Effective irrationality measures for quotients of logarithms of rational numbers

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Abstract. We establish uniform irrationality measures for the quotients of the logarithms of two rational numbers which are very close to 1. Our proof is based on a refinement in the theory of linear forms in logarithms which goes back to a paper of Shorey.

1. Introduction and result

Let $\xi$ be an irrational real number. The real number $\mu$ is an irrationality measure of $\xi$ if, for every positive $\varepsilon$, there are a positive number $C(\xi, \varepsilon)$ and at most finitely many rational numbers $p/q$ with $q \geq 1$ and

$$|\xi - \frac{p}{q}| < \frac{C(\xi, \varepsilon)}{q^{\mu+\varepsilon}}.$$  

If, moreover, the constant $C(\xi, \varepsilon)$ is effectively computable for every positive $\varepsilon$, then $\mu$ is an effective irrationality measure of $\xi$. We denote by $\mu(\xi)$ (resp. $\mu_{\text{eff}}(\xi)$) the infimum of the irrationality measures (resp. effective irrationality measures) of $\xi$. It follows from the theory of continued fractions that $\mu(\xi) \geq 2$ for every irrational real number $\xi$ and an easy covering argument shows that there is equality for almost all $\xi$, with respect to the Lebesgue measure.

The following statement is a straightforward consequence of Baker’s theory of linear forms in logarithms (see e.g. [10] and the references therein). By definition, two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational.

Theorem 1.1. Let $a_1, a_2, b_1, b_2$ be positive integers with $a_1 > a_2$ and $b_1 > b_2$. Assume that $a_1/a_2$ and $b_1/b_2$ are multiplicatively independent. There exists an absolute, effectively computable, constant $C$ such that

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq C(\log a_1) (\log b_1).$$

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The purpose of this note is to show how a known refinement in the theory of linear forms in logarithms in the special case where the rational numbers are very close to 1, which goes back to Shorey’s paper [7], allows one to considerably improve Theorem 1.1 in this special case. Several spectacular applications to Diophantine problems and to Diophantine equations of this idea of Shorey have already been found; see for example [8, 10] and the survey [1]. Quite surprisingly, it seems that it has not yet been noticed that it can be used to give uniform upper bounds for irrationality measures of roots of rational numbers (see [2]) and of quotients of logarithms of rational numbers, under some suitable assumptions.

Our main result is the following.

**Theorem 1.2.** Let \( a_1, a_2, b_1, b_2 \) be positive integers such that
\[\max\{16, a_2\} < a_1 < 6a_2/5 \quad \text{and} \quad \max\{16, b_2\} < b_1 < 6b_2/5. \quad (1.1)\]
Define \( \eta \) by \( a_1 - a_2 = a_1^{1-\eta} \) and \( \nu \) by \( b_1 - b_2 = b_1^{1-\nu} \). If \( a_1/b_1 \) and \( a_2/b_2 \) are multiplicatively independent, then we have
\[\mu_{\text{eff}} \left( \frac{\log(a_1/a_2)}{\log(b_1/b_2)} \right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}. \quad (1.2)\]

We display an immediate corollary of Theorem 1.2 which deals with the case \( \eta > 1/2 \) and \( \nu > 1/2 \). It illustrates the strength of the theory of linear forms in logarithms.

**Corollary 1.3.** Let \( a_1, a_2, b_1, b_2 \) be positive integers such that
\[36 \leq a_2 < a_1 < a_2 + \sqrt{a_1}, \quad 36 \leq b_2 < b_1 < b_2 + \sqrt{b_1}, \quad \text{and} \quad \sqrt{b_1} < a_1 < b_1^2. \]
If \( a_1/b_1 \) and \( a_2/b_2 \) are multiplicatively independent, then we have
\[\mu_{\text{eff}} \left( \frac{\log(a_1/a_2)}{\log(b_1/b_2)} \right) \leq 221105. \quad (1.3)\]

It is apparent from the proof of Theorem 1.2 that the numerical constants in (1.2) and (1.3) can be reduced (roughly, divided by 3) if \( a_1 \) and \( b_1 \) are sufficiently large. No particular significance has to be attached to the numerical constant 6/5 in (1.1).

