show how the theory of linear forms in two logarithms allows one to get very good effective irrationality measures for n-th roots of rational numbers $\frac{a}{b}$, when a is very close to b. We give a p-adic analogue of this result under the assumption that a is p-adically very close to b, that is, that a large power of p divides $a-b$. As an application, we solve completely certain families of Thue–Mahler equations. Our results illustrate, admittedly in a very special situation, the strength of the known estimates for linear forms in logarithms.*

1. Introduction

Let ξ be an irrational real number. The real number μ is an irrationality measure for ξ if, for every positive ε, there is a positive number $C(\xi, \varepsilon)$ such that every rational number $\frac{p}{q}$ with $q \geq 1$ satisfies

$$\left|\xi - \frac{p}{q}\right| > \frac{C(\xi, \varepsilon)}{q^{\mu+\varepsilon}}.$$

If, moreover, the constant $C(\xi, \varepsilon)$ is effectively computable for every positive ε, then μ is an effective irrationality measure for ξ. We denote by $\mu(\xi)$ (resp. $\mu_{\text{eff}}(\xi)$) the infimum of the irrationality measures (resp. effective irrationality measures) for ξ. Clearly, $\mu_{\text{eff}}(\xi)$ is larger than or equal to $\mu(\xi)$.

Every real algebraic number ξ of degree $n \geq 2$ satisfies $\mu_{\text{eff}}(\xi) \leq n$, by Liouville's theorem, and $\mu(\xi) = 2$, by Roth's theorem. This shows that $\mu_{\text{eff}}(\xi) = 2$ if ξ is quadratic, but the value of $\mu_{\text{eff}}(\xi)$ remains unknown for every ξ of degree $n \geq 3$. In this case, by using the theory of linear forms in logarithms, Feldman [14] proved that there exists a (small) positive real number $\tau(\xi)$ such that $\mu_{\text{eff}}(\xi) \leq n - \tau(\xi)$; see [9, 6] for more recent results. An alternative proof of Feldman's result, which does not depend on Baker's theory, was subsequently given by Bombieri [7]. This upper bound for $\mu_{\text{eff}}(\xi)$, valid for every real algebraic number ξ, can be considerably improved for some particular real algebraic

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numbers ξ, including n-th roots of rational numbers sufficiently close to 1. This is the content of the following theorem of Bombieri and Mueller [8].

Theorem BM. *Let a, b, n be positive integers with $b \geq 2$ and $n \geq 3$. Set*

$$\eta := 1 - \frac{\log|a-b|}{\log b}.$$

If $\sqrt[n]{a/b}$ is of degree n and $n > 2/\eta$, then

$$\mu_{\text{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq \frac{2}{\eta} + 6\Big(\frac{n^5 \log n}{\log b}\Big)^{1/3}. \tag{1.1}$$

It follows from (1.1) that, for any positive real number ε and any $n \geq 3$, we have $\mu_{\text{eff}}(\sqrt[n]{a/b}) < 2 + \varepsilon$ if b is sufficiently large in terms of n and if $\frac{a}{b}$ is sufficiently close to 1.

In the same paper, Bombieri and Mueller observed that the theory of linear forms in logarithms implies that, for every positive real algebraic number ξ, there exists an effectively computable constant $C(\xi)$, depending only on ξ, such that

$$\mu_{\text{eff}}(\sqrt[n]{\xi}) < C(\xi)(\log n), \tag{1.2}$$

for every $n \geq 3$. When n is large, (1.2) considerably improves Liouville's theorem. Furthermore, if ξ is the rational number $\frac{a}{b}$, where $a > b \geq 1$, then there exists an absolute, effectively computable constant C such that

$$\mu_{\text{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq C(\log a)(\log n). \tag{1.3}$$

The aim of the present note is to show how a known refinement in the theory of linear forms in logarithms in the special case where the rational numbers involved are very close to 1, which goes back to Shorey's paper [20], allows one to remove the dependence on $\log a$ in (1.3) when a is very close to b. Several spectacular applications to Diophantine problems and to Diophantine equations of this idea of Shorey have already been found; see for example [22, 23] and the survey [11]. Quite surprisingly, it seems that it has not yet been noticed that it can be used to give uniform, effective irrationality measures for roots of rational numbers and for quotients of logarithms of rational numbers (see [12]), under some suitable assumptions.

