Mahler's classification of numbers compared with Koksma's, II

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À Wolfgang Schmidt, pour son soixante-dixième anniversaire

1. Introduction

Mahler [7], in 1932, and Koksma [6], in 1939, introduced two related measures of the degree of approximation of a complex transcendental number ξ by algebraic numbers. Following Mahler [7], for any integer $n \geq 1$, we denote by $w_n(\xi)$ the supremum of the exponents w for which

$$0 < |P(\xi)| < \mathrm{H}(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most n. Here, $\mathrm{H}(P)$ stands for the naïve height of the polynomial $P(X)$, that is, the maximum of the absolute values of its coefficients. Following Koksma [6], for any integer $n \geq 1$, we denote by $w_n^*(\xi)$ the supremum of the exponents w for which

$$0 < |\xi - \alpha| < \mathrm{H}(\alpha)^{-w-1}$$

has infinitely many solutions in complex algebraic numbers α of degree at most n. Here, $\mathrm{H}(\alpha)$ stands for the naïve height of α, that is, the naïve height of its minimal defining polynomial over $\mathbf{Z}$. Clearly, the functions w_1 and w_1^* coincide.

For any integer $n \geq 2$ and any complex transcendental number ξ we have

$$w_n^*(\xi) \leq w_n(\xi) \leq w_n^*(\xi) + n - 1. \tag{1}$$

The first inequality in (1) is easy (see e.g. [4, Section 3.4]), and the second one is due to Wirsing [11]. Furthermore, Sprindžuk [10] established that $w_n(\xi) = w_n^*(\xi) = (n-1)/2$ holds for all $n \geq 1$ and almost all ξ (in the sense of the Lebesgue measure on the complex plane). This raises the question whether there exist complex numbers ξ such that $w_n^*(\xi) < w_n(\xi)$ for some integer $n \geq 2$. In 1976, R. C. Baker [1] gave a positive answer to this problem by proving that for any integer $n \geq 2$ the function $w_n - w_n^*$ can take any value in the interval $[0, (n-1)/n]$. This has been subsequently improved upon by Bugeaud [2], who showed that, for any integer $n \geq 3$, the set of values taken by the function $w_n - w_n^*$ contains the interval $[0, n/4]$. Like Baker's, his approach originates in two papers by Schmidt [8, 9],

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where the existence of T-numbers is established (these are transcendental numbers ξ for which $\limsup_{n\to+\infty} w_n(\xi)/n = +\infty$ and $w_n(\xi)$ is finite for any $n \ge 1$). The main novelty introduced in [2] is the use in the inductive construction of integer polynomials having two zeros very close to each other.

The above quoted results from [1] and [2] have been obtained by constructing suitable *real* numbers ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ differ. In the present paper, we are mainly interested in the approximation of *complex non-real* transcendental numbers ξ. In this case, (1) can be replaced by the sharper inequalities

$$w_n^*(\xi) \le w_n(\xi) \le w_n^*(\xi) + \frac{n-1}{2}, \tag{2}$$

see [11] or Section 9.1 of [4]. Furthermore, we have $w_n(\xi) = w_n^*(\xi)$ for $1 \le n \le 3$. Now, we may ask whether there are complex non-real numbers ξ such that $w_n^*(\xi) < w_n(\xi)$ for some integer $n \ge 4$. A positive answer has been given by Baker [1] when n is even, but his method does not seem to extend to the case of odd n. In the present paper, we show that the approach followed in [2] can be adapted to prove, for any odd integer $n \ge 5$, the existence of complex non-real transcendental numbers ξ with $w_n^*(\xi) < w_n(\xi)$. This confirms a guess made at the end of [2]. Our main tool is the construction of families of irreducible integer polynomials of odd degree having two complex non-real roots very close to each other.

We further show how to use the integer polynomials introduced in [5] to improve upon some of the results established in [2].

2. Results

We begin by pointing out a direct consequence of our main result, given in Theorem 2 below.

