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# Irreducibility criteria for sums of two relatively prime multivariate polynomials

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**Abstract.** We provide irreducibility conditions for some classes of multivariate polynomials over a field $K$, namely for polynomials of the form $f + p^k g$, where $f, g \in K[X_1, \ldots, X_r]$, $\deg_r f < \deg_r g$, $p \in K[X_1, \ldots, X_{r-1}]$ is irreducible over $K(X_1, \ldots, X_{r-2})$, and $k \geq 1$ is an integer. More precisely, we prove that if $f$ and $g$ regarded as polynomials in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$ are relatively prime over $K(X_1, \ldots, X_{r-1})$, $k$ is prime to $\deg_r g - \deg_r f$, and $\deg_{r-1} p^k$ is sufficiently large, then the polynomial $f + p^k g$ is irreducible over $K(X_1, \ldots, X_{r-1})$.

## 1. Introduction

Given two relatively prime polynomials with coefficients in a unique factorization domain, to decide whether their sum is irreducible or not, is in general a difficult problem, and no general answer in this respect seems to be available. The problem seems to be a little bit easier if we consider linear combinations of the two relatively prime polynomials, say $n_1 f(X) + n_2 g(X)$, instead of their sum. Such a linear combination proves to be irreducible, provided some conditions on the factorization of $n_1$ and $n_2$ are satisfied. In this respect, several recent results provide irreducibility criteria for polynomials of the form $f(X) + pg(X)$, where $f$

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and $g$ are relatively prime polynomials with rational coefficients, and $p$ is a sufficiently large prime number. In [7] Cavachi proved that given two relatively prime polynomials $f(X), g(X) \in \mathbb{Q}[X]$ with $\deg f < \deg g$, the polynomial $f(X)+pg(X)$ must be irreducible over $\mathbb{Q}$ for all but finitely many prime numbers $p$. This result has been improved in [8] by providing an explicit lower bound $p_0$ depending on the coefficients of $f$ and $g$, such that for all primes $p > p_0$, the polynomial $f(X)+pg(X)$ is irreducible over $\mathbb{Q}$. The method in [8] was adapted in [5] in order to obtain sharper bounds $p_0$ as well as explicit upper bounds for the total number of factors over $\mathbb{Q}$ of a linear combination of the form $n_1 f(X) + n_2 g(X)$, where $f$ and $g$ are relatively prime polynomials with $\deg f \le \deg g$, and $n_1$ and $n_2$ are non-zero integers with absolute value of $n_2/n_1$ sufficiently large. Similar results have been also provided for compositions of polynomials with integer coefficients [3] and for multiplicative convolutions of polynomials with integer coefficients [2]. Some analogous results for multivariate polynomials over an arbitrary field have been also obtained for linear combinations of relatively prime polynomials [9], for compositions of multivariate polynomials [4], and for multiplicative convolutions of multivariate polynomials [1].

In [6] we provided irreducibility criteria for polynomials of the form $f(X) + p^k g(X)$ with $f, g \in \mathbb{Z}[X]$, $f, g$ relatively prime, $\deg g > \deg f$, $p$ a prime number that divides none of the leading coefficients of $f$ and $g$, and $k$ a positive integer prime to $\deg g - \deg f$. More precisely, we proved the following result.

**Theorem 1.1.** ([6, Theorem 1.1]) *Let $f, g \in \mathbb{Z}[X]$ be two relatively prime polynomials with $\deg g = n$ and $\deg f = n - d$, $d \ge 1$. Then for any prime number $p$ that divides none of the leading coefficients of $f$ and $g$, and any positive integer $k$ prime to $d$ such that*

$$p^k \ge \left(2 + \frac{1}{2^{n+1-d} H(g)^{n+1}}\right)^{n+1-d} H(f)H(g)^n - \frac{H(f)}{H(g)},$$

*the polynomial $f(X) + p^k g(X)$ is irreducible over $\mathbb{Q}$.*

Here, for a polynomial $f \in \mathbb{Z}[X]$, $H(f)$ stands for the usual height of $f$, that is the maximum of the absolute values of its coefficients. For the proof of this result we used the following lemma, which might be of independent interest.

