Approximation by algebraic integers and Hausdorff dimension

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Abstract. *We compute the Hausdorff dimension of sets of real numbers which are close to infinitely many real algebraic integers of bounded degree. We also investigate the distribution of real algebraic integers of bounded degree, which we prove to be evenly spaced.*

1. Introduction

Around 1930, Mahler and Koksma proposed to classify real and complex numbers according to the quality of their approximation by algebraic numbers. Details may be found in the book of Schneider [21]. Many authors have contributed to that topic and, although some open questions remain, there are quite a few definitive results, most of which are given in the books of Baker [2] and of Bernik & Dodson [7]. Before stating some of these results we need to introduce some notation. We emphasize that, in the present work, we restrict our attention to sets of real numbers, and we do not investigate the approximation of complex non-real numbers, nor specific questions related to the approximation of algebraic numbers by algebraic numbers.

Let ξ be a transcendental real number and let $n \geq 1$ be an integer. We denote by $\mathbf{A}_n$ (*resp.* $\tilde{\mathbf{A}}_n$) the set of real algebraic numbers of degree less than or equal to n (*resp.* of degree exactly n). We define the number $w_n(\xi)$ (*resp.* $\tilde{w}_n(\xi)$) as the supremum of the w such that the equation $|\xi - \alpha| \leq \mathrm{H}(\alpha)^{-w-1}$ has infinitely many solutions $\alpha \in \mathbf{A}_n$ (*resp.* $\alpha \in \tilde{\mathbf{A}}_n$). Notice that $w_n(\xi)$ and $\tilde{w}_n(\xi)$ can be equal to $+\infty$. Here $\mathrm{H}(\alpha)$ denotes the naïve height of α, i.e. the maximum of the absolute values of the coefficients of its minimal defining polynomial over $\mathbf{Z}$. Accordingly, we denote by $\mathrm{H}(P)$ the maximum of the absolute values of the coefficients of a polynomial $P(X) \in \mathbf{Z}[X]$. We point out that the quantity denoted by $w_n(\xi)$ in the present work is usually denoted by $w_n^*(\xi)$, referring to Koksma's classification of complex numbers. We choose to remove the * to simplify the notation.

Sprindžuk [22] was first to prove that $w_n(\xi) = n$ for almost all (in the sense of Lebesgue measure) real numbers ξ. His result was complemented by Baker & Schmidt [3] who proved that, for $w > n$, the set of real numbers ξ such that $w_n(\xi) = w$ (*resp.* $w_n(\xi) \geq w$) has

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Hausdorff dimension $(n+1)/(w+1)$. The same result holds true with $w_n(\xi)$ replaced by $\tilde{w}_n(\xi)$ and provides a generalization of the Jarník–Besicovitch theorem ([8], [16]), which deals with the case $n=1$. Recently, using a result of Beresnevich [5] on the distribution of real algebraic numbers of fixed degree, Bugeaud [9] was able to refine Baker & Schmidt's theorem, using generalized Hausdorff measures.

The aim of the present note is to study the analogous problem in a non-homogeneous setting, indeed in the case of approximation by algebraic integers. We recall that an algebraic integer is an algebraic number, whose minimal defining polynomial over $\mathbf{Z}$ is monic. This is a rather different question which deserves alternative methods, since, for instance, Minkowski's theorem on systems of linear forms is useless in the present context. The first work on approximation by algebraic integers is due to Davenport & Schmidt [12], who studied the analogue of Wirsing's problem (see [24], [20, Chapter VIII]). Their approach for constructing good approximants has been recently reworked by Bugeaud & Teulié [10], and we shall combine ideas from [10], [12] and [3] in order to prove the analogue of Baker & Schmidt's theorem for the approximation by algebraic integers. In the course of the proof, we are led to study the distribution of real algebraic integers of bounded degree, and we show that these are evenly spaced. Such a precise result yields many other applications ([5], [9]). As for the case of quadratic integers, we display a completely explicit statement.

2. Statement of our results

For $n \geq 2$, we denote by $\mathbf{A}'_n$ (*resp.* $\tilde{\mathbf{A}}'_n$) the set of real algebraic integers of degree less than or equal to n (*resp.* of degree exactly n). For a real transcendental number ξ and an integer $n \geq 2$, let $w'_n(\xi)$ (*resp.* $\tilde{w}'_n(\xi)$) be the supremum of the w such that the equation $|\xi - \alpha| \leq \mathrm{H}(\alpha)^{-w}$ has infinitely many solutions $\alpha \in \mathbf{A}'_n$ (*resp.* $\alpha \in \tilde{\mathbf{A}}'_n$). Further, it has been proved (see [10], Corollaire 1) that $w'_n(\xi) = n$ for almost all real numbers ξ.

We briefly recall the definition of the Hausdorff dimension of a real set E. We denote by $|I|$ the length of a bounded real interval I. Let s and δ be positive real numbers and set

$$h(s, \delta, E) := \inf \sum_j |U_j|^s, \tag{1}$$

where the infimum is taken over all enumerable families of intervals U_j of length $|U_j| < \delta$ which cover E. As δ decreases to 0, $h(s, \delta, E)$ increases to a finite or infinite limit $\mathcal{H}^s(E)$, called the s-measure of E. Then, the Hausdorff dimension of E, which we denote by $\dim E$, is the real number $s_0 \in [0,1]$ defined by

$$s_0 = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

We postpone to Section 3 complementary definitions and results on Hausdorff measures.

Our first result is the analogue of the main result of [3].