Theorem 1. *Let $n \ge 4$ be an integer. Then, there exist complex non-real numbers ξ such that $w_n^*(\xi) < w_n(\xi)$.*

For even values of n, Theorem 1 is due to Baker [1]. However, his method does not seem to extend (at least straightforwardly) to the case of odd values of n.

Theorem 1 is an immediate consequence of [1] and Theorem 2 below, which asserts the existence of complex non-real numbers with special properties.

Theorem 2. *Let $n \ge 5$ be an odd integer and set $F(n) = (5n^3 + 5n^2 + 5n - 3)/2$. Let w_n and w_n^* be real numbers such that*

$$\begin{aligned} w_n^* + \frac{1}{2} \le w_n \le w_n^* + \frac{n+5}{16}, \quad w_n > F(n), \ \ (n = 5, 7), \\ w_n^* + 1 - \frac{9}{2n} \le w_n \le w_n^* + \frac{n+5}{16}, \quad w_n > F(n), \ \ (n \ge 9). \end{aligned} \tag{3}$$

Then there exist complex non-real numbers ξ such that

$$w_n^*(\xi) = w_n^* \quad \text{and} \quad w_n(\xi) = w_n.$$

The main tool for establishing Theorem 2 is the construction of integer polynomials of odd degree having two complex non-real roots very close to each other, and not close to the real axis (thus, two pairs of complex non-real roots). Then, denoting by γ_j one of these roots, we construct ξ as the limit of a suitable sequence $\xi_j = (c_j + \gamma_j)/g_j$, where the c_j's and the g_j's are positive integers tending to infinity. To ensure that ξ_j is sufficiently far away from the real axis, we have to choose γ_j with a strong dependence on g_j. The integer polynomials we are using are the polynomials

$$P_{n,a,b}(X) = X\bigl(X^2 - 2X + b^2 + 1\bigr)^{(n-1)/2} + 2\bigl(a^2X^2 - 2a(a+1)X + 2a + 1 + a^2 + a^2b^2\bigr)^2$$

defined in Lemma 1 below, which have two roots very close to $1 + ib + a^{-1}$. They have been constructed by suitably modifying the polynomials $X^n - 2(aX-1)^2$ used in [2]. As far as we know, no example of integer polynomials with two complex non-real roots very close to each other appeared previously in the literature.

A result similar to Theorem 2 can be proved for even integers $n \geq 6$ by using in the inductive construction the polynomials

$$\bigl(X^2 - 2X + b^2 + 1\bigr)^{n/2} + 2\bigl(a^2X^2 - 2a(a+1)X + 2a + 1 + a^2 + a^2b^2\bigr)^2,$$

instead of the polynomials $P_{n,a,b}(X)$. However, a slightly sharper result follows by combining ideas from [1], [2] and results from [5], see Theorem 4 below.

We take the opportunity of the present paper to point out how the results from [2] can be improved by using families of polynomials introduced in [5], where Theorem A below is established.

Theorem A. *Let $n \geq 6$ and a be integers with n even. Set*

$$\tilde{P}_{n,a}(X) := (X^{n/2} - aX + 1)^2 - 2X^{n-2}\,(aX-1)^2.$$

If a is large enough, then the polynomial $\tilde{P}_{n,a}(X)$ is irreducible and has two real roots in the disc of center $a^{-1} + a^{-1-n/2}$ and of radius $3a^{-n}$.

Using Theorem A instead of Lemma 3 from [2], it is possible to improve Theorem 1 from [2] as follows.

Theorem 3. *Let $n \geq 6$ be an even integer. Let Δ be in $[1 - 1/n, n/2)$. Set $\mu := (n\Delta - n + 1)/(n - 2\Delta)$ and $G(n) = n(n+1)(n+2\mu) + 3n - 1$. Let w_n be a real number with $w_n > G(n)$. Then there exist real numbers ξ such that*

$$w_n(\xi) = w_n \quad \text{and} \quad w_n(\xi) = w_n^*(\xi) + \Delta.$$

The proof of Theorem 3 follows exactly the same lines as that of Theorem 3 from [2] combined with Section 6 of that paper. However, we work with algebraic numbers γ_j having large heights, thus we have to modify the lower bound given in inequality (6) from [2]. This is the reason why $G(n)$ in Theorem 3 above is much larger than $F(n)$ occurring in Theorem 1 from [2].