**Lemma 1.2.** ([6, Lemma 1.4]) *Let $f, g \in \mathbb{Z}[X]$ be two polynomials with $\deg g = n$ and $\deg f = n-d$, $d \geq 1$. Let also $p$ be a prime number that divides none of the leading coefficients of $f$ and $g$, and let $k$ be any positive integer prime to $d$. If $f(X) + p^k g(X)$ may be written as a product of two non-constant polynomials with integer coefficients, say $f_1$ and $f_2$, then one of the leading coefficients of $f_1$ and $f_2$ must be divisible by $p^k$.*

The aim of this paper is to complement the results in [6] and [9], by providing irreducibility conditions for some classes of multivariate polynomials over a field $K$, namely for polynomials of the form $f + p^k g$, where $f, g \in K[X_1, \ldots, X_r]$ regarded as polynomials in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$ are relatively prime over $K(X_1, \ldots, X_{r-1})$, $p \in K[X_1, \ldots, X_{r-1}]$ is irreducible over the field $K(X_1, \ldots, X_{r-2})$, and $k$ is a positive integer. For any $f \in K[X_1, \ldots, X_r]$ we will denote by $\deg_r f$ the degree of $f$ regarded as a polynomial in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$. We will prove the following effective result that provides an explicit condition on $k$ and an explicit lower bound for $\deg_{r-1} p^k$ depending on the degrees of the coefficients of $f$ and $g$, that ensure the irreducibility of the polynomial $f + p^k g$ over $K(X_1, \ldots, X_{r-1})$.

**Theorem 1.3.** *Let $K$ be a field, $r \geq 2$, $n \geq d \geq 1$ integers, and let $f = \sum_{i=0}^{n-d} a_i X_r^i$ and $g = \sum_{i=0}^{n} b_i X_r^i$ in $K[X_1, \ldots, X_r]$ with $a_0, \ldots, a_{n-d}, b_0, \ldots, b_n \in K[X_1, \ldots, X_{r-1}]$, $a_{n-d} b_n \neq 0$. Assume that $f$ and $g$ as polynomials in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$ are relatively prime over $K(X_1, \ldots, X_{r-1})$. Then for any polynomial $p \in K[X_1, \ldots, X_{r-1}]$ irreducible over $K(X_1, \ldots, X_{r-2})$, that does not divide $a_{n-d} b_n$, and any positive integer $k$ prime to $d$ such that*

$$k \deg_{r-1} p > \max_{0 \leq i \leq n-d} \deg_{r-1} a_i + n \max_{0 \leq i \leq n} \deg_{r-1} b_i,$$

*the polynomial $f + p^k g$ is irreducible over $K(X_1, \ldots, X_{r-1})$.*

We note that for $k = 1$ we do not need to ask $a_{n-d} b_n$ to be not divisible by $p$, because this condition will be automatically satisfied since $p$ is irreducible over $K(X_1, \ldots, X_{r-2})$, and our hypothesis on $\deg_{r-1} p$ obviously implies $\deg_{r-1} p > \max\{\deg_{r-1} b_n,\ \deg_{r-1} a_{n-d}\}$. So for $k = 1$ one obtains the following result.

**Theorem 1.4.** *Let $K$ be a field, $r \geq 2$, $n \geq d \geq 1$ integers, and let $f = \sum_{i=0}^{n-d} a_i X_r^i$ and $g = \sum_{i=0}^{n} b_i X_r^i$ in $K[X_1, \ldots, X_r]$ with $a_0, \ldots, a_{n-d}, b_0, \ldots, b_n \in K[X_1, \ldots, X_{r-1}]$, $a_{n-d} b_n \neq 0$. Assume that $f$ and $g$ as polynomials in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$ are relatively prime over $K(X_1, \ldots, X_{r-1})$. Then for any polynomial $p \in K[X_1, \ldots, X_{r-1}]$ that is irreducible over $K(X_1, \ldots, X_{r-2})$ and satisfies*

$$\deg_{r-1} p > \max_{0 \leq i \leq n-d} \deg_{r-1} a_i + n \max_{0 \leq i \leq n} \deg_{r-1} b_i,$$

*the polynomial $f + pg$ is irreducible over $K(X_1, \ldots, X_{r-1})$.*

In particular, we deduce from Theorem 1.3 the following corollary.