Theorem 1. *Let $n \geq 2$ be an integer and let $w \geq n$ be real. Then any of the four sets*

$$\mathcal{K}_n(w) := \{\xi \in \mathbf{R} \mid w'_n(\xi) \geq w\}, \quad \tilde{\mathcal{K}}_n(w) := \{\xi \in \mathbf{R} \mid \tilde{w}'_n(\xi) \geq w\}$$

and

$$\mathcal{K}_n^*(w) := \{\xi \in \mathbf{R} \mid w_n'(\xi) = w\}, \quad \tilde{\mathcal{K}}_n^*(w) := \{\xi \in \mathbf{R} \mid \tilde{w}_n'(\xi) = w\}$$

has Hausdorff dimension equal to n/w.

Theorem 1 shows that $\mathcal{K}_n(w)$ and the set of real numbers ξ for which $w_{n-1}(\xi) \geq w-1$ have the same Hausdorff dimension. This is indeed not surprising, since integral polynomials of degree $n-1$ and monic integral polynomials of degree n both have precisely n free coefficients. However, our method of proof, and in particular the strength of Lemma 1 below, allows us to say a little more and to investigate sets of real numbers ξ with prescribed values for both $w_{n-1}(\xi)$ and $w_n'(\xi)$.

For any real ξ, we set

$$\theta(\xi) = \sup_{n\geq 1} \frac{\max\{w_n(\xi)+1, w_{n+1}'(\xi)\}}{n+1} \quad \text{and} \quad \omega(\xi) = \limsup_{n\geq 1} \frac{\max\{w_n(\xi)+1, w_{n+1}'(\xi)\}}{n+1}$$

and, for any $\lambda \geq 1$, we denote by $\Theta(\lambda)$ (*resp.* $\Omega(\lambda)$) the set of real numbers ξ with $\theta(\xi) \geq \lambda$ (*resp.* with $\omega(\xi) \geq \lambda$).

Theorem 2. *For any $\lambda \geq 1$, the Hausdorff dimensions of the sets $\Theta(\lambda)$ and $\Omega(\lambda)$ are equal to $1/\lambda$.*

One of the main tools for the proof of Theorem 1 is the notion of *regular system*, first introduced by Baker & Schmidt [3].

Definition 1. *Let Γ be a countable set of real numbers and $N : \Gamma \to \mathbf{R}$ be a positive function. The pair (Γ, N) is called a regular system if there exists a positive constant $c_1 = c_1(\Gamma, N)$ with the following property: For any bounded interval $J \subset \mathbf{R}$, there exists a positive number $K_0 = K_0(\Gamma, N, J)$ and, for any $K \geq K_0$, there are $\gamma_1, \ldots, \gamma_t$ in $\Gamma \cap J$ such that*

$$N(\gamma_j) \leq K, \quad |\gamma_j - \gamma_k| \geq K^{-1} \quad (1 \leq j < k \leq t)$$

and

$$t \geq c_1 |J| K.$$

Baker & Schmidt [3], using an improvement due to Baker [1] of a result of Sprindžuk [22], proved that, for any $n \geq 1$, the set $\mathbf{A}_n$ together with the function

$$N_n^{(1)} : \alpha \mapsto \left(\frac{\mathrm{H}(\alpha)}{\log_*^{3n} \mathrm{H}(\alpha)} \right)^{n+1}$$

is a regular system (For $x > 0$, the notation $\log_* x$ stands for $\max\{1, \log x\}$). As shown by Bernik & Dodson ([7], page 101), by means of a further improvement due to Bernik [6], this is also true with the exponent of $\log_* \mathrm{H}(\alpha)$ equal to $n+1$ instead of $3n(n+1)$. Very recently, this has been considerably improved upon by Beresnevich [4, 5], who proved that

$$N_n^{(2)} : \alpha \mapsto \left(\frac{\mathrm{H}(\alpha)}{(1+|\alpha|)^n} \right)^{n+1}$$

is indeed convenient, and that $(\tilde{\mathbf{A}}_n, N_n^{(2)})$ is also a regular system. This definitive result means that real algebraic numbers of given degree are evenly spaced. This yields many interesting applications ([5], [9]).

Combining some ideas of [5] together with the method of [10, 12], we obtain an analogue of Beresnevich's result for the distribution of algebraic integers of bounded degree.

Theorem 3. *Let $n \geq 2$. The sets $\mathbf{A}'_n$ and $\tilde{\mathbf{A}}'_n$ together with the function*

$$N_{n-1}^{(2)} : \alpha \mapsto \left(\frac{\mathrm{H}(\alpha)}{(1 + |\alpha|)^{n-1}} \right)^n$$

are regular systems.

The full strength of Theorem 3 is not needed to get Theorem 1. Indeed, using the above mentioned result of Bernik [6] and some arguments of [10, 12], it is not too difficult to prove that, for any $n \geq 2$, the set $\mathbf{A}'_n$ together with the function

$$N_{n-1}^{(3)} : \alpha \mapsto \left(\frac{\mathrm{H}(\alpha)}{\log_* \mathrm{H}(\alpha)} \right)^n$$

is a regular system. Such a result appears to be sufficient to derive Theorem 1, but not to get Theorem 5 below.

We would like to point out that our proof of Theorem 3 does not yield an effectively computable value for the number $K_0(\Gamma, N, J)$ occurring in the definition of a regular system: we are using Proposition 2 below, whose proof is of an ineffective nature. The same remark also applies for Theorem 3 of [5]. However, the main result of [4], which concerns the distribution of real quadratic numbers, is completely explicit, and his proof does not rely on Proposition 2 below. In the case of quadratic integers, the same is true. Indeed, in the course of the proof of Theorem 3, we have obtained the following result, which may have some interesting consequences.

