On the Littlewood conjecture in fields of power series

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Abstract. *Let* $\mathbf{k}$ *be an arbitrary field. For any fixed badly approximable power series* Θ *in* $\mathbf{k}((X^{-1}))$, *we give an explicit construction of continuum many badly approximable power series* Φ *for which the pair* (Θ, Φ) *satisfies the Littlewood conjecture. We further discuss the Littlewood conjecture for pairs of algebraic power series.*

1. Introduction

A famous problem in simultaneous Diophantine approximation is the Littlewood conjecture [17]. It claims that, for any given pair (α, β) of real numbers, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{1.1}$$

where $\| \cdot \|$ denotes the distance to the nearest integer. Despite some recent remarkable progress [24,12], this remains an open problem.

The present Note is devoted to the analogous question in fields of power series. Given an arbitrary field $\mathbf{k}$ and an indeterminate X, we define a norm $| \cdot |$ on the field $\mathbf{k}((X^{-1}))$ by setting $|0| = 0$ and, for any non-zero power series $F = F(X) = \sum_{h=-m}^{+\infty} f_h X^{-h}$ with $f_{-m} \neq 0$, by setting $|F| = 2^m$. We write $||F||$ to denote the norm of the fractional part of F, that is, of the part of the series which comprises only the negative powers of X. In analogy with (1.1), we ask whether we have

$$\inf_{q \in \mathbf{k}[X] \setminus \{0\}} |q| \cdot \|q\Theta\| \cdot \|q\Phi\| = 0 \tag{1.2}$$

for any given Θ and Φ in $\mathbf{k}((X^{-1}))$. A negative answer to this question has been obtained by Davenport and Lewis [11] (see also [3,6,9,10,13] for explicit counter-examples) when the field $\mathbf{k}$ is infinite. As far as we are aware, the problem is still unsolved when $\mathbf{k}$ is a finite field (the papers by Armitage [2], dealing with finite fields of characteristic greater than or equal to 5, are erroneous, as kindly pointed out to us by Bernard de Mathan).

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A first natural question regarding this problem can be stated as follows:

Question 1. Given a badly approximable power series Θ, does there exist a power series Φ such that the pair (Θ, Φ) satisfies *non-trivially* the Littlewood conjecture?

First, we need to explain what is meant by *non-trivially* and why we restrict our attention to badly approximable power series, that is, to power series from the set

$$\boldsymbol{Bad} = \big\{\Theta \in \mathbf{k}((X^{-1})) : \inf_{q \in \mathbf{k}[X]\setminus\{0\}} |q| \cdot \|q\Theta\| > 0\big\}.$$

Obviously, (1.2) holds as soon as Θ or Φ does not belong to $\boldsymbol{Bad}$. This is also the case when 1, Θ and Φ are linearly dependent over $\mathbf{k}[X]$. Hence, by *non-trivially*, we simply mean that both of these cases are excluded.

In the present paper, we answer positively Question 1 by using the constructive approach developed in [1]. Our method rests on the basic theory of continued fractions and works without any restriction on the field $\mathbf{k}$. Actually, our result is much more precise and motivates the investigation of a stronger question, introduced and discussed in Section 2. Section 3 is concerned with the Littlewood conjecture for pairs of algebraic power series. When $\mathbf{k}$ is a finite field, we provide several examples of such pairs for which (1.2) holds. In particular, we show that there exist infinitely many pairs of quartic power series in $\mathbf{F}_3((X^{-1}))$ that satisfy non-trivially the Littlewood conjecture. It seems that no such pair was previously known. The proofs are postponed to Sections 5 and 6, after some preliminaries on continued fractions gathered in Section 4.

2. Main results

The real analogue of Question 1 was answered positively by Pollington and Velani [24] by using metric theory of Diophantine approximation, as a consequence of a much stronger statement. Some years later, Einsiedler, Katok and Lindenstrauss [12] proved the outstanding result that the set of pairs of real numbers for which the Littlewood conjecture does not hold has Hausdorff dimension zero. Obviously, this implies a positive answer to Question 1. However, it is unclear that either of these methods could be transposed in the power series case. Furthermore, both methods are not constructive, in the sense that they do not yield explicit examples of pairs of real numbers satisfying (1.1).

A new, explicit and elementary approach to solve the real analogue of Question 1 is developed in [1]. It heavily rests on the theory of continued fractions and it can be quite naturally adapted to the function field case. Actually, our Theorem 1 answers a strong form of Question 1.

Theorem 1. *Let φ be a positive, non-increasing function defined on the set of positive integers and with $\varphi(1) = 1$ and $\lim_{d\to+\infty} \varphi(d) = 0$. Given Θ in $\boldsymbol{Bad}$, there is an uncountable subset $B_\varphi(\Theta)$ of $\boldsymbol{Bad}$ such that, for any Φ in $B_\varphi(\Theta)$, the power series 1, Θ, Φ are linearly independent over $\mathbf{k}[X]$ and there exist polynomials q in $\mathbf{k}[X]$ with arbitrarily large degree and satisfying*

$$|q|^2 \cdot \|q\Theta\| \cdot \|q\Phi\| \le \frac{1}{\varphi(|q|)}\cdot \tag{2.1}$$

In particular, if

$$\lim_{\deg q \to +\infty} |q|\varphi(|q|) = +\infty,$$

the Littlewood conjecture holds non-trivially for the pair (Θ, Φ) *for any* Φ *in* $B_\varphi(\Theta)$. *Furthermore, the set* $B_\varphi(\Theta)$ *can be effectively constructed.*

Although the proof closely follows that of Theorem 1 from [1], we give it with full detail. Actually, some steps are even slightly easier than in the real case.