Let \( a, b \) and \( d \) be positive integers with \( a \neq b \) and \( \max\{a, b\} < d \). Under certain conditions, Rhin [5] (see also [6]) obtained explicit upper bounds for
\[\mu_{\text{eff}} \left( \frac{\log(1 + a/d)}{\log(1 + b/d)} \right). \]
His approach, which gives better numerical results than ours, heavily uses the fact that the two rational numbers \( a/d \) and \( b/d \) have the same denominator. It seems to us that Theorem 1.2, which applies without any specific restriction on the denominators \( a_2 \) and \( b_2 \) of the rational numbers, is new and cannot be straightforwardly derived from the methods of [5, 6].
2. Proof of Theorem 1.2

We reproduce with some simplification Corollary 2.4 of Gouillon [3] in the special case where the algebraic numbers involved are rational numbers. We replace his assumption \( E \geq 2 \) by \( E \geq 15 \), to avoid trouble with the quantity \( \log \log \log E \) occurring in the definition of \( E^* \) in Corollary 2.4 of [3], which is not defined if \( E \) is too small.

**Theorem G.** Let \( a_1, a_2, b_1, b_2 \) be positive integers such that \( a_1/a_2 \) and \( b_1/b_2 \) are multiplicatively independent and greater than 1. Let \( A \) and \( B \) be real numbers such that

\[
A \geq \max\{a_1, e\}, \quad B \geq \max\{b_1, e\}.
\]

Let \( x \) and \( y \) be positive integers and set

\[
X' = \frac{x}{\log A} + \frac{y}{\log B}.
\]

Set

\[
E = 1 + \min\left\{ \log \frac{A}{\log(a_1/a_2)}, \frac{B}{\log(b_1/b_2)} \right\}
\]

and

\[
\log X = \max\{\log X' + \log E, 265 \log E, 600 + 150 \log E\}.
\]

Assume furthermore that \( 15 \leq E \leq \min\{A^{3/2}, B^{3/2}\} \). Then,

\[
\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -8550(\log A) (\log B)(\log X)(4 + \log E)(\log E)^{-3}.
\]

It is crucial for our proof that the dependence on \( X \) comes through the factor \( \log X \) and not through \( (\log X)^2 \), as in [4]. Instead of Theorem G we could use an earlier result of Waldschmidt [9] (see also Theorem 9.1 of [10]), but the numerical constants in (1.2) and (1.3) would then be slightly larger.

**Proof of Theorem 1.2.** Our aim is to estimate from below the quantity

\[
\left| \frac{\log(a_1/a_2)}{\log(b_1/b_2)} - \frac{x}{y} \right|,
\]

for large positive integers \( x, y \). We will establish a lower bound of the form

\[
\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -C_1(C_2 + \log \max\{x, y\}), \tag{2.1}
\]

for some quantities \( C_1, C_2 \) which depend at most on \( a_1, a_2, b_1 \) and \( b_2 \). This will show that \( 1 + C_1 \) is an irrationality measure for \( \log(a_1/a_2)/\log(b_1/b_2) \).

We apply Theorem G and set \( A = a_1 \) and \( B = b_1 \).

Observe that

\[
\log(a_1/a_2) = \log(1 + ((a_1 - a_2)/a_2)) \leq (a_1 - a_2)/a_2 = a_1^{-\eta}a_2^{-1},
\]

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thus

\[
\frac{\log a_1}{\log(a_1/a_2)} \geq (\log a_1) \frac{a_2}{a_1} \geq \max\{14, a_1^\eta\},
\]

since \(a_1 \geq 17\) and \(a_1^\eta = a_1/(a_1 - a_2) \geq 6\). Likewise, we check that

\[
\frac{\log b_1}{\log(b_1/b_2)} \geq \max\{14, b_1^\nu\}.
\]

Furthermore, since \(a_2 \geq 14\) and \(a_1 \geq 17\), we get

\[
1 + \frac{\log a_1}{\log(a_1/a_2)} \leq 1 + \frac{\log a_1}{\log(1 + 1/a_2)} \leq 1 + 1.1a_2 \log a_1 \leq a_1^{3/2},
\]

and a similar upper bound holds for \(1 + (\log b_1)/(\log(b_1/b_2))\), thus we have proved that

\[
E := 1 + \min\left\{\frac{\log a_1}{\log(a_1/a_2)}, \frac{\log b_1}{\log(b_1/b_2)}\right\}
\]

satisfies

\[
15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}.
\]

Note also that the quantity

\[
E' = \min\{a_1^\eta, b_1^\nu\}
\]

satisfies \(6 \leq E' \leq E\), thus \((4 + \log E')/(\log E') \leq (4 + \log 6)/(\log 6)\). It then follows from Theorem G that

\[
\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -27638(\log A)(\log B)(\log \max\{x, y\} + \log E)(\log E')^{-2},
\]

when \(\max\{x, y\}\) is sufficiently large. This lower bound is of the form (2.1), with explicit values for \(C_1\) and \(C_2\), thus we have proved that

\[
\mu_{\text{eff}} \left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}.
\]

This completes the proof of Theorem 1.2.
References


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