Theorem 4. *Let J be a bounded real interval of length $|J| \leq 1$. For any integer $K \geq 24|J|^{-1} \log(1 + |J|^{-1})$, there exist $\gamma_1, \ldots, \gamma_t$ quadratic integers in $\Gamma \cap J$ such that*

$$\mathrm{H}(\gamma_j) \leq K^{1/2}, \quad |\gamma_j - \gamma_k| \geq K^{-1} \quad (1 \leq j < k \leq t)$$

and

$$t \geq 10^{-6} |J| K.$$

We remark that the lower bound for K in Theorem 4 is sharp, since one cannot hope to have anything better than $K \geq |J|^{-1}$.

Let $n \geq 1$ be an integer, let Ψ be a decreasing sequence of positive numbers, and set

$$\mathcal{K}_n(\Psi) := \{\xi \in \mathbf{R} : |\xi - \alpha| < \mathrm{H}(\alpha)^{-n} \Psi(\mathrm{H}(\alpha)) \text{ holds for infinitely many } \alpha \in \mathbf{A}_n\}.$$

A well-known theorem of Khintchine ([17], Theorem 32) asserts that the measure of $\mathcal{K}_1(\Psi)$ is null or full, according as the series $\sum_{h \geq 1} \Psi(h)$ converges or diverges. Applying his Theorem 3, Beresnevich [5] showed that the same result holds true for the sets $\mathcal{K}_n(\Psi)$, with $n \geq 2$. Thanks to our Theorem 3, we have an analogous statement for the sets

$$\mathcal{K}'_n(\Psi) := \{x \in \mathbf{R} : |x - \alpha| < \mathrm{H}(\alpha)^{-n+1} \Psi(\mathrm{H}(\alpha)) \text{ holds for infinitely many } \alpha \in \mathbf{A}'_n\},$$

for any integer $n \geq 2$.

Theorem 5. *Let Ψ be a decreasing sequence of positive numbers and let $n \geq 2$ be an integer. Then the Lebesgue measure of $\mathcal{K}'_n(\Psi)$ is null if the series $\sum_{i=1}^{\infty} \Psi(h)$ converges, while $\mathcal{K}'_n(\Psi)$ has full Lebesgue measure if the series $\sum_{i=1}^{\infty} \Psi(h)$ diverges.*

Proceeding exactly as in [9], we can compute precisely the generalized Hausdorff dimension of the sets $\mathcal{K}'_n(\Psi)$, under some (not very restrictive) hypotheses on Ψ.

As a consequence of Theorem 5, we derive the following statement related to a conjecture of Wirsing [24] (see [20], Chapter VIII), which has been investigated by Davenport & Schmidt [12] in the case of the approximation by algebraic integers.

Corollary 1. *Let $n \geq 2$ be an integer. Then, for almost every real transcendental number ξ, there exist a positive constant $c(\xi)$ and infinitely many algebraic integers α of degree n such that*

$$|\xi - \alpha| \leq c(\xi)\, \mathrm{H}(\alpha)^{-n}.$$

Corollary 1 sharpens Corollaire 1 of [10]. A challenging problem is to show that the conclusion of Corollary 1 holds true for *every* real transcendental number ξ. This is true for $n = 2$ (see [12]), but the general case remains open.

Combining ideas of Teulié [23] with the arguments used in the proof of Theorem 3, it is not difficult to show that real algebraic units of fixed degree are evenly distributed. We recall that an algebraic number α is a unit if both α and α^{-1} are algebraic integers. For any $n \geq 2$, we denote by $\mathbf{U}_n$ (*resp.* $\tilde{\mathbf{U}}_n$) the set of real algebraic units of degree less than or equal to n (*resp.* of degree exactly n).

Theorem 6. *For any integer $n \geq 3$, the sets $\mathbf{U}_n$ and $\tilde{\mathbf{U}}_n$ together with the function*

$$N_{n-2}^{(2)} : \alpha \mapsto \left(\frac{\mathrm{H}(\alpha)}{1 + |\alpha|^{n-2}} \right)^{n-1}$$

are regular systems.

Proceeding as explained above, one can derive from Theorem 6 results analogous to Theorems 1 and 5.

Dirichlet's theorem on Diophantine approximation has been studied by Davenport & Schmidt [13, 14]. The proof of Theorem 3 enables one to extend Theorem 3 of [13].

Theorem 7. *Let $n \geq 2$ be an integer. There exists an effectively computable positive constant $c = c(n)$ such that for almost every real numbers ξ there are infinitely many integers N for which the inequalities*

$$|a_n \xi^n + \ldots + a_1 \xi + a_0| < c(n)\, N^{-n}, \quad 1 \leq \max\{|a_1|, \ldots, |a_n|\} \leq N$$

are insoluble in integers $a_0, \ldots, a_n$. One can take the value $c(n) = 2^{-3n-3}$.

According to Davenport & Schmidt [13], one can take for $c(2)$ any positive real number less than $1/4$. Their proof extensively uses the exact expression of the roots of a quadratic polynomial, thus it cannot be generalized for greater values of n.