Shorey's idea has been incorporated in the recent lower bounds for linear forms in Archimedean logarithms through a term usually denoted by $\log E$. Roughly speaking, the development of the theory of linear forms in non-Archimedean logarithms followed the one of its Archimedean analogue. For instance, the paper [13] can be regarded as the p-adic analogue of [17], although [17] includes a parameter $\log E$ while [13] does not. A parameter also called $\log E$ appeared for the first time in the p-adic setting in [10] and allows one to get better estimates when the rational numbers involved in the linear form are p-adically close to 1. Some applications of these refined estimates have been given in [10], a more spectacular one can be found in [2, 3]. Here, we apply it to get explicit uniform, effective irrationality measures for p-adic n-th roots of certain rational numbers.

Mignotte [19] was the first to observe that the introduction of the parameter $\log E$ in the estimates of linear forms in logarithms has a striking application to parametric families of Thue equations $ax^n - by^n = c$, when a and b are positive integers very close to each other. A precise statement is given in Section 4. We extend Mignotte's result and solve completely multi-parametric families of Thue–Mahler equations.

As will be clear in the proofs, the main results of the present note are nearly immediate consequences of known lower bounds for linear forms in logarithms. However, we believe that the results are striking enough to deserve to be pointed out. They show, admittedly in a very special situation, the strength of these estimates.

2. Effective irrationality measures for real roots of rational numbers

Our first result gives effective irrationality measures for n-th roots of rational numbers sufficiently close to 1.

Theorem 2.1. *Let a, b, n be integers with $n \geq 3$ and $16 < b < a < \frac{6b}{5}$. Define η in $(0, 1]$ by $a - b = a^{1-\eta}$. Then, we have*

$$\mu_{\text{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq \frac{35.1}{\eta} \max\Big\{\frac{\log 2n}{\eta \log a}, 10\Big\}^2. \tag{2.1}$$

In view of Liouville's theorem, Theorem 2.1 gives nothing new for small values of n and is only interesting for $n \geq 3511$.

The reader may wonder whether the dependence on n in (2.1) involves $(\log n)^2$ and not only $\log n$, as in (1.3). The reason for this is that our proof uses estimates from [17]. We could have applied Gouillon's lower bounds [15] and would have then obtained a dependence in $\log n$, but with much larger numerical constants; see (5.3) at the end of the proof of Theorem 2.1.

We cannot deduce from Theorem 2.1 that, for any positive real number ε and any integer $n \geq 3$, there exist positive integers a, b such that $\sqrt[n]{\frac{a}{b}}$ is of degree n and has an effective irrationality measure less than $2 + \varepsilon$. In this respect, our result is much less interesting than Theorem BM. However, we stress that Theorem BM gives a non-trivial bound only when b is very large compared to n; namely, one requires that b satisfies

$$b > n^{216n^2}. \tag{2.2}$$

Theorem 2.1 is much stronger for smaller values of b.

We point out an immediate consequence of Theorem 2.1 in the particular case $\eta = \frac{1}{2}$.

Corollary 2.2. *Let a, b, n be integers with $n \geq 3$ and $30 < b < a < b + \sqrt{a}$. If*

$$a \geq (2n)^{1/5} \tag{2.3}$$

then we have

$$\mu_{\mathrm{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \le 7020.$$

The assumption (2.3) is fulfilled if $b > (2n)^{1/5}$, which is a considerably weaker condition than (2.2).

3. Effective irrationality measures for p-adic roots of rational numbers

Let p be a prime number and $|\cdot|_p$ denote the absolute value on $\mathbf{Q}_p$ normalized such that $|p|_p = p^{-1}$. Let ξ be an irrational element of $\mathbf{Q}_p$. The real number μ is an irrationality measure for ξ if, for every positive ε, there is a positive number $C(\xi, \varepsilon)$ such that every rational number p/q with $q \ge 1$ satisfies

$$\Big|\xi - \frac{p}{q}\Big|_p > \frac{C(\xi,\varepsilon)}{q^{\mu+\varepsilon}}.$$

If, moreover, the constant $C(\xi, \varepsilon)$ is effectively computable for every positive ε, then μ is an effective irrationality measure for ξ. We denote by $\mu(\xi)$ (resp. $\mu_{\mathrm{eff}}(\xi)$) the infimum of the irrationality measures (resp. effective irrationality measures) for ξ.