**Corollary 1.5.** *Let $K$ be a field, $r \geq 2$, let $f, g \in K[X_1, \ldots, X_r]$ be two polynomials with $\deg_r g = n$ and $\deg_r f = n - d$, $d \geq 1$, and assume that $f$ and $g$ as polynomials in $X_r$ with coefficients in $K[X_1, \ldots, X_{r-1}]$ are relatively prime over $K(X_1, \ldots, X_{r-1})$. Then $f + p^k g$ is irreducible over $K(X_1, \ldots, X_{r-1})$ provided one of the following holds:*

*(i) $p \in K[X_1, \ldots, X_{r-1}]$ is a fixed polynomial that is irreducible over the field $K(X_1, \ldots, X_{r-2})$ and that divides none of the leading coefficients of $f$ and $g$, and $k$ is any sufficiently large positive integer that is prime to $d$;*

*(ii) $k$ is a fixed positive integer that is prime to $d$, and $p \in K[X_1, \ldots, X_{r-1}]$ is an arbitrary polynomial that is irreducible over $K(X_1, \ldots, X_{r-2})$, and whose degree with respect to $X_{r-1}$ is sufficiently large.*

We note here that we will only need to prove the statement in Theorem 1.3 in the bivariate case, that is for polynomials $f, g \in K[X, Y]$ and $p \in K[X]$, since the result for $r \geq 3$ will follow from this particular case by writing $Y$ for $X_r$, $X$ for $X_{r-1}$, and by replacing $K$ with the field generated by $K$ and the variables $X_1, \ldots, X_{r-2}$.

For the proof of our results we will need the following lemma, which is an analogue of Lemma 1.4 in [6] for the bivariate case.

**Lemma 1.6.** *Let $K$ be a field and let $f, g \in K[X, Y]$ be two polynomials with $\deg_Y g = n$ and $\deg_Y f = n - d$, $d \geq 1$. Regard $f$ and $g$ as polynomials in*

*$Y$ with coefficients in $K[X]$, and let $p(X) \in K[X]$ be an irreducible polynomial that divides none of the leading coefficients of $f$ and $g$. Let $k$ be any positive integer prime to $d$. If $f(X,Y)+p(X)^k g(X,Y)$ may be written as a product of two polynomials $f_1, f_2 \in K[X,Y]$ with $\deg_Y f_1 \geq 1$ and $\deg_Y f_2 \geq 1$, then one of the leading coefficients of $f_1$ and $f_2$, regarded as polynomials in $Y$ with coefficients in $K[X]$, must be divisible by $p(X)^k$.*

## 2. Proof of the main results

As we shall see in this section, the proof of the main result has in some sense a $p$-adic nature, requiring on one hand a Newton polygon argument, and on the other hand the study of a $\rho$-adic absolute value of the resultant of $g$ and a hypothetical non-trivial factor of $f+p^k g$, whose leading coefficient is not divisible by $p$ (whose existence is guaranteed by Lemma 1.6).

We will first recall some facts about Newton polygons (see for instance Prasolov [23], or Gouvêa [18] for $p$-adic Newton polygons), that will be required in the proof of Lemma 1.6. So let $R$ be a unique factorization domain, $p$ a fixed prime element of $R$, and let $f(Y) = \sum_{i=0}^{n} a_i Y^i \in R[Y]$, $a_0 a_n \neq 0$. Represent the non-zero coefficients of $f$ in the form $a_i = \alpha_i p^{\beta_i}$, where $\alpha_i$ is an element of $R$ that is not divisible by $p$, and let us assign to each non-zero coefficient $\alpha_i p^{\beta_i}$ a point in the plane with integer coordinates $(i, \beta_i)$. The Newton polygon of $f$ corresponding to the prime element $p$ is constructed from these points in the following way. Let $A_0 = (0, \beta_0)$ and let $A_1 = (i_1, \beta_{i_1})$, where $i_1$ is the largest integer for which there are no points $(i, \beta_i)$ below the segment $A_0A_1$. Next, let $A_2 = (i_2, \beta_{i_2})$, where $i_2$ is the largest integer for which there are no points $(i, \beta_i)$ below the segment $A_1A_2$, and so on (see Figure 1). The very last segment that we will draw will be $A_{m-1}A_m$, say, where $A_m = (n, \beta_n)$. Observe that the broken line constructed so far is the lower convex hull of the points $(i, \beta_i)$, $i = 0, \ldots, n$. Now, if some segments of the broken line $A_0A_1 \ldots A_m$ are passing through points in the plane that have integer coordinates, then such points in the plane will be also considered to be vertices of the broken line. In this way, to the vertices

$A_0, A_1, \ldots, A_m$ plotted in the first phase, we might need to add a number of $t \geq 0$ more vertices.