3. Auxiliary results

Like the proof of Theorem 3 of [5], the one of our Theorem 3 rests on the following two propositions. Let I be a sub-interval of $[-1/2, 1/2]$ and let $n \geq 2$ be an integer. Given $\varepsilon > 0$ and a polynomial $P(X) \in \mathbf{Z}[X]$ of degree $\leq n$, we denote by $\sigma_1(P)$ the set of real numbers $\xi \in I$ satisfying

$$|P(\xi)| \leq \varepsilon, \quad |P'(\xi)| \geq 2|I|^{-1},$$

and by $\sigma_2(P)$ the set of real numbers $\xi \in I$ satisfying

$$|P(\xi)| < \mathrm{H}(P)^{-n}, \quad |P'(\xi)| < 2|I|^{-1}.$$

For any $Q > 0$, let $B_{1,n}(I, Q, \varepsilon)$ denote the union of the sets $\sigma_1(P)$ over all polynomials $P(X) \in \mathbf{Z}[X]$ of degree $\leq n$ and height $\leq Q$. Let also $B_{2,n}(I, Q)$ denote the union of the sets $\sigma_2(P)$ over all polynomials $P(X) \in \mathbf{Z}[X]$ of degree $\leq n$ and height $> Q$.

Denoting by μ the Lebesgue measure, Proposition 1 asserts that $\mu(B_{1,n}(I, Q, \varepsilon))$ is small when ε is sufficiently small, while Proposition 2 asserts that $\mu(B_{2,n}(I, Q))$ is small when Q is large.

Proposition 1. *Let I and n be as above and set $Q_1 = \max\{(2^{n-1}|I|)^{-1/n}, 4n^2\}$. Then for any $Q > Q_1$ and any $\varepsilon < n^{-1}2^{-n-2}Q^{-n}$, we have*

$$\mu(B_{1,n}(I, Q, \varepsilon)) \leq n2^{n+2}\varepsilon Q^n |I|.$$

Proof : This is Proposition 1 in [5] with a slight change in the notation. □

Proposition 2. *Let I and n be as above. Then for almost all $\xi \in I$ the system*

$$|P(\xi)| < \mathrm{H}(P)^{-n}, \quad |P'(\xi)| < 2|I|^{-1} \tag{2}$$

has at most finitely many solutions.

Proof : This is Proposition 2 in [5] with a slight change in the notations. We point out that this result is of an ineffective nature. This means that, given a ξ in I such that (2) has no solution, one cannot compute an H such that (2) has no solution $P(X) \in \mathbf{Z}[X]$ with $\mathrm{H}(P) > H$. □

In Section 2, we have defined the usual Hausdorff dimension, which is based on the family of functions $x \mapsto x^s$ for $0 < s \leq 1$. But we can as well define $h(f, \delta, E)$, where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is any continuous, increasing function defined on a neighborhood of 0 and satisfying $f(0) = 0$. Instead of summing up $|U_j|^s$ as in (1), we sum up $f(|U_j|)$. Then, the measure of E relative to the function f, also called the $\mathcal{H}^f$-measure of E, is

$$\mathcal{H}^f(E) := \sup_{\delta > 0} h(f, \delta, E),$$

which belongs to $[0, +\infty]$. Further, by the Corollary following Theorem 40 of [19], if $g : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous, increasing function defined on a neighborhood of 0, which satisfies $g(0) = 0$ and

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0,$$

then

$$\mathcal{H}^g(E) = +\infty \quad \text{as soon as} \quad \mathcal{H}^f(E) > 0. \tag{3}$$

In the course of the proof of Theorem 1, we need the following result of Baker & Schmidt [3].

Lemma 1. *Let f and g be positive functions defined for $x > 0$ such that f decreases and $f(x) \le 1/(2x)$ for large x. Suppose also that g and $x \mapsto x/g(x)$ both increase and tend to zero with x, and that $xg(f(x)/2)$ tends to infinity with x. Let (Γ_i, N_i) be an at most countable family of regular systems. Then, for any $i \ge 1$, the set $(\Gamma_i, N_i; f)$ of all real numbers ξ for which there exist infinitely many γ in Γ_i such that*

$$|\xi - \gamma| < f(N_i(\gamma))$$

satisfies

$$\mathcal{H}^g(\Gamma_i, N_i; f) > 0.$$

Moreover, we have

$$\mathcal{H}^g\Big(\bigcap_{i \ge 1} (\Gamma_i, N_i; f)\Big) > 0.$$

Proof : This is Lemma 1 of [3], rewritten according to the notation of the present paper. □

4. Proofs

Since Theorems 1 and 2 both depend on Theorem 3, we first give the proof of Theorem 3 before those of Theorems 1, 2, 5 and 7.

Proof of Theorem 3.

Let $I \subset [-1/2, 1/2]$ be a finite interval and let $n \ge 2$ be an integer. Let $\xi \in I$ and $Q > 30n$ be a (large) real number. Assume that there exists a polynomial $P(X) \in \mathbf{Z}[X]$ of degree $\le n$ such that

$$|P(\xi)| \le n^{-1} 2^{-n-5} Q^{-n} \quad \text{and} \quad \mathrm{H}(P) \le Q. \tag{4}$$

We have either $|P'(\xi)| < 2|I|^{-1}$ or $|P'(\xi)| \ge 2|I|^{-1}$.

- Assume first that $|P'(\xi)| < 2|I|^{-1}$.