Observe that, for any given Θ and Φ in $\boldsymbol{Bad}$, there exists a positive constant $c(\Theta, \Phi)$ such that

$$|q|^2 \cdot \|q\Theta\| \cdot \|q\Phi\| \geq c(\Theta, \Phi)$$

holds for any non-zero polynomial q in $\mathbf{k}[X]$. In view of this and of Theorem 1, we propose the following problem in which we ask whether the above inequality is best possible.

Question 2. Given a power series Θ in $\boldsymbol{Bad}$, does there exist a power series Φ such that the pair (Θ, Φ) satisfies *non-trivially* the Littlewood conjecture and such that we moreover have

$$\liminf_{\deg q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot \|q\Phi\| < +\infty \ ? \tag{2.2}$$

The restriction 'non-trivially' in the statement of Question 2 is needed, since (2.2) clearly holds when the power series $1, \Theta, \Phi$ are linearly dependent over $\mathbf{k}[X]$. There are, however, non-trivial examples for which (2.2) holds. Indeed, if the continued fraction expansion of a power series Θ begins with infinitely many palindromes and if $\Phi = 1/\Theta$, then (2.2) is true for the pair (Θ, Φ). This can be seen by working out in the power series case the arguments from Section 4 of [1].

Theorem 2. *Let* Θ *be an element of the field* $\mathbf{k}((X^{-1}))$ *whose continued fraction expansion begins with infinitely many palindromes. Then, the Littlewood conjecture is true for the pair* (Θ, Θ^{-1}) *and, furthermore, we have*

$$\liminf_{\deg q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot \|q\Theta^{-1}\| \leq 1.$$

Moreover, if $\mathbf{k}$ *has characteristic zero, then* Θ *is transcendental over* $\mathbf{k}(X)$.

We can weaken the assumption that the continued fraction expansion of Θ begins with infinitely many palindromes to get additional examples of pairs (Θ, Θ^{-1}) that satisfy the Littlewood conjecture. Before stating our next result, we need to introduce some notation. It is convenient to use the terminology from combinatorics on words. We identify any finite or infinite word $W = w_1 w_2 \ldots$ on the alphabet $\mathbf{k}[X] \setminus \mathbf{k}$ with the sequence of partial quotients $w_1, w_2, \ldots$ Further, if $U = u_1 \ldots u_m$ and $V = v_1 v_2 \ldots$ are words on $\mathbf{k}[X] \setminus \mathbf{k}$, with V finite or infinite, and if u_0 is in $\mathbf{k}[X]$, then $[u_0, U, V]$ denotes the continued fraction $[u_0, u_1, \ldots, u_m, v_1, v_2, \ldots]$. The mirror image of any finite word $W = w_1 \ldots w_m$ is denoted by $\overline{W} := w_m \ldots w_1$. Recall that a palindrome is a finite word W such that $\overline{W} = W$. Furthermore, we denote by $|W|$ the number of letters composing W (here, we clearly have $|W| = m$). There should not be any confusion between $|W|$ and the norm $|F|$ of the power series F.

Theorem 3. *Let Θ be in **Bad** such that $|\Theta| \neq 1$. Denote by $(p_n/q_n)_{n\geq 1}$ the sequence of its convergents. Assume that there exist a positive real number x, a sequence of finite words $(U_k)_{k\geq 1}$, and a sequence of palindromes $(V_k)_{k\geq 1}$ such that, for every $k \geq 1$, the continued fraction expansion of Θ is equal to $[U_k, V_k \ldots]$ and $|V_{k+1}| > |V_k| \geq x|U_k|$. Set further*

$$M = \limsup_{\ell\to+\infty} \frac{\deg q_\ell}{\ell} \quad \textit{and} \quad m = \liminf_{\ell\to+\infty} \frac{\deg q_\ell}{\ell}.$$

If we have

$$x > 3 \cdot \frac{M}{m} - 1, \tag{2.3}$$

then the Littlewood conjecture is true for the pair (Θ, Θ^{-1}). Moreover, if $\mathbf{k}$ has characteristic zero, then Θ is transcendental over $\mathbf{k}(X)$.

From now on, we make use of the following notation: if ℓ is a positive integer, then $W^{[\ell]}$ denotes the word obtained by concatenation of ℓ copies of the word W.

Theorem 4. *Let $\Theta = [a_0, a_1, a_2, \ldots]$ be in **Bad**. Denote by $(p_n/q_n)_{n\geq 1}$ the sequence of its convergents. Assume that there exist a finite word V, a sequence of finite words $(U_k)_{k\geq 1}$, an increasing sequence of positive integers $(n_k)_{k\geq 1}$ and a positive real number x such that, for every $k \geq 1$, the continued fraction expansion of Θ is equal to $[U_k, V^{[n_k]} \ldots]$ and $|V^{[n_k]}| \geq x|U_k|$. Let Φ be the quadratic power series defined by*

$$\Phi := [\overline{V}, \overline{V}, \overline{V}, \ldots].$$

Set further

$$M = \limsup_{\ell\to+\infty} \deg a_\ell \quad \textit{and} \quad m = \liminf_{\ell\to+\infty} \deg a_\ell.$$

If we have

$$x > \frac{M}{m}, \tag{2.4}$$

then the pair (Θ, Φ) satisfies the Littlewood conjecture. Moreover, if $\mathbf{k}$ has characteristic zero, then Θ is transcendental over $\mathbf{k}(X)$.