As in the real case, every p-adic algebraic number ξ of degree $n \ge 2$ satisfies $\mu_{\mathrm{eff}}(\xi) \le n$, by Liouville's theorem, and $\mu(\xi) = 2$, by Ridout's theorem. This shows that $\mu_{\mathrm{eff}}(\xi) = 2$ if ξ is quadratic, but the value of $\mu_{\mathrm{eff}}(\xi)$ remains unknown for every ξ of degree $n \ge 3$.

Let a, b, n be integers with $a > b \ge 1$ and $n \ge 3$. Let p be a prime number. The theory of linear forms in p-adic logarithms implies that there exists an absolute, effectively computable C such that every n-th root ζ of $\frac{a}{b}$ in $\mathbf{Q}_p$ satisfies

$$\mu_{\mathrm{eff}}(\zeta) \le Cp(\log a)(\log n). \tag{3.1}$$

The factor p in (3.1) can be removed if p divides $a - b$ and $|\zeta - 1|_p < 1$. If p divides $a - b$ but does not divide abn, then it follows from Hensel's lemma that the polynomial $bX^n - a$ has a root ζ in $\mathbf{Q}_p$ such that $|\zeta - 1|_p < 1$. In the sequel, we denote this root by $\sqrt[n]{\frac{a}{b}}$.

Our results concern n-th roots of rational numbers $\frac{a}{b}$ which are p-adically close to 1, that is, such that a large power of p divides $a - b$. We considerably improve the (p-adic) Liouville inequality for a class of algebraic numbers.

Theorem 3.1. *Let p be a prime number. Let a, b be integers with $1 \le b < a$ and assume that p divides $a - b$ but does not divide ab. Define η in $(0, 1)$ by $|a - b|_p^{-1} = a^\eta$. Assume that $a^\eta \ge 4$. For any integer $n \ge 3$ which is not divisible by p, we have*

$$\mu_{\mathrm{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) < \frac{53.8}{\eta}\,\max\Big\{\frac{\log 2n}{\eta\log a}, 4\Big\}^2 \tag{3.2}$$

As in Theorem 2.1, the dependence on n occurs in (3.2) through the factor $(\log n)^2$. It is theoretically possible to reduce it to $\log n$.

We highlight an immediate consequence of Theorem 3.1 in the case $b = 1$ and $\eta = \frac{1}{2}$.

Corollary 3.2. *Let p be a prime number. Let c, k, n be positive integers with $1 \leq c < p^k$. If*

$$p^k > \sqrt{2n}$$

and p does not divide n, then we have

$$\mu_{\mathrm{eff}}\left(\sqrt[n]{1 + cp^k} \right) < 1722.$$

4. Parametric families of Thue–Mahler equations

Starting with a seminal paper of Emery Thomas [21], there is now an extensive literature on parametric families of Thue equations; see for example the survey of Heuberger [16]. Much less is known for parametric families of Thue–Mahler equations. The first results on this question were obtained in 2012 by Levesque and Waldschmidt [18], who constructed parametric families of Thue–Mahler equations of arbitrary degree having only finitely many solutions, and in 2013 by Bennett and Dahmen [4], who solved completely some specific families of Thue–Mahler equations of small degree and where the corresponding set of prime numbers is unbounded.

Mignotte [19] was the first to observe that the introduction of the parameter $\log E$ in the estimates of linear forms in logarithms has a striking application to parametric families of Thue equations. He established, among others, the following result.

Theorem M. *Let b and n be positive integers. If n exceeds 600, then the only solution in positive integers to the Thue equation*

$$(b+1)x^n - by^n = 1$$

is given by $x = y = 1$.

Theorem M was subsequently improved and extended by Bennett and de Weger [5] in 1998. Three years later appeared a remarkable paper of Bennett [1], who managed to solve completely the remaining few hundreds of Thue equations left over in [5].

Theorem Be. *Let a, b and n be integers with $a > b \geq 1$ and $n \geq 3$. Then, the equation*

$$|ax^n - by^n| = 1$$

has at most one solution in positive integers x and y.

Theorems M and Be are closely related to Theorem 2.1, since there is a connection between effective irrationality measures for a given algebraic number ξ and effective upper bounds for the solutions of the Thue equation $F(X, Y) = 1$, where $F(X, 1)$ denotes the minimal defining polynomial of ξ over the rational integers.

By means of (the proof of) Theorem 3.1 we can go a step forward and solve completely parametric families of Thue–Mahler equations.