We infer from (4) that $|P(\xi)| < \mathrm{H}(P)^{-n}$, thus (2) is fulfilled. However, Proposition 2 asserts that the measure of the set $\mathcal{A}(\tilde{Q})$ of real numbers x for which there is a polynomial

$\tilde{P}(X) \in \mathbf{Z}[X]$ of degree $\leq n$ and height $\geq \tilde{Q}$ such that (2) is satisfied tends to zero when $\tilde{Q}$ tends to infinity. Thus, there exists a real Q_1 such that $\mu(\mathcal{A}(\tilde{Q})) \leq |I|/16$ as soon as $\tilde{Q} \geq Q_1$. If $\xi \notin \mathcal{A}(Q_1)$, then $\mathrm{H}(P) \leq Q_1$ and $|P(\xi)| \leq n^{-1}2^{-n-5}Q^{-n}$. Let us define the set

$$\mathcal{B}(Q, Q_1) := \bigcup_{\deg P_1 \leq n, \mathrm{H}(P_1) \leq Q_1} \{x \in I : |P_1(x)| \leq n^{-1}2^{-n-5}Q^{-n}\}.$$

Since this is a finite union, there exists $Q_2 > Q_1$ such that $\mu(\mathcal{B}(\tilde{Q}, Q_1)) \leq |I|/16$ as soon as $\tilde{Q} \geq Q_2$. Consequently, if $Q \geq \max\{Q_1, Q_2\}$ and if there exists a polynomial P such that $P(\xi)$ satisfies (4), then $\xi \in \mathcal{A}(Q_1)$ or $\xi \in \mathcal{B}(Q, Q_1)$, according as $\mathrm{H}(P) \geq Q_1$ or not. Thus, ξ belongs to a set of measure $\leq |I|/8$.

- Assume now that $|P'(\xi)| \geq 2|I|^{-1}$.

We may suppose that $Q \geq Q_3 := \max\{(2^{n-1}|I|)^{-1/n}, 4n^2\}$. We apply Proposition 1 with $\varepsilon_Q = n^{-1}2^{-n-5}Q^{-n}$, whence, with the notation of Section 3, we have $\xi \in B_{1,n}(I, Q, \varepsilon_Q)$ and $\mu(B_{1,n}(I, Q, \varepsilon_Q)) \leq |I|/8$.

Consequently, we have proved that for $Q \geq \max\{Q_1, Q_2, Q_3\}$, the set of real numbers ξ such that (4) has no solution has measure $\geq 3|I|/4$.

This will be sufficient in order to prove Theorem 3 and Theorem 7.

Assume now that ξ and Q are such that (4) has no solution. That means that the first successive minima τ_1 of the compact convex defined by the inequalities

$$\begin{gathered} |a_n\xi^n + \ldots + a_1\xi + a_0| \leq Q^{-n}, \\ |a_1|, \ldots, |a_n| \leq Q \end{gathered}$$

satisfies $\tau_1 > 1/8$. Indeed, otherwise there would exist a polynomial P with $|P(\xi)| \leq 2^{-3}Q^{-n} = 2^{-3(n+1)}(2^{-3}Q)^{-n}$ and $\mathrm{H}(P) \leq 2^{-3}Q$, and (4) would thus have a solution. Since $\tau_1 \ldots \tau_{n+1} \leq 1$ (see Cassels [11], page 203), we deduce that $\tau_{n+1} \leq 8^n$, whence there exist $n+1$ linearly independent polynomials $P_j(X) := x_n^{(j)}X^n + \ldots + x_1^{(j)}X + x_0^{(j)}$, $1 \leq j \leq n+1$, of degree $\leq n$ and whose coefficients $x_i^{(j)}$ are in $\mathbf{Z}$, satisfying

$$|P_j(\xi)| \leq 8^n Q^{-n} \quad \text{and} \quad \mathrm{H}(P_j) \leq 8^n Q, \tag{5}$$

for any $1 \leq j \leq n+1$, and (by Cassels [11], page 219) $\delta := |\det(x_i^{(j)})| \leq (n+1)!$. Let p be a prime which does not divide δ. By Bertrand's postulat, one can assume that

$$p \leq 2(n+1)!. \tag{6}$$

The upper bound (6) is used in [23] instead of the upper estimate obtained in [10] (see inequality (7) in [10]). It enables one to get rid of the logarithmic factors in Théorème 4 and in Théorème 5 of [10].

Now, we argue exactly as Davenport & Schmidt [12] in order to construct a monic polynomial of degree $n+1$, small when evaluated at ξ. As in [10], we denote by d one of the integers n or $n+1$. Taking $d = n$, we recover Theorem 3 of [5], while, taking $d = n+1$, we prove our Theorem 3.

Since p does not divide δ, there exists a superscript j such that p does not divide $x_0^{(j)}$. Without loss of generality, we assume that $j = 1$.

We consider the following system of $n+1$ equations in the $n+1$ unknowns $(\theta_i)_{1\le i\le n+1}$

$$\begin{aligned} \xi^d + p(\theta_1 P_1(\xi) + \ldots + \theta_{n+1} P_{n+1}(\xi)) &= p(n+2)8^n Q^{-n} \\ d\xi^{d-1} + p(\theta_1 P_1'(\xi) + \ldots + \theta_{n+1} P_{n+1}'(\xi)) &= pQ + p \sum_{1\le i\le n+1} |P_i'(\xi)| \\ \theta_1 x_m^{(1)} + \ldots + \theta_{n+1} x_m^{(n+1)} &= 0 \ (m = 2, \ldots, n). \end{aligned} \tag{7}$$

Since the polynomials P_i are linearly independent, this is a Cramer system, which thus has one and only one solution $(\theta_1, \ldots, \theta_{n+1})$. We take an $(n+1)$-tuple $(t_1, \ldots, t_{n+1})$ of integers such that $|\theta_i - t_i| \le 1$ for every i with $1 \le i \le n+1$, and we set

$$x_m = t_1 x_m^{(1)} + \ldots + t_{n+1} x_m^{(n+1)}, \text{ for } m = 0, \ldots, n.$$

We consider the polynomial

$$\begin{aligned} P(X) &= X^d + p(x_n X^n + \ldots + x_1 X + x_0) \\ &= X^d + p(t_1 P_1(X) + \ldots + t_{n+1} P_{n+1}(X)). \end{aligned}$$

In order to prove that P is irreducible by using Eisenstein's Criteria, it is sufficient to check that its constant coefficient, namely $p(t_1 x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)})$, is not divisible by p^2, since its leading coefficient is congruent to 1 modulo p. We fix an n-tuple $(t_2, \ldots, t_{n+1})$ and there remain two possible choices for t_1, which we denote by $t_{1,0}$ and $t_{1,1} = t_{1,0} + 1$. Since p does not divide $x_0^{(1)}$, at least one of the integers $t_{1,0} x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)}$ or $t_{1,1} x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)}$ is not divisible by p. This enables us to choose t_1 such that the polynomial P is p-Eisensteinian, whence irreducible.