The last assertion of Theorems 2, 3 and 4 follows from the analogue of the Schmidt Subspace Theorem in fields of power series over a field of characteristic zero, worked out by Ratliff [25]. It is well-known that the analogue of the Roth theorem (and, *a fortiori*, the analogue of the Schmidt Subspace Theorem) does not hold for fields of power series over a finite field. For $\mathbf{k} = \mathbf{F}_p$, a celebrated example given by Mahler [18] is recalled in Section 3.

Theorems 2, 3 and 4 will be used in the next section to provide new examples of pairs of algebraic power series satisfying the Littlewood conjecture.

3. On the Littlewood conjecture for pairs of algebraic power series

It is of particular interest to determine whether the Littlewood conjecture holds for pairs of algebraic real numbers. To the best of our knowledge, only two results are known in this direction. First, if (α, β) is a pair of real numbers lying in a same quadratic field, then 1, α and β are linearly dependent over $\mathbf{Q}$ and the Littlewood conjecture is thus easily satisfied. This was for instance remarked in [7]. The other result is due to Cassels and Swinnerton-Dyer [8] who proved that the Littlewood conjecture is satisfied for pairs of real numbers lying in a same cubic field. However, it is generally believed that no algebraic number of degree greater than or equal to 3 is badly approximable. At present, no pair of algebraic numbers is known to satisfy non-trivially the Littlewood conjecture.

In this Section, we discuss whether the (function field analogue of the) Littlewood conjecture holds for pairs of algebraic power series defined over a finite field $\mathbf{k}$. Our knowledge is slightly better than in the real case, especially thanks to works of Baum and Sweet [4] and of de Mathan [19,20,21,22] that we recall below.

First, we observe that, as in the real case, (1.2) holds when Θ and Φ are in a same quadratic extension of $\mathbf{k}[X]$, since 1, Θ and Φ are then linearly dependent over $\mathbf{k}[X]$. We further observe that the existence of the Frobenius automorphism (that is, the p-th power map) yield many examples of well-approximated algebraic power series. For instance, for any prime number p, the power series $\Theta_p = [0, X, X^p, X^{p^2}, X^{p^3}, \ldots]$ is a root in $\mathbf{F}_p((X^{-1}))$ of the polynomial $Z^{p+1} + XZ - 1$, and Θ_p is well-approximated by rational functions. Indeed, there exist infinitely many rational functions p_n/q_n such that

$$\left| \Theta_p - \frac{p_n}{q_n} \right| \leq \frac{1}{|q_n|^{p+1}}.$$

Clearly, for any (algebraic or transcendental) power series Φ in $\mathbf{F}_p((X^{-1}))$, the Littlewood conjecture holds for the pair (Θ_p, Φ).

On the other hand, there are several results on pairs of algebraic functions that satisfy non-trivially the Littlewood conjecture. De Mathan [21] established that (1.2) holds for any pair (Θ, Φ) of quadratic elements when $\mathbf{k}$ is any finite field of characteristic 2 (see also [19,20] for results when $\mathbf{k}$ is any finite field). Furthermore, he proved in [22] the analogue of the Cassels and Swinnerton-Dyer theorem when $\mathbf{k}$ is a finite field. We stress that, when $\mathbf{k}$ is finite, there do exist, unlike in the real case, algebraic power series in ***Bad*** that are of degree greater than or equal to 3 over $\mathbf{k}(X)$. The first example was given by Baum and Sweet [4] who proved that, for $\mathbf{k} = \mathbf{F}_2$, the unique Θ in $\mathbf{F}_2((X^{-1}))$ which satisfies $X\Theta^3 + \Theta + X = 0$ is in ***Bad***. Thus, it follows from [22] that the pair of algebraic power series (Θ, Θ^{-1}) satifies non-trivially the Littlewood conjecture.

Further examples of badly approximable algebraic power series were subsequently found by several authors. It turns out that many of these examples contain some symmetric patterns in their continued fraction expansion. In the sequel of this Section, this property is used in order to apply Theorems 2, 3 and 4 to provide new examples of pairs of algebraic power series satisfying non-trivially the Littlewood conjecture. These examples also illustrate the well-known fact that there is no analogue to the Schmidt Subspace Theorem for power series over finite fields.

We keep on using the terminology from combinatorics on words. For sake of readability we sometimes write commas to separate the letters of the words we consider.

3.1. A first example of a badly approximable quartic in $\mathbf{F}_3((X^{-1}))$

Mills and Robbins [23] established that the polynomial

$$X(X+2)Z^4 - (X^3 + 2X^2 + 2X + 2)Z^3 + Z - X - 1$$

has a root Θ in $\mathbf{F}_3((X^{-1}))$ whose continued fraction expansion is expressed as follows. For every positive integer n, set

$$H_n = X^{[3^n-2]}, X + \varepsilon, 2X + \varepsilon, (2X)^{[3^n-2]}, 2X + \varepsilon, X + \varepsilon,$$

where $\varepsilon = 2$ if n is odd and $\varepsilon = 1$ otherwise. Then, the continued fraction expansion of the quartic power series Θ is given by

$$\Theta = [X, 2X + 2, X + 1, H_1, H_2, H_3, \ldots].$$

It turns out that the continued fraction expansion of Θ contains some symmetric patterns that we can use to apply Theorem 3. This gives rise to the following result.

Theorem 5. *The pair (Θ, Θ^{-1}) satisfies the Littlewood conjecture. In particular, there exists a pair of quartic power series in $\mathbf{F}_3((X^{-1}))$ satisfying non-trivially the Littlewood conjecture.*

To our knowledge, this is the first known example of a pair of algebraic power series of degree greater than 3 for which the Littlewood conjecture is non-trivially satisfied.