The following corollary is an immediate consequence of our Theorem 3 and Theorem 1 from [1].

Corollary 1. *For any even integer $n \geq 6$, the set of values taken by the function $w_n - w_n^*$ contains the interval* $[0, n/2)$.

Using similar ideas as in [1], it is easy to adapt the proof of [2] to construct complex non-real numbers ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ differ for some *even* integer $n \geq 6$. Indeed, we construct ξ as a limit of algebraic numbers of degree n of the form $\xi_j = (c_j + id_j + \gamma_j)/g_j$, where $i^2 = -1$, the c_j's, d_j's and g_j's are positive integers and the γ_j's are real algebraic numbers of degree $n/2$. As in [2], we can take for the γ_j's roots of the polynomials $X^{n/2} - 2(aX - 1)^2$ or $X^{n/2} - 2a^{n/2}$, for a suitable positive integer a. To improve upon the results obtained in this way when n is divisible by 4 (as Theorem 3 above improves Theorem 1 from [2]), we can take for the γ_j's roots of the polynomials $\tilde{P}_{n/2,a}(X)$. Baker [1] established that, for any even integer $n \geq 4$, the set of values taken by the function $w_n - w_n^*$ on complex non-real numbers includes the interval $[0, (n-2)/(2n)]$. Proceeding as described above, we improve upon Baker's result for $n \geq 6$.

Theorem 4. *For any even integer $n \geq 6$, the set of values taken by the function $w_n - w_n^*$ on complex non-real numbers contains the interval* $[0, n/16]$. *If, moreover, n is divisible by 4, then this set contains the interval* $[0, n/8]$.

Theorems 2 to 4 show that inequalities (1) and (2) are close to be best possible. However, we are unable to solve completely the following question.

Problem. *Let $n \geq 2$ be an integer, and let Δ be in* $[0, n-1]$. *Does there exist a real number ξ with $w_n(\xi) = w_n^*(\xi) + \Delta$? Let $n \geq 4$ be an integer, and let Δ be in* $[0, (n-1)/2]$. *Does there exist a complex non-real number ξ with $w_n(\xi) = w_n^*(\xi) + \Delta$?*

The above Problem has been (nearly completely) solved for $n = 2$ and $n = 3$. Namely, it is established in [3] that the set of values taken by the function $w_2 - w_2^*$ (resp. $w_3 - w_3^*$) contains the half-open interval $[0, 1)$ (resp. $[0, 2)$).

Presumably, it is possible to extend the range of values in (3) and to prove that, when $n \geq 6$ is congruent to 2 modulo 4, the set of values taken by $w_n - w_n^*$ on complex non-real numbers include the interval $[0, n/8]$. Indeed, by a suitable modification of the polynomials $\tilde{P}_{n,a}(X)$ defined in Theorem A above, we can construct families of integer polynomials of even degree having two complex non-real roots very close to each other, even closer than in Lemma 1 below. However, we failed to prove that these polynomials are irreducible.

It is apparent from [1, 2] and the present work that Schmidt's construction offers much flexibility to confirm the existence of transcendental numbers with special properties of approximation. In this respect, there remain many interesting unanswered problems, including the following one. Recall that, for almost all real numbers ξ, the quality of approximation to ξ by algebraic numbers of degree at most n is the same as the quality of approximation to ξ by algebraic *integers* of degree at most $n+1$. Is it possible to construct real numbers which are strictly better approximable by algebraic integers of degree $n+1$ than by algebraic numbers of degree at most n?

The sequel of the paper is organized as follows. In Section 3, we establish the key lemma. Afterwards, in Section 4, we formulate Theorem 5. Its proof is given in the same

Section. In Section 5 we derive Theorem 2 from Theorem 5. Finally, Section 6 is devoted to the proof of Theorem 3, whereas Theorem 4 is established in Section 7.