Further, the polynomial P satisfies

$$P(\xi) = \xi^d + p(t_1 P_1(\xi) + \ldots + t_{n+1} P_{n+1}(\xi)),$$

hence, by (5) and the first equation of the system (7), we have

$$0 < p8^n Q^{-n} \le P(\xi) \le p(2n+3)8^n Q^{-n} < 1, \tag{8}$$

since we have assumed $Q > 30n$. On the other hand,

$$P'(\xi) = d\xi^{d-1} + p(t_1 P_1'(\xi) + \ldots + t_{n+1} P_{n+1}'(\xi)),$$

whence, by (5) and the second equation of (7), we get

$$P'(\xi) \le pQ + 2p \sum_{i=1}^{n+1} |P_i'(\xi)| \le pQ(1 + 2n(n+1)8^n), \tag{9}$$

since $|\xi| \leq 1/2$, and

$$P'(\xi) \geq pQ. \tag{10}$$

Finally, by (5) and the last equations of (7), we get

$$|x_m| \leq (n+1)8^n Q, \text{ for } m = 2, \ldots, n \tag{11}$$

and we infer from (9), (10), (11) and $|\xi| \leq 1/2$ that

$$|x_1| \leq |P'(\xi)| + \sum_{j=2}^{n} |x_j| \leq (n^2 - 1 + 2n(2n+1)p)8^n Q + pQ. \tag{12}$$

Since it follows from (8), (11) and (12) that

$$|x_0| \leq P(\xi) + |x_1| + \sum_{j=2}^{n} |x_j| \leq (2n(2n+1)p + 2n^2)8^n Q + pQ,$$

we finally derive from (6) that

$$\mathrm{H}(P) \leq (n+3)! 8^{n+1} Q. \tag{13}$$

Now, we take $d = n+1$, and we denote by $\gamma_1, \ldots, \gamma_{n+1}$ the roots of P, numbered such that $|\xi - \gamma_1|$ is minimal. Since

$$\left| \frac{P'(\xi)}{P(\xi)} \right| = \sum_{i=1}^{n+1} \frac{1}{|\xi - \gamma_i|} \leq \frac{n+1}{|\xi - \gamma_1|},$$

we infer from (8) and (10) that

$$0 < |\xi - \gamma_1| \leq (n+1) \left| \frac{P(\xi)}{P'(\xi)} \right| \leq (n+1)(2n+3)8^n Q^{-n-1}.$$

If γ_1 would be non-real, then its conjugate would also be a root of P, and, by (13) and the upper bound $|\gamma_i| \leq \mathrm{H}(P) + 1$ (for $1 \leq i \leq n+1$), we would have

$$|P(\xi)| = \prod_{i=1}^{n+1} |\xi - \gamma_i| \leq (n+1)^2 (2n+3)^2 64^n Q^{-2n-2} \times \left(2 + (n+3)! 8^{n+1} Q\right)^{n-1},$$

which contradicts (8). Thus, γ_1 is real.

To conclude the proof, we proceed exactly as in [5]. We choose a maximal collection $\{\alpha_1, \ldots, \alpha_t\} \subset I$ of real algebraic integers α_i of degree $n+1$, height $\leq (n+3)! 8^{n+1} Q$ and such that, for any $1 \leq i \neq j \leq t$, we have

$$|\alpha_i - \alpha_j| \geq (n+1)(2n+3)8^n Q^{-n-1}.$$

Arguing as in [5], we deduce that

$$t \geq \frac{3}{16(n+1)(2n+3)8^n}\, Q^{n+1}|I|.$$

Now, we let J be any real interval of length ≤ 1. There exists an integer m such that $J_m := (J+m)\cap[-1/2,1/2]$ has measure $\geq |J|/2$. Applying the above result with $I := J_m$, we get that there are $\beta_1,\dots,\beta_t \in \tilde{\mathbf{A}}'_{n+1}\cap J$ such that

$$\begin{aligned} \mathrm{H}(\beta_i) &\leq (1+|\beta_i|)^n (n+3)!16^{n+1}Q, \quad (1\leq i\leq t)\\ |\beta_i-\beta_j| &\geq (n+1)(2n+3)8^n Q^{-n-1}, \quad (1\leq i\neq j\leq t)\\ t &\geq \frac{3}{32(n+1)(2n+3)8^n}\, Q^{n+1}|J|. \end{aligned} \tag{14}$$

Setting

$$K := \big((n+3)!16^{n+1}Q\big)^{n+1},$$

the three inequalities in (14) imply that we have

$$\begin{aligned} N_n^{(2)}(\beta_i) &\leq K, \quad (1\leq i\leq t)\\ |\beta_i-\beta_j| &\geq K^{-1}, \quad (1\leq i\neq j\leq t)\\ t &\geq \frac{3}{32(n+1)(2n+3)8^n\big((n+3)!16^{n+1}\big)^{n+1}}\,|J|K. \end{aligned}$$

Consequently, the set $\tilde{\mathbf{A}}'_{n+1}$ together with the function $N_n^{(2)}$ is a regular system, and one may take for the constant c_1 occurring in Definition 1 the value

$$\Big((n+3)^{n^2+4n+5}\,2^{3n^2+7n+7}\Big)^{-1}$$

Thus, Theorem 3 is proved, and we see that, by taking $d=n$, we obtain another proof of Theorem 3 of [5], with however a slightly worse value for c_1 than the one given in [5].