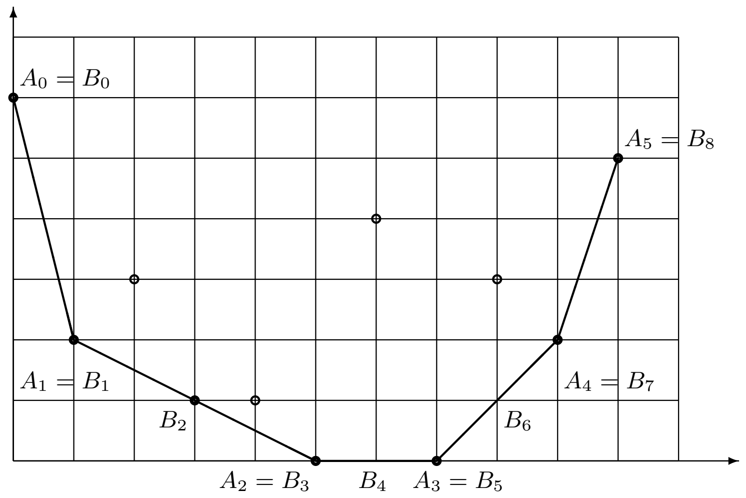


**Figure 1.** The Newton polygon of $f(X,Y) = X^6 + X^2Y + X^3Y^2 + XY^3 + XY^4 + Y^5 + X^4Y^6 - Y^7 + X^3Y^8 + X^2Y^9 + X^5Y^{10}$ in $\mathbb{Z}[X][Y]$ with respect to the prime element $X$.

The *Newton polygon of* $f$ (with respect to the prime element $p$) is defined to be the resulting broken line $B_0B_1 \ldots B_{m+t}$ that is obtained after relabelling all these points from left to the right, as they appear in this construction (here $B_0 = A_0$ and $B_{m+t} = A_m$). With this notation, $A_jA_{j+1}$ and $B_iB_{i+1}$ are called *edges* and *segments* of the Newton polygon, respectively, while the vectors $\overrightarrow{B_iB_{i+1}}$ are called the *vectors of the segments* of the Newton polygon. Therefore, a segment $B_iB_{i+1}$ of the Newton polygon will contain no points with integer coordinates other than its endpoints $B_i$ and $B_{i+1}$, while by this procedure, no two edges are allowed to have the same slope. The collection of all the vectors of the segments of the Newton polygon, considering each vector with its multiplicity, that is as many times as it appears, is called *the system of vectors* for the Newton polygon with respect to $p$. With these definitions, we have the following famous result of Dumas [10], which is a key factor in the theory of Newton polygons.

**Theorem (Dumas).** *Let $R$ be a unique factorization domain, $p$ be a prime element of $R$, and let $f = gh$, where $f, g$ and $h$ are non-constant polynomials in $R[Y]$. Then the system of vectors for the Newton polygon of $f$ with respect to $p$ is the union of the systems of vectors for the Newton polygons of $g$ and $h$ with respect to $p$.*

Here the union of the two systems of vectors contains all the vectors in each system, the vectors appearing multiple times counted to the total multiplicity that they occur in both systems. Thus, the edges in the Newton polygon of $f = gh$ with respect to $p$ may be formed by constructing a polygonal path composed by translates of all the edges that appear in the Newton polygons of $g$ and $h$ with respect to $p$, using exactly one translate for each edge, in such a way as to form a polygonal path with increasing slopes.

Many fundamental results concerning the irreducibility of some special classes of polynomials, such as Bessel polynomials and Laguerre polynomials, rely on the use of Newton polygon method. For such recent applications, we refer the reader to the work of Filaseta [11], [12], Filaseta, Finch and Leidy [13], Filaseta, Kidd and Trifonov [14], Filaseta and Lam [15], Filaseta and Trifonov [16], Filaseta and Williams [17], Hajir [19], [20], [21], and Sell [24]. For an excellent survey on the use of Newton polygons to test irreducibilty we refer the reader to the paper of Mott [22].