Proof. For every integer $n \geq 2$, set

$$U_n := X, 2X + 2, X + 1, H_1, H_2, H_3, \ldots, H_{n-1}$$

and

$$V_n := H_n, X^{[3^n-2]}.$$

Since $X^{[3^n-2]}$ is a prefix of H_{n+1}, the continued fraction expansion of Θ is equal to $[U_n V_n \ldots]$. Furthermore, V_n is a palindrome and the length of H_n (resp. of U_n, of V_n) is equal to $2 \cdot 3^n$ (resp. to 3^n, to $3^{n+1} - 2$). In particular, we have $|V_n| > 2.5|U_n|$ for every $n \geq 2$, and, since all the partial quotients of Θ are linear, the assumption (2.3) is satisfied. We apply Theorem 3 to complete the proof. $\square$

3.2. An infinite family of badly approximable quartics in $\mathbf{F}_3((X^{-1}))$

We now consider the infinite family of badly approximable quartics in $\mathbf{F}_3((X^{-1}))$ introduced by Lasjaunias in [15]. Let k be a non-negative integer. For any non-negative integer n, set

$$u_n = (k+2)3^n - 2,$$

and define the finite word $H_n(X)$ on $\mathbf{F}_3[X] \setminus \mathbf{F}_3$ by

$$H_n(X) := (X+1)X^{[u_n]}(X+1).$$

Then, consider the power series

$$\Theta(k) := [0, H_0(X), H_1(-X), H_2(X), \ldots, H_n((-1)^n X)), \ldots]. \tag{3.1}$$

This definition obviously implies that $\Theta(k)$ is badly approximable by rational functions, since all of its partial quotients are linear. Lasjaunias [15] established that $\Theta(k)$ is a quartic power series. More precisely, if $(p_n(k)/q_n(k))_{n\geq 0}$ denotes the sequence of convergents to $\Theta(k)$, he proved that

$$q_k(k)\Theta^4(k) - p_k(k)\Theta^3(k) + q_{k+3}(k)\Theta(k) - p_{k+3}(k) = 0.$$

The description of the continued fraction expansion of $\Theta(k)$ given in (3.1) makes transparent the occurrences of some palindromic patterns. This can be used to apply Theorem 3 and yields the following result.

Theorem 6. *For any non-negative integer k, the pair $(\Theta(k), \Theta(k)^{-1})$ satisfies the Littlewood conjecture. In particular, there exist infinitely many pairs of quartic power series in $\mathbf{F}_3((X^{-1}))$ satisfying non-trivially the Littlewood conjecture.*

Proof. For any even positive integer n, set

$$U_n := H_0(X)H_1(-X)H_2(X)\ldots H_{n-2}(X)(-X+1)$$

and

$$V_n := (-X)^{[u_{n-1}]}(-X+1)(X+1)X^{[u_n]}(X+1)(-X+1)(-X)^{[u_{n-1}]}.$$

Observe that the continued fraction expansion of $\Theta(k)$ is equal to $[0, U_n V_n \ldots]$ and that

$$|U_n| = \left(\frac{k+2}{2}\right)3^{n-1} - \frac{k}{2} \quad \text{and} \quad |V_n| = 5(k+2)3^{n-1} - 2.$$

Furthermore, V_n is a palindrome. We have $|V_n| \geq 3|U_n| + 3$ for $n \geq 2$, and, since all the partial quotients of $\Theta(k)$ are linear, the assumption (2.3) is satisfied. We apply Theorem 3 to complete the proof. $\square$

3.3. Badly approximable power series in $\mathbf{F}_p((X^{-1}))$ with $p \geq 5$

Let $p \geq 5$ be a prime number. For any positive integer k, consider the polynomial f_k in $\mathbf{F}_p[X]$ defined by

$$f_k = \sum \binom{k-j}{j} X^{k-2j},$$

where the sum is over all integers j such that $0 \leq 2j \leq k$. Then, Mills and Robbins [23] showed that the polynomial of degree $p+1$

$$XZ^{p+1} - (X^2-3)Z^p + (Xf_{p-2} - 3f_{p-1})Z - f_{p-2}(X^2-3) + f_{p-1}X$$

has a root Θ_p in $\mathbf{F}_p((X^{-1}))$ with a nice continued fraction expansion. Let $V(-1) = -X, -X$ and $V(3) = X/3, 3X$ and, for $k \geq 1$, set

$$L_k(-1) = V(-1)^{[(p^k-1)/2]} \quad \text{and} \quad L_k(3) = V(3)^{[(p^k-1)/2]}.$$

Mills and Robbins proved that the continued fraction expansion of Θ_p is given by

$$\Theta_p = [X, L_0(3), -X/3, L_0(-1), X, L_1(3), -X/3, L_1(-1), X, L_2(3), -X/3, L_2(-1), \ldots],$$

where $L_0(3)$ and $L_0(-1)$ are equal to the empty word. It follows that Θ_p is badly approximable by rational functions, all of its partial quotients being linear. Moreover, Θ_p is not quadratic since its continued fraction expansion is not eventually periodic.

As a consequence of Theorem 3, we get the following result.

Theorem 7. *For any prime number $p \geq 7$, the pair $(\Theta_p, \Theta_p^{-1})$ of algebraic power series in $\mathbf{F}_p((X^{-1}))$ satisfies non-trivially the Littlewood conjecture.*

Moreover, we can apply Theorem 4 to provide pairs of algebraic power series of distinct degrees satisfying non-trivially the Littlewood conjecture. To the best of our knowledge, no such pair was previously known.