It now remains for us to consider the case $n=1$, i.e. the approximation by quadratic integers. Let again $I\subset[-1/2,1/2]$ be a finite interval, and let $Q\geq 24|I|^{-1}\log(1+|I|^{-1})$ be a real number. Let ξ be a transcendental real number in I and denote by τ_1 the first successive minima of the compact convex body defined by the inequalities

$$\begin{aligned} |a_1\xi+a_0| &\leq Q^{-1},\\ |a_1| &\leq Q. \end{aligned} \tag{15}$$

It is easy to see that

$$\xi \in \Bigg(\bigcup_{1\leq q\leq[\tau_1 Q],\, p\in\mathbf{Z}} \left[\frac{p}{q}-\frac{\tau_1}{qQ}, \frac{p}{q}+\frac{\tau_1}{qQ}\right]\Bigg)\cap I.$$

For any $q \geq 1$, there are at most $q|I|+2$ such intervals centered on a rational with denominator q which have non empty intersection with I. Consequently, fixing $\tau_1 = 1/2$, the Lebesgue measure of the set of real numbers ξ such that the first successive minima of (15) is $\leq 1/2$ is not greater than

$$\sum_{q=1}^{[\tau_1 Q]} (q|I|+2)\frac{2\tau_1}{qQ} \leq \frac{|I|}{2} + \frac{\log(1+Q/2)}{Q} \leq \frac{2|I|}{3}.$$

This means that, on a subset of I of Lebesgue measure $\geq |I|/3$, the first successive minima of (15) is greater than $1/2$. To conclude, we proceed exactly as above. $\square$

Proof of Theorem 1.

Let $n \geq 2$ be an integer, and let $w > n$. A covering of the set $\mathcal{K}_n(w)$ is given by the union of all intervals centered at algebraic integers α and of radius $\mathrm{H}(\alpha)^{-w}$. For any $s > 0$, the series

$$\sum_{H \geq 1} 2n(2H+1)^{n-1}(2H)^{-sw}$$

has the same behaviour as the series of h^{n-1-sw}. Hence, it converges if and only if $s > n/w$. Consequently, the union $\bigcup_{n\geq 1} I_n$ of all the intervals centered at algebraic integers α and of radius $\mathrm{H}(\alpha)^{-w}$ covers $\mathcal{K}_n(w)$ and satisfies $\sum_n |I_n|^s < +\infty$ for any $s > n/w$. Hence, the dimension of $\mathcal{K}_n(w)$ is not greater than n/w.

Define the function f for $x > 1$ by $f(x) = x^{-w/n}$. By Theorem 3, $(\tilde{\mathbf{A}}'_n, N_{n-1}^{(2)})$ is a regular system and we easily check that the set $(\tilde{\mathbf{A}}'_n, N_{n-1}^{(2)}; f)$, which is defined in Lemma 1, is included in $\tilde{\mathcal{K}}_n(w)$. The hypotheses of Lemma 1 are satisfied with g being defined for $u > 0$ by $g(u) = u^{n/w}\log(1+1/u)$. Thus $\mathcal{H}^g(\tilde{\mathcal{K}}_n(w)) > 0$ and, since for any $s < n/w$ we have

$$\lim_{u\to 0} \frac{g(u)}{u^s} = 0,$$

it follows from (3) that $\mathcal{H}^s(\tilde{\mathcal{K}}_n(w)) = +\infty$, whence $\dim \tilde{\mathcal{K}}_n(w) \geq n/w$. Since $\tilde{\mathcal{K}}_n(w) \subset \mathcal{K}_n(w)$, we are done.

To prove the second assertion of Theorem 1, write

$$\{\xi \in \mathbf{R} \mid w'_n(\xi) = w\} = \{\xi \in \mathbf{R} \mid w'_n(\xi) \geq w\} \setminus \bigcup_{k=1}^{\infty} \{\xi \in \mathbf{R} \mid w'_n(\xi) \geq w + 1/k\},$$

i.e.

$$\mathcal{K}_n^*(w) = \mathcal{K}_n(w) \setminus \bigcup_{k=1}^{\infty} \mathcal{K}_n(w+1/k),$$

and observe that for any $k > 0$ we have $\mathcal{H}^g(\mathcal{K}_n(w+1/k)) = 0$. Since $\mathcal{H}^g(\mathcal{K}_n(w)) > 0$, we obtain that $\mathcal{H}^g(\mathcal{K}_n^*(w)) > 0$, and we conclude exactly as above. The same argument allows us to prove that the set $\tilde{\mathcal{K}}_n^*(w)$ has dimension n/w, and also to treat the case $w = n$. $\square$

Proof of Theorem 2.