Theorem 4.1. *Let s be a positive integer and $p_1, \dots, p_s$ be distinct prime numbers. Let η be a real number in $(0, \frac{1}{s+1})$. Let $b \geq 2$ and $c \geq 1$ be integers such that*

$$\frac{\log c}{\log b} < 1 - \eta \quad \text{and} \quad \frac{\log |c|_{p_j}^{-1}}{\log(b+c)} > \eta, \quad \text{for } j = 1, \dots, s.$$

There exists an effectively computable constant κ such that, for any integer d with $|d| \leq b$ and any integer n satisfying

$$n \geq \kappa \frac{s}{\eta} \left(\log \frac{s}{\eta} \right)^2 \quad \text{and} \quad \gcd(n, p_1 \cdots p_s (p_1 - 1) \cdots (p_s - 1)) = 1,$$

all the solutions to the Thue–Mahler equation

$$(b+c)x^n - by^n = dp_1^{z_1} \cdots p_s^{z_s},$$

in integers $x, y, z_1, \dots, z_s$ with $\gcd(x, y) = 1$, satisfy $|xy| \leq 1$.

This is apparently the first example of a complete resolution of a multi-parametric family of Thue–Mahler equations.

5. Proof of Theorem 2.1

We reproduce Corollaire 3 of [17] and Corollary 2.4 of [15], with minor simplification, in the special case where the algebraic numbers involved are rational.

Theorem LMNG. *Let a_1, a_2, b_1, b_2 be positive integers such that a_1/a_2 and b_1/b_2 are multiplicatively independent and greater than 1. Let A and B be real numbers such that*

$$A \geq \max\{a_1, \mathrm{e}\}, \quad B \geq \max\{b_1, \mathrm{e}\}.$$

Let u and v be positive integers and set

$$U = \frac{u}{\log A} + \frac{v}{\log B}.$$

Set

$$E = 1 + \min\left\{ \frac{\log A}{\log(a_1/a_2)}, \frac{\log B}{\log(b_1/b_2)} \right\},$$

$$\log U_1 = \max\{\log U + \log E, 600 + 150 \log E\},$$

and

$$\log U_2 = \max\{\log U + \log\log E + 0.47, 10 \log E\}.$$

Assume furthermore that $15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}$. Then,

$$\log\left| v \log \frac{a_1}{a_2} - u \log \frac{b_1}{b_2} \right| \geq -8550 (\log A)(\log B)(\log U_1)(4 + \log E)(\log E)^{-3} \tag{5.1}$$

and

$$\log\left| v \log \frac{a_1}{a_2} - u \log \frac{b_1}{b_2} \right| \geq -35.1 (\log A)(\log B)(\log U_2)^2 (\log E)^{-3}. \tag{5.2}$$

The numerical constant in (5.2) is much smaller than the one in (5.1), but the dependence on U occurs through the factor $(\log U)^2$ in (5.2), while it only occurs through the factor $\log U$ in (5.1).

Proofs of Theorem 2.1 and Corollary 2.2.

Let x, y be coprime integers such that $10^{10n}a < y < x \leq 2y$ and $|(a/b)^{1/n} - x/y| < 1/x^2$. Since

$$\Big|\zeta\Big(\frac{a}{b}\Big)^{1/n} - \frac{x}{y}\Big| \leq 4,$$

for every n-th root of unity ζ, we get

$$\begin{aligned} 4^n \left|\Big(\frac{a}{b}\Big)^{1/n} - \frac{x}{y}\right| &\geq 4\left|\frac{a}{b} - \Big(\frac{x}{y}\Big)^n\right| \\ &\geq \left|n\log\frac{x}{y} - \log\frac{a}{b}\right| =: \Lambda. \end{aligned}$$

We apply Theorem LMNG to bound Λ from below.

Recall that η is defined by $a - b = a^{1-\eta}$. We check that

$$\frac{\log a}{\log(a/b)} \geq (\log a)\frac{b}{a}a^\eta \geq 2.36a^\eta,$$

since $a \geq 17$ and $b > 5a/6$. Since $a^\eta = a/(a-b) \geq 6$, we get $2.36a^\eta > 14$. Noticing that $x/y < a/b$, we obtain

$$\frac{\log x}{\log(x/y)} \geq 2.36a^\eta.$$

Set

$$E := 1 + 2.36a^\eta \quad \text{and} \quad U = \frac{n}{\log a} + \frac{1}{\log x}.$$

Observe that $15 \leq E \leq a^{3/2}$. It then follows from (5.2) and the lower bound $\log E \geq \eta\log a$ that

$$\begin{aligned} \log\Lambda &\geq -35.1(\log x)(\log a)(\log E)^{-1}\Big(\max\Big\{\frac{\log U + \log\log E + 0.47}{\log E}, 10\Big\}\Big)^2. \\ &\geq -\frac{35.1}{\eta}\,(\log x)\,\Big(\max\Big\{\frac{\log 2n}{\eta\log a}, 10\Big\}\Big)^2, \end{aligned}$$

since x is assumed to be sufficiently large.