Theorem 8. *Let $p \geq 5$ be a prime number. Let Θ_p be as above. Let Φ_p be the quadratic power series in $\mathbf{F}_p((X^{-1}))$ defined by*

$$\Phi_p := [3X, X/3, 3X, X/3, 3X, X/3, 3X, \ldots].$$

Then the pair (Θ_p, Φ_p) satisfies non-trivially the Littlewood conjecture.

Proof of Theorems 7 and 8. For any even positive integer n, set

$$U_n := X, -X/3, X, L_1(3), -X/3, L_1(-1), X, L_2(3), -X/3, L_2(-1), X \ldots, L_{n-1}(-1), X$$

and

$$V_n := (X/3, 3X)^{[(p^n-3)/2]}, X/3.$$

Observe that the continued fraction expansion of Θ_p is equal to $[U_n V_n \ldots]$ and that

$$|U_n| = 1 + \left(2 \cdot \frac{p^n - 1}{p - 1}\right) \quad \text{and} \quad |V_n| = p^n - 2.$$

Furthermore, V_n is a palindrome and $|V_n| \geq 2.5|U_n|$ holds for $p \geq 7$ and $n \geq 2$. Since all the partial quotients of Θ_p are linear, the assumption (2.3) is then satisfied. We apply Theorem 3 to complete the proof of Theorem 7.

To get Theorem 8, we observe that $L_n(3)$ is the concatenation of $(p^n - 1)/2$ copies of the word $V(3)$, and we check that $|L_n(3)| \geq 1.5|U_n|$ holds for $p \geq 5$ and $n \geq 2$. Since all the partial quotients of Θ_p are linear, the assumption (2.4) is then satisfied. We then apply Theorem 4 to complete the proof of Theorem 8. □

3.4. A normally approximable quartic in $\mathbf{F}_3((X^{-1}))$

We end this Section with another quartic power series in $\mathbf{F}_3((X^{-1}))$ found by Mills and Robbins [23]. Unlike the previous examples, this quartic is not badly approximable but we will see that it has some interesting Diophantine properties.

Mills and Robbins pointed out that the polynomial

$$Z^4 + Z^2 - XZ + 1$$

has a unique root Θ in $\mathbf{F}_3((X^{-1}))$. They observed empirically that Θ has a particularly simple continued fraction expansion. Define recursively a sequence $(\Omega_n)_{n \geq 0}$ of words on the alphabet $\mathbf{F}_3[X] \setminus \mathbf{F}_3$ by setting $\Omega_0 = \varepsilon$, the empty word, $\Omega_1 = X$, and

$$\Omega_n = \Omega_{n-1}(-X)\Omega_{n-2}^{(3)}(-X)\Omega_{n-1}, \quad \text{for } n \geq 2. \tag{3.2}$$

Here, if $W = w_1 w_2 \ldots w_r = w_1, w_2, \ldots, w_r$ with $w_i \in \mathbf{F}_3[X] \setminus \mathbf{F}_3$, then $W^{(3)}$ denotes the word obtained by taking the cube of every letter of W, that is, $W^{(3)} := w_1^3, w_2^3, \ldots, w_r^3$. Set

$$\Omega_\infty = \lim_{n \to +\infty} \Omega_n. \tag{3.3}$$

Buck and Robbins [5] confirmed a conjecture of Mills and Robbins [23] asserting that the continued fraction expansion of Θ is $[0, \Omega_\infty]$ (note that their proof was later simplified by Schmidt [26], and that Lasjaunias [14] gave an alternative proof).

The quartic power series Θ does not lie in $\boldsymbol{Bad}$. Lasjaunias [14, Theorem A] proved that Θ is normally approximable (this terminology is explained in [16]) in the following sense: there exist infinitely many rational functions p/q such that

$$|\Theta - p/q| \leq |q|^{-(2+2/\sqrt{3 \deg q})},$$

while for any positive real number ε there are only finitely many rational functions p/q such that

$$|\Theta - p/q| \leq |q|^{-(2+2/\sqrt{3 \deg q}+\varepsilon)}.$$

Note that an easy induction based on (3.2) shows that for every positive integer n the word Ω_n is a palindrome. By (3.3), we thus get that the continued fraction expansion of Θ^{-1} begins with infinitely many palindromes. The following consequence of Theorem 2 and of Theorem A from [14] is worth to be pointed out.

Theorem 9. *Let Θ be the unique root in $\mathbf{F}_3((X^{-1}))$ of the polynomial Z^4+Z^2-XZ+1. Then,*

$$\inf_{q\in\mathbf{k}[X]\setminus\{0\}} |q|^2\cdot\|q\Theta\|\cdot\|q\Theta^{-1}\|<+\infty$$

and for any positive real number ε we have

$$|q|^{2+4/\sqrt{3\deg q}+\varepsilon}\cdot\|q\Theta\|\cdot\|q\Theta^{-1}\|\geq 1,$$

for any q in $\mathbf{F}_3[X]$ with $\deg q$ large enough.

4. Preliminaries on continued fraction expansions of power series

It is well-known that the continued fraction algorithm can as well be applied to power series. The partial quotients are then elements of $\mathbf{k}[X]$ of positive degree. We content ourselves to recall some basic facts, and we direct the reader to Schmidt's paper [26] and to Chapter 9 of Thakur's book [27] for more information.