*Proof of Lemma 1.6.* Let $f(X,Y) = a_0(X) + a_1(X)Y + \cdots + a_{n-d}(X)Y^{n-d}$ and $g(X,Y) = b_0(X) + b_1(X)Y + \cdots + b_n(X)Y^n$, $a_{n-d}b_n \neq 0$, and let us write

$$f(X,Y) + p(X)^k g(X,Y) = c_0(X) + c_1(X)Y + \cdots + c_n(X)Y^n,$$

where $c_i(X) = a_i(X) + p(X)^k b_i(X)$ for $i = 0, \ldots, n-d$, while $c_i(X) = p(X)^k b_i(X)$ for $i = n-d+1, \ldots, n$. Since by our assumption $p \nmid a_{n-d}b_n$, we deduce that $c_{n-d}$ is not divisible by $p$, while the coefficients $c_{n-d+1}, \ldots, c_n$ are all divisible by $p^k$, and moreover, $p^{k+1} \nmid c_n$. Therefore, in the Newton polygon of our polynomial $f + p^k g$ with respect to the prime element $p$, the right-most edge will join the points $(n-d, 0)$ and $(n, k)$, points that are labelled in Figure 2 below by $B_{m-1}$ and $B_m$, respectively.

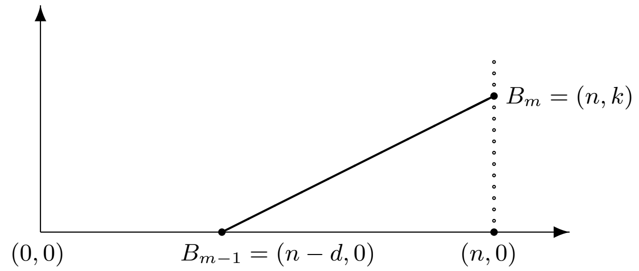


**Figure 2.** The right-most segment in the Newton polygon of $f(X,Y)+p(X)^k g(X,Y)$ with respect to $p(X)$.

Moreover, since $k$ is prime to $d$, we may easily deduce that $B_{m-1}B_m$ is in fact a segment of the Newton polygon, since it contains no points with integer coordinates other than its endpoints $B_{m-1}$ and $B_m$. To see this, we consider the equation of the straight line passing through $B_{m-1}$ and $B_m$, which is given by

$$y = \frac{k}{d} \cdot x + \frac{k}{d} \cdot (d-n),$$

and observe that since $k$ and $d$ are coprime, for each $j = 1, \ldots, d-1$ the $y$-coordinate of the point $(n-j, k(d-j)/d)$ on this line is never a rational integer.

Now, since the sequence of the slopes of the edges of the Newton polygon, when considered from left to the right, is strictly increasing, we deduce that $B_{m-1}B_m$ is the only segment with positive slope. On the other hand, if $f(X,Y) + p(X)^k g(X,Y)$ may be written as a product of two polynomials $f_1, f_2 \in K[X,Y]$ with $\deg_Y f_1 \geq 1$ and $\deg_Y f_2 \geq 1$, then each of the factors $f_1$ and $f_2$ regarded as polynomials in $Y$ with coefficients in $K[X]$, must have at least one coefficient which is not divisible by $p(X)$. If we assume now that each of the leading coefficients of $f_1$ and $f_2$ is divisible by $p(X)$, then each of the Newton polygons of $f_1$ and $f_2$ with respect to $p$ would contain at least one segment with positive slope. Therefore, by Dumas' Theorem each factor $f_1$ and $f_2$ will contribute to the Newton polygon of $f + p^k g$ at least one segment with positive slope, giving a contradiction. Therefore, one of the leading coefficients of $f_1$ and $f_2$ must be divisible by $p^k$, while the other must be coprime with $p$. This completes the proof of our lemma. □