We apply the last statement of Lemma 1 with the functions $f : x \mapsto x^{-\lambda}$ and $g : u \mapsto u^{1/\lambda} \log(1 + 1/u)$, defined respectively for $x > 1$ and $u > 0$, and with the regular systems $(\Gamma_i, \tilde{N}_i)$, where, for $i \geq 1$, we set

$$\Gamma_{2i-1} = \mathbf{A}_i, \tilde{N}_{2i-1} = N^{(2)}_{2i-1}, \Gamma_{2i} = \mathbf{A}'_i \quad \text{and} \quad \tilde{N}_{2i} = N^{(2)}_{2i}.$$

We observe that $\Theta(\lambda)$ and $\Omega(\lambda)$ both contain $\bigcap_{i \geq 1} (\Gamma_i, \tilde{N}_i; f)$, which, by arguing as in the proof of Theorem 1, has dimension at least equal to $1/\lambda$. The reverse inequality follows easily from Theorem 1. □

Proof of Theorem 5.

It suffices to reproduce the proof of Theorem 2 of [5] with the regular system arising from Theorem 3. □

Proof of Theorem 7.

We proceed as in [12], using a classical argument in metrical number theory (see for instance [15], page 44). For any positive integer Q and any bounded real interval $I \subset [-1/2, 1/2]$, we denote by $S(Q, I)$ the set of real numbers ξ in I such that the system

$$\begin{aligned} &|a_n \xi^n + \ldots + a_1 \xi + a_0| \leq 2^{-3n-3} Q^{-n}, \\ &1 \leq \max\{|a_1|, \ldots, |a_n|\} \leq Q \end{aligned} \tag{16}$$

has a solution. It easily follows from the beginning of the proof of Theorem 1 that the Lebesgue measure of $S(Q, I)$ is at most $|I|/4$, when Q is large enough. Denote by $S(I)$ the set of real numbers ξ in I such that $\xi \in S(Q, I)$ for all sufficiently large Q, i.e.

$$S(I) = \bigcup_{N \geq 1} \bigcap_{Q \geq N} S(Q, I).$$

The Lebesgue measure of $S([-1/2, 1/2])$ is at most $1/4$, and this set can be covered by a finite number of intervals $I_1, \ldots, I_t$, with total length at most $1/2$. We take Q large enough in order that the measure of each $S(Q, I_j)$, $1 \leq j \leq t$, is at most $|I_j|/4$. We deduce that the measure of $S([-1/2, 1/2])$ is at most $1/8$. Proceeding inductively, we infer that, for any integer $n \geq 1$, the measure of $S([-1/2, 1/2])$ is at most 2^{-n}, hence is 0. Consequently, for almost all real numbers ξ in $[-1/2, 1/2]$, there are infinitely many integers Q such that (16) has no solution. □

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References

[1] A. Baker, *On a theorem of Sprindžuk*, Proc. Roy. Soc. London Ser. A 292 (1966), 92–104.

[2] A. Baker, Transcendental number theory, Cambridge University Press, 1974.

[3] A. Baker and W. M. Schmidt, *Diophantine approximation and Hausdorff dimension*, Proc. London Math. Soc. 21 (1970), 1–11.

[4] V. Beresnevich, *Effective measure estimates for sets of real numbers with a given error of approximation by quadratic irrationalities*, Vestsi Akad. Navuk Belarusi Ser. Fiz. Mat. Navuk 1996 (4), 10–15 (in Russian).

[5] V. Beresnevich, *On approximation of real numbers by real algebraic numbers*, Acta Arith. 90 (1999), 97–112.

[6] V. I. Bernik, *On the best approximation of zero by values of integral polynomials*, Acta Arith. 53 (1989), 17–28 (in Russian).

[7] V. I. Bernik and M. M. Dodson, Metric Diophantine approximation on manifolds, Cambridge Tracts in Mathematics 137, Cambridge University Press, 1999.

[8] A. S. Besicovitch, *Sets of fractional dimension (IV): on rational approximation to real numbers*, J. London Math. Soc. 9 (1934), 126–131.

[9] Y. Bugeaud, *Approximation par des nombres algébriques de degré borné et dimension de Hausdorff*. Submitted.

[10] Y. Bugeaud et O. Teulié, *Approximation d'un nombre réel par des nombres algébriques de degré donné*, Acta Arith. 93 (2000), 77–86.

[11] J. W. S. Cassels, An Introduction to the Geometry of Numbers, Classics in Mathematics, Reprint of the 1971 Edition, Springer, Berlin, 1997.

[12] H. Davenport and W. M. Schmidt, *Approximation to real numbers by algebraic integers*, Acta Arith. 15 (1969), 393–416.

[13] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on Diophantine approximation*, Inst. Alt. Math. Symp. Math. 4 (1970), 113–132.

[14] H. Davenport and W. M. Schmidt, *Dirichlet's theorem on Diophantine approximation. II*, Acta Arith. 16 (1970), 413–424.

[15] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, 5th. edition, Clarendon Press, 1979.

[16] V. Jarník, *Über die simultanen Diophantische Approximationen*, Math. Z. 33 (1931), 505–543.

[17] A. Khintchine, Continued Fractions, University of Chicago Press, 1964.

[18] J. F. Koksma, Diophantische Approximationen, Ergebnisse d. Math. u. ihrer Grenzgebiete, vol. 4, Springer, 1936.

[19] C. A. Rogers, Hausdorff Measures, Cambridge University Press, Cambridge, 1970.

[20] W. M. Schmidt, Diophantine Approximation, Lecture Notes in Math. 785, Springer, Berlin, 1980.

[21] T. Schneider, Introduction aux nombres transcendants, Paris, Gauthier-Villars, 1959.

[22] V. G. Sprindžuk, Mahler's problem in metric number theory, Amer. Math. Soc., Transl. math. monogr. 25, Providence, R. I., 1969.

[23] O. Teulié, *Approximation d'un nombre réel par des unités algébriques*, Monatsh. Math. 132 (2001), 169–176.

[24] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades*, J. reine angew. Math. 206 (1961), 67–77.

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