Specifically, given a power series $F=F(X)$ in $\mathbf{k}((X^{-1}))$, which we assume not to be a rational function, we define inductively the sequences $(F_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ by $F_0=F$ and $F_{n+1}=1/(F_n-a_n)$, where F_n-a_n is the fractional part of F_n. Plainly, for $n\geq 1$, the a_n are polynomials of degree at least one. We then have

$$F=[a_0,a_1,a_2,\ldots]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots}}}.$$

The truncations $[a_0,a_1,a_2,\ldots,a_n]:=p_n/q_n$, with relatively prime polynomials p_n and q_n, are rational functions and are called the *convergents* to F. It is easily seen that

$$\deg q_{n+1}=\deg a_{n+1}+\deg q_n,$$

thus

$$\deg q_n=\sum_{j=1}^{n}\deg a_j. \tag{4.1}$$

Furthermore, we have

$$\deg(q_nF-p_n)=-\deg q_{n+1}<-\deg q_n,$$

that is,

$$\|q_nF\|=|q_{n+1}|^{-1}=2^{-\deg q_{n+1}}<2^{-\deg q_n}. \tag{4.2}$$

We stress that F is in ***Bad*** if and only if the degrees of the polynomials a_n are uniformly bounded. We also point out that $|\cdot|$ is an ultrametric norm, that is, $|F+G|\leq\max\{|F|,|G|\}$ holds for any F and G in $\mathbf{k}((X^{-1}))$, with equality if $|F|\neq|G|$.

We end this Section by stating three basic lemmas on continued fractions in $\mathbf{k}((X^{-1}))$.

Lemma 1. *Let* $\Theta = [a_0, a_1, a_2, \ldots]$ *be an element of* $\mathbf{k}((X^{-1}))$ *and let* $(p_n/q_n)_{n\geq 1}$ *be its convergents. Then, for any* $n \geq 2$, *we have*

$$\frac{q_{n-1}}{q_n} = [0, a_n, a_{n-1}, \ldots, a_1].$$

Proof. As in the real case, this easily follows from the recursion formula $q_{n+1} = a_{n+1}q_n + q_{n-1}$. □

Lemma 2. *Let* $\Theta = [a_0, a_1, a_2, \ldots]$ *and* $\Phi = [b_0, b_1, b_2, \ldots]$ *be two elements of* $\mathbf{k}((X^{-1}))$. *Assume that there exists a positive integer* n *such that* $a_i = b_i$ *for any* $i = 0, \ldots, n$. *We then have* $|\Theta - \Phi| \leq |q_n|^{-2}$, *where* q_n *denotes the denominator of the* n*-th convergent to* Θ.

Proof. Let p_n/q_n be the n-th convergent to Θ. By assumption, p_n/q_n is also the n-th convergent to Φ and we have

$$|\Theta - \Phi| \leq \max\{|\Theta - p_n/q_n|,\ |\Phi - p_n/q_n|\} \leq |q_n|^{-2},$$

since the norm $|\cdot|$ is ultrametric. □

Lemma 3. *Let* M *be a positive real number. Let* $\Theta = [a_0, a_1, a_2, \ldots]$ *and* $\Phi = [b_0, b_1, b_2, \ldots]$ *be two elements of* $\mathbf{k}((X^{-1}))$ *whose partial quotients are of degree at most* M. *Assume that there exists a positive integer* n *such that* $a_i = b_i$ *for any* $i = 0, \ldots, n$ *and* $a_{n+1} \neq b_{n+1}$. *Then, we have*

$$|\Theta - \Phi| \geq \frac{1}{2^{2M}|q_n|^2},$$

where q_n *denotes the denominator of the* n*-th convergent to* Θ.

Proof. Set $\Theta' = [a_{n+1}, a_{n+2}, \ldots]$ and $\Phi' = [b_{n+1}, b_{n+2}, \ldots]$. Since $a_{n+1} \neq b_{n+1}$, we have

$$|\Theta' - \Phi'| \geq 1. \tag{4.3}$$

Furthermore, since the degrees of the partial quotients of both Θ and Φ are bounded by M, we immediately obtain that

$$|\Theta'| \leq 2^M \quad \text{and} \quad |\Phi'| \leq 2^M. \tag{4.4}$$

Denote by $(p_j/q_j)_{j\geq 1}$ the sequence of convergents to Θ. Then, the theory of continued fractions gives that

$$\Theta = \frac{p_n\Theta' + p_{n-1}}{q_n\Theta' + q_{n-1}} \quad \text{and} \quad \Phi = \frac{p_n\Phi' + p_{n-1}}{q_n\Phi' + q_{n-1}},$$

since the first n-th partial quotients of Θ and Φ are assumed to be the same. We thus obtain

$$|\Theta - \Phi| = \left|\frac{p_n\Theta' + p_{n-1}}{q_n\Theta' + q_{n-1}} - \frac{p_n\Phi' + p_{n-1}}{q_n\Phi' + q_{n-1}}\right| = \left|\frac{\Theta' - \Phi'}{(q_n\Theta' + q_{n-1})(q_n\Phi' + q_{n-1})}\right| = \left|\frac{\Theta' - \Phi'}{\Theta'\Phi' q_n^2}\right|.$$

Together with (4.3) and (4.4), this yields

$$|\Theta - \Phi| \geq \frac{1}{2^{2M}|q_n|^2},$$

concluding the proof of the lemma. □

5. Proof of Theorem 1

Without any loss of generality, we may assume that $|\Theta| \leq 1/2$ and we write $\Theta = [0, a_1, a_2, \ldots, a_k, \ldots]$. Let M be an upper bound for the degrees of the polynomials a_k. We first construct inductively a rapidly increasing sequence $(n_j)_{j\geq 1}$ of positive integers. We set $n_1 = 1$ and we proceed with the inductive step. Assume that $j \geq 2$ is such that $n_1, \ldots, n_{j-1}$ have been constructed. Then, we choose n_j sufficiently large in order that

$$\varphi(2^{m_j}) \leq 2^{-2(M+2)(m_{j-1}+1)}, \tag{5.1}$$

where $m_j = n_1 + n_2 + \ldots + n_j + (j-1)$. Such a choice is always possible since φ tends to zero at infinity and since the right-hand side of (5.1) only depends on $n_1, n_2, \ldots, n_{j-1}$.