*Proof of Theorem 1.3.* As mentioned in the previous section, we will only need to prove the bivariate case. So let $f(X,Y)=\sum_{i=0}^{n-d}a_i(X)Y^i$, $g(X,Y)=\sum_{i=0}^{n}b_i(X)Y^i\in K[X,Y]$ with $a_0,\dots,a_{n-d},b_0,\dots,b_n\in K[X]$, $a_{n-d}b_n\neq 0$ and $d\geq 1$, and assume that $f$ and $g$ as polynomials in $Y$ with coefficients in $K[X]$ are relatively prime over $K(X)$. Let also $p(X)\in K[X]$ be an irreducible polynomial that does not divide $a_{n-d}b_n$, and let $k$ be a positive integer prime to $d$ such that

$$k\deg p>\max_{0\leq i\leq n-d}\deg a_i+n\max_{0\leq i\leq n}\deg b_i. \tag{1}$$

We will adapt the ideas in [6] and [9]. First of all we notice that for $d=n$ the conclusion in Theorem 1.3 holds without restrictions on the degree of $p(X)$. Indeed, in this case $\deg_Y f=0$, so $f\in K[X]$ and $p$ is not a divisor of $f$, hence the Newton polygon of $f(X)+p(X)^kg(X,Y)$ with respect to $p$ consists of a single segment joining the points $A_0=(0,0)$ and $A_1=(n,k)$. The fact that $A_0A_1$ contains no points with integer coordinates other than $A_0$ and $A_1$, and hence is a segment of the Newton polygon, follows by the same argument used in the proof of Lemma 1.6, since $k$ and $d$ are coprime. The irreducibility of $f+p^kg$ is then an immediate consequence of Dumas' Theorem. So in what follows we may obviously assume that $n-d\geq 1$.

Now we will introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, in the following way. We fix an arbitrary real number $\rho>1$, and for any polynomial $F(X)\in K[X]$ we define $|F(X)|$ by the equality

$$|F(X)|=\rho^{\deg F(X)}.$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity, that is, for $F(X),G(X)\in K[X]$, $G(X)\neq 0$, we let $\left|\frac{F(X)}{G(X)}\right|=\frac{|F(X)|}{|G(X)|}$. Let us note here that for any non-zero element $F$ of $K[X]$ one has $|F|\geq 1$.

Let now $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$, and let us fix an extension of our absolute value $|\cdot|$ to $\overline{K(X)}$, which we will also denote by $|\cdot|$.

As in [6], the proof will be obtained by contradiction and will consist of two parts. The first part consists in proving that if our polynomial factorizes as

$$f(X,Y)+p(X)^kg(X,Y)=f_1(X,Y)f_2(X,Y) \tag{2}$$

with $f_1, f_2 \in K[X,Y]$, $\deg_Y f_1 \geq 1$ and $\deg_Y f_2 \geq 1$, and we regard $f_1$ and $f_2$ as polynomials in $Y$ with coefficients in $K[X]$, then one of the leading coefficients of the hypothetical factors $f_1$ and $f_2$, say the leading coefficient of $f_1$, will not exceed in absolute value the absolute value of the leading coefficient of $g$. The second part consists in proving that the conclusion of the first part forces the resultant $\mathrm{Res}(g, f_1)$, as a non-zero polynomial in $K[X]$, to be less than 1 in absolute value, which obviously can not hold.

So let $f(X,Y) = a_0(X) + a_1(X)Y + \cdots + a_{n-d}(X)Y^{n-d}$ and $g(X,Y) = b_0(X) + b_1(X)Y + \cdots + b_n(X)Y^n \in K[X,Y]$ with $a_0, \ldots, a_{n-d}, b_0, \ldots, b_n \in K[X]$, $a_{n-d}b_n \neq 0$, and assume that $f$ and $g$ are algebraically relatively prime, that is they can only share common factors in $K[X]$. Now let us assume that the polynomial $f(X,Y) + p(X)^k g(X,Y)$ is reducible over $K(X)$, so it satisfies (2) with $f_1, f_2$ given by

$$\begin{aligned} f_1(X,Y) &= c_0(X) + c_1(X)Y + \cdots + c_s(X)Y^s, \\ f_2(X,Y) &= d_0(X) + d_1(X)Y + \cdots + d_t(X)Y^t, \end{aligned}$$