3. An auxiliary result

The key ingredient for the proof of Theorem 2 is the following lemma, which asserts the existence of irreducible, monic integer polynomials having two (pairs of) complex roots very close to each other.

Lemma 1. *Let $n \geq 5$ be an odd integer. Let a and b be positive integers with $a \geq b/10$ and b odd. If a is sufficiently large, then the polynomial*

$$P_{n,a,b}(X) := X\big(X^2 - 2X + b^2 + 1\big)^{(n-1)/2} + 2\big(a^2X^2 - 2a(a+1)X + 2a + 1 + a^2 + a^2b^2\big)^2$$

is irreducible and has two complex non-real roots very close to each other, namely

$$\delta^+(n,a,b) := 1 + ib + a^{-1} + \frac{\sqrt{1+ib}}{2\sqrt{2}}\,(1+i)^{(n-1)/2}\,a^{-(n+7)/4}\,b^{(n-5)/4} + \varepsilon^+(n,a,b)$$

and

$$\delta^-(n,a,b) := 1 + ib + a^{-1} - \frac{\sqrt{1+ib}}{2\sqrt{2}}\,(1+i)^{(n-1)/2}\,a^{-(n+7)/4}\,b^{(n-5)/4} + \varepsilon^-(n,a,b),$$

where $|\varepsilon^+(n,a,b)|, |\varepsilon^-(n,a,b)| \leq c_1(n)\,a^{-(n+9)/4}\,b^{(n-5)/4}$ for some constant $c_1(n)$, depending only on n. Furthermore, we have

$$\big|P'_{n,a,b}\big(\delta^+(n,a,b)\big)\big| \asymp a^{-(n-9)/4}\,b^{(n+5)/4}. \tag{4}$$

Proof : Since b is odd, the irreducibility of $P_{n,a,b}(X)$ follows from the Eisenstein criterion applied with the prime number 2. Then, we study the function $x \mapsto P_{n,a,b}(1+ib+a^{-1}+x)$ in a neighbourhood of the origin and we use the inequality $a \geq b/10$ to show that $P_{n,a,b}(X)$ has two roots which can be expressed as stated above. The estimate (4) is a straightforward calculation. $\square$

Actually, Lemma 1 holds under a slightly weaker condition than $a \geq b/10$, namely it is enough to assume that $a \geq c_2(n) b^{(n-3)/(n+3)}$, for a suitable positive constant $c_2(n)$. Taking this into consideration yields a (very) slight improvement of Theorem 2. However, to avoid additional technical difficulties, we choose to keep the weaker assumption $a \geq b/10$.

We direct the reader to our previous work [2] for the other auxiliary results used in the proof of Theorem 2.

4. The inductive construction

Theorem 5 below gives an explicit inductive construction of sequences $(\xi_j)_{j\geq 1}$ of complex non-real algebraic numbers of odd degree n. It will be proved in Section 5 that such sequences converge to complex non-real numbers having the property stated in Theorem 2. We use in Theorem 5 the same notation as in Lemma 1, namely we denote by $\delta^+(n,a,b)$ the root of the polynomial $P_{n,a,b}(X)$ defined in this lemma.

For any real numbers a, b, c, d with $a < b$ and $c < d$, the set of complex points

$$\{x + iy : a < x < b, c < y < d\}$$

will often be called the rectangle $(a, b) \times (c, d)$ and will be denoted by $(a, b) \times (c, d)$.

The norm of an algebraic number means the product of its conjugates over $\mathbf{Q}$.

Theorem 5. *Let $n \geq 5$ be an odd integer. Let μ, ν be real numbers with $1 \leq \mu \leq n-1$ and $\nu > 5$. Set $H(n) = (5n^3 + 5n^2 + 3n + 1)/2$ and let $\chi > H(n)$ be a real number. Then, there exist a positive real number $\lambda < 1/3$, prime numbers $g_1 \geq 11$, $g_2, \ldots$, positive integers c_1, $c_2, \ldots$, and positive even integers $d_1, d_2, \ldots$ such that the following conditions are satisfied. Writing $\gamma_j := \delta^+(n, [g_j^\mu], g_j + d_j)$ for $j = 1, 2, \ldots$, we have*

(I_j) $\qquad g_j$ *does not divide the norm of* $c_j + \gamma_j \qquad (j \geq 1)$.