Our sequence $(n_j)_{j\geq 1}$ being now constructed, for an arbitrary sequence $\mathbf{t} = (t_j)_{j\geq 1}$ with values in $\mathbf{k}[X] \setminus \mathbf{k}$, we set

$$\begin{aligned}\Phi_{\mathbf{t}} =& [0, b_1, b_2, \ldots] \\ =& [0, a_{n_1}, \ldots, a_1, t_1, a_{n_2}, \ldots, a_1, t_2, a_{n_3}, \ldots, a_1, t_3, \ldots].\end{aligned}$$

Then, we introduce the set

$$B_\varphi(\Theta) = \left\{\Phi_{\mathbf{t}},\ \mathbf{t} \in (\mathbf{k}_{M+1}[X] \cup \mathbf{k}_{M+2}[X])^{\mathbf{Z}_{\geq 1}}\right\},$$

where $\mathbf{k}_n[X]$ denotes the set of polynomials in $\mathbf{k}[X]$ of degree n. Clearly, the set $B_\varphi(\Theta)$ is uncountable.

Let Φ be in $B_\varphi(\Theta)$. We first prove that (2.1) holds for the pair (Θ, Φ). Denote by $(p_n/q_n)_{n\geq 1}$ (resp. by $(r_n/s_n)_{n\geq 1}$) the sequence of convergents to Θ (resp. to Φ). Let $j \geq 2$ be an integer. We infer from Lemma 1 that

$$\frac{s_{m_j-1}}{s_{m_j}} = [0, a_1, \ldots, a_{n_j}, t_{j-1}, a_1, \ldots, a_{n_{j-1}}, t_{j-2}, \ldots, t_1, a_1, \ldots, a_{n_1}].$$

By (4.2), we have

$$\|s_{m_j}\, \Phi\| \leq |s_{m_j}|^{-1}. \tag{5.2}$$

On the other hand, Lemma 2 implies that

$$\left|\Theta - \frac{s_{m_j-1}}{s_{m_j}}\right| \leq \frac{1}{|q_{n_j}|^2} = 2^{-2\deg q_{n_j}}.$$

Consequently, we get

$$\|s_{m_j}\,\Theta\| \le 2^{\deg s_{m_j} - 2\deg q_{n_j}}. \tag{5.3}$$

It follows from (4.1) that

$$\sum_{k=1}^{m_j-n_j} \deg b_k \le (M+2)(m_j - n_j) \tag{5.4}$$

and

$$\deg s_{m_j} = \sum_{k=1}^{m_j} \deg b_k = \deg q_{n_j} + \sum_{k=1}^{m_j-n_j} \deg b_k. \tag{5.5}$$

We infer from (5.3), (5.4) and (5.5) that

$$\|s_{m_j}\Theta\| \le \frac{1}{|s_{m_j}| \cdot 2^{-2(M+2)(m_j-n_j)}}. \tag{5.6}$$

Since φ is a non-increasing function and $m_{j-1}+1 = m_j - n_j$, we deduce from (5.1) that

$$\varphi(|s_{m_j}|) \le \varphi(2^{m_j}) \le 2^{-2(M+2)(m_{j-1}+1)} = 2^{-2(M+2)(m_j-n_j)}. \tag{5.7}$$

From (5.2), (5.6) and (5.7), we thus obtain that

$$|s_{m_j}| \cdot \|s_{m_j}\,\Phi\| \cdot \|s_{m_j}\,\Theta\| \le \|s_{m_j}\,\Theta\| \le \frac{1}{|s_{m_j}| \cdot \varphi(|s_{m_j}|)}.$$

Since $j \ge 2$ is arbitrary, we have established that (2.1) holds.

It now remains to prove that 1, Θ and Φ are independent over $\mathbf{k}[X]$. Therefore, we assume that they are dependent and we aim at deriving a contradiction. Let (A, B, C) be a non-zero triple of polynomials in $\mathbf{k}[X]$ satisfying

$$A\Theta + B\Phi + C = 0.$$

Then, for any non-zero polynomial q in $\mathbf{k}[X]$, we have

$$\|qA\Theta\| = \|qB\Phi\|.$$

In particular, we get

$$\|s_{m_j}A\Theta\| = \|s_{m_j}B\Phi\| \le |B| \cdot \|s_{m_j}\Phi\| \ll |s_{m_j}|^{-1}, \tag{5.8}$$

for any $j \ge 2$. Here and below, the constants implied by $\ll$ depend (at most) on A, B, C, Θ and M, but do not depend on j.