say, with $c_0, \ldots c_s, d_0, \ldots d_t \in K[X]$, $c_s d_t \neq 0$, and $s \geq 1$, $t \geq 1$, $s + t = n$. By Lemma 1.6 we see that one of the leading coefficients of $f_1$ and $f_2$ will be divisible by $p^k$, while the other one will be coprime with $p$. Without loss of generality we may assume that $p(X)^k \mid d_t(X)$ and $p(X) \nmid c_s(X)$, hence $c_s(X)$ must be a divisor of $b_n(X)$. In particular, we must have $\deg c_s \leq \deg b_n$, which, using the definition of our absolute value, reads

$$|c_s| \leq |b_n|. \tag{3}$$

Now we are going to estimate the resultant $\mathrm{Res}(g(X,Y), f_1(X,Y))$. Since $f$ and $g$ are relatively prime as polynomials in $Y$, $g$ and $f_1$ must also be relatively prime as polynomials in $Y$, hence the resultant $\mathrm{Res}(g(X,Y), f_1(X,Y))$ must be a non-zero element of $K[X]$, so in particular we have

$$|\mathrm{Res}(g(X,Y), f_1(X,Y))| \geq 1. \tag{4}$$

If we decompose $f_1$, say $f_1(X,Y) = c_s(X)(Y - \theta_1) \cdots (Y - \theta_s)$, with $\theta_1, \ldots, \theta_s \in \overline{K(X)}$, then

$$|\mathrm{Res}(g(X,Y), f_1(X,Y))| = |c_s(X)|^n \prod_{1 \leq j \leq s} |g(X, \theta_j)|. \tag{5}$$

Since each root $\theta_j$ of $f_1$ is also a root of $f(X,Y)+p(X)^k g(X,Y)$, we have

$$g(X,\theta_j) = -\frac{f(X,\theta_j)}{p(X)^k} \tag{6}$$

and moreover, since $f$ and $g$ are relatively prime, $f(X,\theta_j)\neq 0$ and $g(X,\theta_j)\neq 0$ for any index $j\in\{1,\ldots,s\}$. Using now (5) and (6), we obtain

$$|\mathrm{Res}(g(X,Y),f_1(X,Y))| = \frac{|c_s(X)|^n}{|p(X)|^{ks}}\prod_{1\leq j\leq s}|f(X,\theta_j)|\,. \tag{7}$$

We now proceed to find an upper bound for $|f(X,\theta_j)|$. The equality

$$f(X,\theta_j)+p(X)^k g(X,\theta_j)=0$$

yields

$$|p^k b_n\theta_j^n| = \left|\sum_{i=0}^{n-d}(a_i+p^k b_i)\theta_j^i + \sum_{i=n-d+1}^{n-1}p^k b_i\theta_j^i\right|,$$

which by the fact that our absolute value is nonarchimedean shows that

$$|p^k b_n\theta_j^n| \leq \max\{\max_{0\leq i\leq n-d}|a_i|,\ |p|^k\max_{0\leq i\leq n-1}|b_i|\}\cdot\max_{0\leq i\leq n-1}|\theta_j|^i. \tag{8}$$

Since according to our hypothesis on the magnitude of $k\deg p$ we obviously have

$$k\deg p > \max_{0\leq i\leq n-d}\deg a_i - \max_{0\leq i\leq n-1}\deg b_i$$

we deduce that $|p|^k\max_{0\leq i\leq n-1}|b_i| > \max_{0\leq i\leq n-d}|a_i|$, so by (8) we obtain

$$|b_n\theta_j^n| \leq \max_{0\leq i\leq n-1}|b_i|\cdot\max_{0\leq i\leq n-1}|\theta_j|^i. \tag{9}$$

We distinguish now two cases and set

$$M := \frac{\max_{0\leq i\leq n-1}|b_i|}{|b_n|}.$$

**Case 1.** $|\theta_j|\geq 1$. In this case $\max_{0\leq i\leq n-1}|\theta_j|^i = |\theta_j|^{n-1}$, so by (9) we deduce that

$$|\theta_j|\leq M.$$

We note here that if at least one root $\theta_j$ has absolute value greater than or equal to 1, we must have $|b_n| \le \max_{0\le i\le n-1} |b_i|$, thus $M \ge 1$ must hold.