(II_1)

$$\xi_1 = (c_1 + \gamma_1)/g_1 \in (1, 2) \times (5, 6).$$

(II_j) $\qquad \xi_j = (c_j + \gamma_j)/g_j$ *belongs to the rectangle*

$$I_{j-1} = \left(\Re e\, \xi_{j-1} + \frac{1}{2} g_{j-1}^{-\nu}, \Re e\, \xi_{j-1} + \frac{5}{8} g_{j-1}^{-\nu}\right) \times \left(\Im m\, \xi_{j-1} + \frac{1}{2} g_{j-1}^{-\nu}, \Im m\, \xi_{j-1} + \frac{5}{8} g_{j-1}^{-\nu}\right).$$

(III_1)

$$|\xi_1 - \alpha| \geq 2\lambda\, \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number $\alpha \neq \xi_1$ of degree $\leq n$.

(III_j)

$$|\xi_j - \alpha| \geq \lambda\, \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number $\alpha \notin \{\xi_1, \ldots, \xi_j\}$ of degree $\leq n$ $\qquad (j \geq 2)$.

Observe that, under the assumption of Theorem 5, we have $g_j + d_j \leq 10[g_j^\mu]$ and $g_j + d_j$ is odd for any $j \geq 1$. Thus, we can indeed apply the results established in Lemma 1. Furthermore, setting $s = 4n$, we check that there exists a positive constant $c_3(n)$ such that

$$\mathrm{H}(\gamma_j) \leq c_3(n)\, g_j^s, \qquad \text{for any } j \geq 1. \tag{5}$$

Since we will also deal in the present work with other families of algebraic numbers playing the same role as the γ_j's, it is convenient to introduce the parameter s, see Section 6 and the remark at the end of the present section.

To simplify the notation, in what follows we denote by α a complex algebraic number of degree less than or equal to n. By the definitions of $H(n)$ and of s, there exists a positive real number ε such that

$$2\chi > n(n+1)(n+s) + 3n + 1 + 5n^2 s\varepsilon. \tag{6}$$

In order to prove Theorem 5, we add three extra conditions $(IV_j), (V_j)$ and (VI_j), which should be satisfied by the numbers ξ_j. We denote by Leb the Lebesgue measure on the complex plane.

Let J_j denote the subset of I_j consisting of the complex numbers $z = x + iy \in I_j$ satisfying

$$\max\{|x - \Re e\,\alpha|, |y - \Im m\,\alpha|\} \geq 2\lambda\, \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number α of degree $\leq n$, distinct from $\xi_1, \ldots, \xi_j, z$ and of height $\mathrm{H}(\alpha)$ satisfying

$$\mathrm{H}(\alpha) \geq (\lambda g_j^{\nu})^{1/\chi}.$$

The supplementary conditions are the following.

(IV_j) $$\xi_j \in J_{j-1} \quad (j \geq 2).$$

(V_j) If $\mathrm{H}(\alpha) \leq g_j^{2/(n+1+\varepsilon)}$, then we have $|\xi_j - \alpha| \geq 1/g_j \quad (j \geq 1)$.

(VI_j) The measure of J_j satisfies $\mathrm{Leb}(J_j) \geq \mathrm{Leb}(I_j)/2 \quad (j \geq 1)$.