We conclude that

$$\mu_{\mathrm{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq \frac{35.1}{\eta}\,\max\Big\{\frac{\log 2n}{\eta\log a}, 10\Big\}^2.$$

In particular, if

$$a \geq (2n)^{1/(10\eta)},$$

then

$$\mu_{\mathrm{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq \frac{3510}{\eta}.$$

Choosing $\eta = \frac{1}{2}$, this gives Corollary 2.2.

Using (5.1), we obtain a better dependence on n, namely we get the upper bound

$$\mu_{\mathrm{eff}}\Big(\sqrt[n]{\frac{a}{b}}\Big) \leq \frac{21180}{\eta}\,\max\Big\{372, \frac{\log(2n/\log a)}{\eta\log a} + 1\Big\}, \tag{5.3}$$

which is linear in $\log n$.

6. Proof of Theorem 3.1

Let p be a prime number. Let x_1/y_1 and x_2/y_2 be non-zero rational numbers and assume that there exists a real number E such that

$$v_p\big((x_1/y_1)-1)\big) \geq E > 1/(p-1).$$

Theorem Bu below, established in [10], gives an explicit upper bound for the p-adic valuation of

$$\Lambda = \Big(\frac{x_1}{y_1}\Big)^u - \Big(\frac{x_2}{y_2}\Big),$$

where u is a positive integer not divisible by p. Let $A_1 > 1, A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{\log|x_i|, \log|y_i|, E\,\log p\},\ \ (i=1,2).$$

and put

$$U = \frac{u}{\log A_2} + \frac{1}{\log A_1}.$$

Theorem Bu. *With the above notation, if x_1/y_1 and x_2/y_2 are multiplicatively independent, then we have the upper estimate*

$$v_p(\Lambda) \leq \frac{53.8}{E^3\,(\log p)^4}\big(\max\{\log U + \log(E\,\log p) + 0.4, 4\,E\,\log p, 5\}\big)^2 \log A_1 \log A_2,$$

if p is odd or if $p=2$ and $v_2(x_2/y_2-1) \geq 2$.

Proofs of Theorem 3.1 and Corollary 3.2.

Recall that, since p does not divide n, every n-th root of unity $\zeta \neq 1$ in $\overline{\mathbf{Q}}_p$ satisfies $v_p(\zeta-1)=0$. Let x/y be a rational number. We wish to bound from above the quantity $v_p(\sqrt[n]{a/b}-x/y)$. Since $v_p(\sqrt[n]{a/b}-1)$ is positive, we may assume that $v_p((x/y)-1)$ is positive. From

$$\gcd\Big(x-y, \frac{x^n-y^n}{x-y}\Big) = \gcd(x-y,n)$$

and the fact that p does not divide n, we deduce that

$$v_p((x/y)-1) = v_p((x/y)^n-1).$$

We apply Theorem Bu to bound from above the p-adic valuation of the quantity

$$\Lambda_p := \frac{a}{b} - \Big(\frac{x}{y}\Big)^n.$$

We introduce the parameter E equal to the largest power of p which divides $a-b$. By assumption, we have $E \geq 1$ and we get

$$v_p((x/y)-1) = v_p((x/y)^n-1) = v_p((a/b)-1) = E.$$

By definition of η, we have

$$\eta \log a = E \log p.$$

Note that $E \geq 2$ if $p = 2$, since we have assumed that $a^\eta \geq 4$. We take $x > 10^{10n} a$ and apply Theorem Bu with

$$\log A_1 = \max\{\log a, \eta \log a\} = \log a, \quad \log A_2 = \max\{\log x, \eta \log a\} = \log x,$$

and

$$U = \frac{n}{\log a} + \frac{1}{\log x}.$$

We then get

$$v_p(\Lambda_p) \leq \frac{53.8 (\log a)(\log x)}{\eta (\log a)(\log p)} \max\Bigl\{ \frac{\log U + \log(\eta \log a) + 0.4}{\eta \log a}, 4, \frac{5}{\eta \log a} \Bigr\}^2.$$