**Case 2.** $|\theta_j| < 1$. In this case $\max_{0\le i\le n-1} |\theta_j|^i = 1$, which in view of (9) yields

$$|\theta_j| \le M^{1/n}. \tag{10}$$

So in either case the roots $\theta_j$ satisfy

$$|\theta_j| \le \max\{M, M^{1/n}\}. \tag{11}$$

Now, in view of (11) we deduce that

$$\begin{aligned} |f(X,\theta_j)| &= \left|\sum_{i=0}^{n-d} a_i\theta_j^i\right| \le \max_{0\le i\le n-d} |a_i\theta_j^i| \\ &\le \max_{0\le i\le n-d} |a_i| \cdot \max_{0\le i\le n-d} |\theta_j^i| \\ &\le \max_{0\le i\le n-d} |a_i| \cdot \max\{M, M^{1/n}\}^{n-d}, \end{aligned}$$

for $j = 1, \ldots, s$.

On combining this upper bound for $|f(X,\theta_j)|$ with (3) and (7) one obtains

$$|\mathrm{Res}(g(X,Y), f_1(X,Y))| \le |b_n|^n \cdot \left(\frac{\max_{0\le i\le n-d} |a_i| \cdot \max\{M, M^{1/n}\}^{n-d}}{|p|^k}\right)^s.$$

Since $s \ge 1$, all we need to prove is that our assumption on the magnitude of $|p^k|$ will force

$$|b_n|^n \cdot \frac{\max_{0\le i\le n-d} |a_i| \cdot \max\{M, M^{1/n}\}^{n-d}}{|p|^k} < 1,$$

or equivalently that

$$|p|^k > |b_n|^n \cdot \max_{0\le i\le n-d} |a_i| \cdot \max\{M, M^{1/n}\}^{n-d}, \tag{12}$$

which will give the desired contradiction. Recalling the definition of our absolute value, and denoting $\max_{0\le i\le n-1} \deg b_i - \deg b_n$ by $A$, we see that

$$\max\{M, M^{1/n}\}^{n-d} = \max\left\{\rho^A, \rho^{A/n}\right\}^{n-d}$$

which shows that (12) is equivalent to

$$k \deg p > n \deg b_n + \max_{0\leq i\leq n-d} \deg a_i + (n-d)\max\left\{A,\ \frac{A}{n}\right\}.$$

If $\max\limits_{0\leq i\leq n-1} \deg b_i \geq \deg b_n$, then $A \geq 0$ and the above condition reduces to

$$k \deg p > \max_{0\leq i\leq n-d} \deg a_i + d \deg b_n + (n-d) \max_{0\leq i\leq n-1} \deg b_i, \tag{13}$$

while for $\max\limits_{0\leq i\leq n-1} \deg b_i < \deg b_n$ we have $A < 0$ and our condition reduces to

$$k \deg p > \max_{0\leq i\leq n-d} \deg a_i + \left(n - \frac{n-d}{n}\right) \cdot \deg b_n + \frac{n-d}{n} \cdot \max_{0\leq i\leq n-1} \deg b_i. \tag{14}$$

We observe now that our hypothesis (1) implies both inequalities (13) and (14), since

$$n \max_{0\leq i\leq n} \deg b_i \geq (n-u) \deg b_n + u \max_{0\leq i\leq n-1} \deg b_i$$

for every real $u$ with $0 \leq u \leq n$.

Therefore, as condition (1) holds, one obtains $|\mathrm{Res}(g(X,Y), f_1(X,Y))| < 1$, which is a contradiction. This completes the proof of the theorem. $\square$

**Remark.** As one can see from the proof of Theorem 1.3, some additional information on the coefficients of $g$ may lead us to sharper conditions on $k \deg p$. More precisely, if $\max\limits_{0\leq i\leq n-1} \deg b_i \geq \deg b_n$, one may replace condition (1) by (13), while for $\max\limits_{0\leq i\leq n-1} \deg b_i < \deg b_n$ the same condition may be replaced by (14). Therefore, instead of (1), one may ask $k \deg p$ to exceed both right hand sides in the inequalities (13) and (14), and the same conclusion on the irreducibility of $f + p^k g$ will hold. In a similar way one may obviously obtain sharper conditions on $k \deg_{r-1} p$ for polynomials in $r \geq 3$ variables over $K$.

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