We construct the sequences $c_1, c_2, \ldots,$ $d_1, d_2, \ldots,$ $g_1, g_2, \ldots$ by induction. At the j-th stage, there are two distinct steps. Step (A_j) consists in building an algebraic number

$$\xi_j = \frac{c_j + \gamma_j}{g_j}$$

of degree n satisfying conditions (I_j) to (V_j). In step (B_j), we show that the number ξ_j constructed in (A_j) satisfies (VI_j) as well, provided that g_j is chosen large enough in terms of

$$n, \mu, \nu, \chi, \varepsilon, \lambda, \xi_1, \ldots, \xi_{j-1}. \tag{7}$$

The symbols o, $\gg$ and $\ll$ used throughout Steps (A_j) and (B_j) mean that the numerical implicit constants depend (at most) on the quantities (7). Furthermore, the symbol o implies 'as g_j tends to infinity'.

Step (A_1) is rather easy. Let $g_1 > \max\{11, n\}$ be a prime number. There are $\gg g_1^2$ numbers $\xi_1 = (c_1 + \gamma_1)/g_1$ in the rectangle $(1, 2) \times (5, 6)$, since for any fixed d_1 there are $\gg g_1$ suitable choices for c_1. Observe that condition (I_1) is satisfied if, and only if, g_1 does not divide $P(-c_1)$, where $P(X)$ denotes the minimal defining polynomial of γ_1. Thus, by Lemma 5 from [2], there are $\gg g_1^2$ numbers $\xi_1 = (c_1 + \gamma_1)/g_1$ in the rectangle

$(1,2)\times(5,6)$ that satisfy condition (I_1). These $\gg g_1^2$ numbers have mutual distances at least g_1^{-1} and, since there are only $o(g_1^2)$ algebraic numbers α of degree at most n satisfying $\mathrm{H}(\alpha)\le g_1^{2/(n+1+\varepsilon)}$, one can choose ξ_1 such that (V_1) is satisfied. Furthermore, we point out that there are $\gg g_1^2$ suitable choices for the pair (c_1,d_1), where the constant implicit in $\gg$ depends only on n. Further, by Lemma 2 from [2], we have

$$|\xi_1-\alpha|\ge 2\lambda\,\mathrm{H}(\alpha)^{-n},$$

with $\lambda=(n+1)^{-2(n+1)}\,\mathrm{H}(\xi_1)^{-n}/2$, for any algebraic number $\alpha\ne\xi_1$ of degree at most n. Thus (I_1), (II_1), (III_1) and (V_1) are satisfied.

Let $j\ge 2$ be an integer and assume that $c_1,\dots,c_{j-1}$, $d_1,\dots,d_{j-1}$, $g_1,\dots,g_{j-1}$ have been constructed. Step (A_j) is much harder to verify, since we have no control on the set J_{j-1}. Thus, it seems difficult to check that the condition (IV_j) holds. To overcome this problem, we follow Schmidt's argument [9], also used by Baker [1]. We set $\xi_j=(c_j+\gamma_j)/g_j$ for some positive integers c_j, d_j (recall that the definition of γ_j requires an integer d_j) and $g_j>8g_{j-1}$ and we introduce the set J'_{j-1} formed by the complex numbers $z=x+iy\in I_{j-1}$ satisfying

$$\max\{|x-\Re e\,\alpha|,|y-\Im m\,\alpha|\}\ge 2\lambda\,\mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number α of degree $\le n$, distinct from $\xi_1,\dots,\xi_j,z$, and whose height $\mathrm{H}(\alpha)$ satisfies the inequalities

$$(\lambda\,g_{j-1}^{\nu})^{1/\chi}\le \mathrm{H}(\alpha)\le (c_4(n)\,g_j^{n(n+s)})^{1/(\chi-n)},\tag{8}$$

for a suitable constant $c_4(n)$ which will be defined just after inequality (13). Since, by (6), we have

$$\chi-n>n(n+1)(n+s)/2,\tag{9}$$

the exponent of g_j in the right member of (8) is strictly less than $2/(n+1)$. Thus, there are $o(g_j^2)$ algebraic numbers α's satisfying (8), and we observe that, unlike for J_{j-1}, the complement in I_{j-1} of the set J'_{j-1} is a finite union of very small rectangles, and, more precisely, a union of $o(g_j^2)$ rectangles.