On the other hand, we have constructed the sequence $(n_j)_{j\ge1}$ in order to guarantee that

$$|s_{m_j}\Theta - s_{m_j-1}| \le \frac{1}{|s_{m_j}| \cdot \varphi(|s_{m_j}|)}. \tag{5.9}$$

This implies that
$$\|s_{m_j}\Theta\| = |s_{m_j}\Theta - s_{m_j-1}|$$
for j jarge enough. Since by assumption the degree of $b_{m_{j-1}+1} = t_{j-1}$ is either $M+1$ or $M+2$, we have $\deg b_{m_{j-1}+1} \neq \deg a_{n_j+1}$ and in particular $b_{m_{j-1}+1} \neq a_{n_j+1}$. Consequently, Lemma 3 implies that
$$\left|\Theta - \frac{s_{m_j-1}}{s_{m_j}}\right| \geq \frac{1}{2^{2(M+2)} \cdot |q_{n_j}|^2} \gg \frac{1}{2^{2\deg q_{n_j}}},$$
thus,
$$\|s_{m_j}\Theta\| \gg 2^{\deg s_{m_j} - 2\deg q_{n_j}}. \tag{5.10}$$
Moreover, we infer from (5.5) that $\deg s_{m_j} \geq \deg q_{n_j} + m_{j-1}$. Combined with (5.10), this gives
$$|s_{m_j}| \cdot \|s_{m_j}\Theta\| \gg 2^{2m_{j-1}}. \tag{5.11}$$
For j large enough, we deduce from (5.9) that
$$|s_{m_j}A\Theta - s_{m_j-1}A| < 2^{-1},$$
thus,
$$\|s_{m_j}A\Theta\| = |s_{m_j}A\Theta - s_{m_j-1}A| = |A| \cdot \|s_{m_j}\Theta\|.$$
By (5.11), this yields
$$|s_{m_j}| \cdot \|s_{m_j}A\Theta\| \gg 2^{2m_{j-1}},$$
which contradicts (5.8). This completes the proof of Theorem 1. □

6. Proof of Theorems 2, 3 and 4

Proof of Theorem 2. Let $\Theta = [a_0, a_1, a_2, \ldots]$. The key observation for the proof of Theorem 2 is Lemma 1. Indeed, assume that the integer $n \geq 3$ is such that $a_0 a_1 \ldots a_n$ is a palindrome. In particular, the degree of a_0 is at most 1 since $a_0 = a_n$, and $1/\Theta = [0, a_0, a_1, a_2, \ldots]$. Let us denote by $(p_k/q_k)_{k\geq 1}$ the sequence of convergents to $1/\Theta$. It then follows from Lemma 1 that q_{n+1}/q_n is very close to Θ. Precisely, we have
$$\|q_n\Theta\| \leq 2^{-\deg q_n},$$
by Lemma 2. Furthermore, (4.2) asserts that
$$\|q_n\Theta^{-1}\| = 2^{-\deg q_{n+1}} < 2^{-\deg q_n}.$$
Consequently, we get
$$|q_n|^2 \cdot \|q_n\Theta\| \cdot \|q_n\Theta^{-1}\| = 2^{2\deg q_n} \cdot \|q_n\Theta\| \cdot \|q_n\Theta^{-1}\| < 1.$$
This ends the proof. □

In the proofs below we assume that $|\Theta| \le 1/2$ (if needed, replace Θ by $1/\Theta$ in Theorem 3, and Θ by $\Theta - a_0$ in Theorem 4). The constants implied by $\ll$ may depend on Θ but not on k.

Proof of Theorem 3. Assume now that Θ is in ***Bad*** and satisfies the assumption of Theorem 3. Let $k \ge 1$ be an integer and let P_k/Q_k be the last convergent to the rational number

$$\frac{P'_k}{Q'_k} := [0, U_k, V_k, \overline{U}_k].$$

Since, by assumption, V_k is a palindrome, we obtain that the word $U_k V_k \overline{U}_k$ is also a palindrome. Then, Lemma 1 implies that $P'_k = Q_k$. Setting $r_k = |U_k|$ and $s_k = |V_k|$, we infer from Lemma 2 that

$$\|Q_k \Theta\| \le 2^{\deg Q_k}\, 2^{-2 \deg q_{r_k+s_k}}. \tag{6.1}$$

Observe that by Lemmas 1 and 2 we have

$$\left| \Theta^{-1} - \frac{Q'_k}{Q_k} \right| \ll 2^{-2 \deg q_{r_k+s_k}}$$

and thus

$$\|Q_k \Theta^{-1}\| \ll 2^{\deg Q_k}\, 2^{-2 \deg q_{r_k+s_k}}. \tag{6.2}$$

Furthermore, it follows from (4.1) that

$$\deg Q_k < \deg q_{r_k} + \deg q_{r_k+s_k}.$$

Then, we get from (6.1) and (6.2) that

$$|Q_k| \cdot \|Q_k \Theta\| \cdot \|Q_k \Theta^{-1}\| \ll 2^{3 \deg q_{r_k}}\, 2^{-\deg q_{r_k+s_k}}.$$

In virtue of (2.3), this concludes the proof. □

Proof of Theorem 4. Assume now that Θ and Φ satisfy the assumption of Theorem 4. Let $k \ge 1$ be an integer and let P_k/Q_k be the last convergent to the rational number

$$\frac{P'_k}{Q'_k} := [0, U_k, V^{[n_k]}].$$

On the one hand, (4.2) gives

$$\|Q_k \Theta\| < \frac{1}{|Q_k|}.$$

On the other hand, Lemma 1 implies that

$$\frac{Q'_k}{Q_k} = [\overline{V}^{[n_k]}, \overline{U}_k].$$

Setting $r_k = |U_k|$ and $s_k = |V^{[n_k]}|$, we thus infer from Lemma 2 and (4.1) that

$$\|Q_k \Phi\| \ll 2^{M r_k - m s_k}.$$

In virtue of (2.4), this concludes the proof. □

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