Since $a^\eta \geq \mathrm{e}^{5/4}$, by assumption, this gives

$$v_p(\Lambda_p) \leq \frac{53.8 \log x}{\eta \log p} \max\Bigl\{ \frac{\log U + \log(\eta \log a) + 0.4}{\eta \log a}, 4 \Bigr\}^2.$$

Thus, we obtain

$$v_p(\Lambda_p) \leq \frac{53.8 \log x}{\eta \log p} \max\Bigl\{ \frac{\log 2n}{\eta \log a}, 4 \Bigr\}^2,$$

since x is sufficiently large. This gives

$$\mu_{\mathrm{eff}}\Bigl(\sqrt[n]{\frac{a}{b}} \Bigr) \leq \frac{53.8}{\eta} \max\Bigl\{ \frac{\log 2n}{\eta \log a}, 4 \Bigr\}^2.$$

In particular, if a satisfies

$$a \geq (2n)^{1/(4\eta)},$$

then we have

$$\Bigl| \sqrt[n]{\frac{a}{b}} - \Bigl(\frac{x}{y}\Bigr) \Bigr|_p \geq x^{-861/\eta},$$

hence,

$$\mu_{\mathrm{eff}}\Bigl(\sqrt[n]{\frac{a}{b}} \Bigr) \leq \frac{861}{\eta}. \tag{6.1}$$

If $b = 1$ and $a = 1 + cp^k$ for integers $k \geq 1$ and c satisfying $1 \leq c < p^k$, then $|a - b|_p^{-1} = p^k > \sqrt{a}$. Corollary 3.2 then follows from (6.1) with $\eta = \frac{1}{2}$.

7. Proof of Theorem 4.1

Let x, y be integers such that

$$(b+c)x^n - by^n = dp_1^{z_1}\cdots p_s^{z_s}.$$

Assume that $|y| = \max\{|x|, |y|\} \geq 2$, the case $|x| = \max\{|x|, |y|\} \geq 2$ being analogous. Observe that

$$\begin{aligned} |d| &= |(b+c)x^n - by^n| \prod_{j=1}^{s} |(b+c)x^n - by^n|_{p_j} \\ &\geq b|y|^n \left|\left(\frac{b+c}{b}\right)\left(\frac{x}{y}\right)^n - 1\right| \prod_{j=1}^{s} |(b+c)x^n - by^n|_{p_j}. \end{aligned} \tag{7.1}$$

We follow the proofs of Theorem 2.1 and 3.1 to bound from below the quantities

$$\left|\left(\frac{b+c}{b}\right)\left(\frac{x}{y}\right)^n - 1\right| \quad \text{and} \quad |(b+c)x^n - by^n|_{p_j}, \quad \text{for } j = 1, \ldots, s.$$

Unlike in the proof of Theorem 3.1, where $\frac{x}{y}$ was assumed to be a good rational approximation to $\sqrt[n]{\frac{a}{b}}$, we have to suppose that n is coprime to $p-1$ to guarantee that $\frac{x}{y}$ is congruent to 1 modulo p, an assumption which is crucial for applying Theorem Bu. This assumption on n implies that the p-adic valuation of $x/y - 1$ is equal to that of $(x/y)^n - 1$.

Below, the constants implied by $\ll, \gg$ are absolute, effectively computable and positive. Proceeding as in the proof of Theorem 2.1, we get that

$$\log\left|\left(\frac{b+c}{b}\right)\left(\frac{x}{y}\right)^n - 1\right| \gg \frac{\log y}{1-\eta}\max\{\log 2n, 10\}^2 \gg \frac{\log y}{\eta}\max\{\log 2n, 10\}^2.$$

Likewise, proceeding as in the proof of Theorem 3.1, we get for $j = 1, \ldots, s$ that

$$v_{p_j}((b+c)x^n - by^n) \ll \frac{\log y}{\eta \log p_j}\max\{\log 2n, 4\}^2.$$

Since $|y| \geq 2$ and $|d| \leq b$, it follows from (7.1) that

$$n \ll \frac{(s+1)(\log 2n)^2}{\eta}.$$

This completes the proof of the theorem.

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