We will prove that for g_j large enough we have $\gg g_j^2$ choices for the pair (c_j,d_j) in order that conditions (I_j) to (V_j) are fulfilled. We stress that if $\xi'_j=(c'_j+\gamma'_j)/g_j$ for some positive integers c'_j and d'_j, then we have

$$\xi'_j-\xi_j=\frac{c'_j-c_j}{g_j}+i\,\frac{d'_j-d_j}{g_j}+O\big(g_j^{-7/2}\big),\tag{10}$$

by Lemma 1.

Let α be an algebraic number of degree $\le n$. Since $g_j+d_j\le 10g_j$, we infer from (5) and Lemmas 2 and 4 from [2] that there exist positive constants $c_5(n)$ and $c_6(n)$ such that

$$|\xi_j-\alpha|\ge c_5(n)\,\mathrm{H}(\xi_j)^{-n}\,\mathrm{H}(\alpha)^{-n}\ge c_6(n)\,g_j^{-n(n+s)}\,\mathrm{H}(\alpha)^{-n}.\tag{11}$$

In particular, using that $2\sqrt{2}\lambda < 1$, we have

$$|\xi_j - \alpha| \geq 2\sqrt{2}\lambda \, \mathrm{H}(\alpha)^{-\chi} \tag{12}$$

as soon as

$$\mathrm{H}(\alpha)^{\chi - n} \geq c_6(n)^{-1} \, g_j^{n(n+s)}. \tag{13}$$

We take $c_4(n) = c_6(n)^{-1}$.

By (VI_{j-1}) and $J'_{j-1} \supset J_{j-1}$, we have $\mathrm{Leb}(J'_{j-1}) \gg 1$. Since the complement in I_{j-1} of the set J'_{j-1} is the union of $o(g_j^2)$ rectangles, if g_j is a sufficiently large prime number, then, using (10) and Lemma 5 from [2] as in step (A_1), we get that there exist $\gg g_j^2$ numbers $\xi_j = (c_j + \gamma_j)/g_j$ in J'_{j-1} such that (I_j) is satisfied. Such ξ_j's also belong to J_{j-1}, since (13) implies (12), and condition (IV_j) is satisfied.

Thus, we are left with $\gg g_j^2$ suitable algebraic numbers ξ_j, mutually distant by at least g_j^{-1}, as follows from (10). Only $o(g_j^2)$ algebraic numbers α of degree at most n satisfy

$$\mathrm{H}(\alpha) \leq g_j^{2/(n+1+\varepsilon)}, \tag{14}$$

thus one can choose ξ_j in such a way that $|\xi_j - \alpha| \geq 1/g_j$ for the numbers α satisfying (14). Consequently, there are $\gg g_j^2$ algebraic numbers ξ_j satisfying (I_j), (II_j), (IV_j) and (V_j).

To prove that such a ξ_j also satisfies (III_j), we argue exactly as in [2]. We omit the details. Thus, the proof of step (A_j) is completed.

Let $j \geq 1$ be an integer. For the proof of step (B_j), we first establish that if g_j is large enough and if z lies in I_j, then we have

$$|z - \alpha| \geq 2\sqrt{2}\lambda \mathrm{H}(\alpha)^{-\chi} \tag{15}$$

for any algebraic number $\alpha \neq \xi_j$ such that

$$(\lambda g_j^{\nu})^{1/\chi} \leq \mathrm{H}(\alpha) \leq g_j^{\nu/(\chi - (n+1)/2 - \varepsilon)}. \tag{16}$$

Let then $\alpha \neq \xi_j$ be an algebraic number of degree $\leq n$ satisfying (16) and let $z = x + iy$ be in I_j, that is, such that

$$\begin{aligned} \frac{1}{2} \, g_j^{-\nu} &< x - \Re e \, \xi_j < \frac{5}{8} \, g_j^{-\nu} \\ \frac{1}{2} \, g_j^{-\nu} &< y - \Im m \, \xi_j < \frac{5}{8} \, g_j^{-\nu}. \end{aligned} \tag{17}$$

If $g_j^{\nu/(\chi-(n+1)/2-\varepsilon)} \leq g_j^{2/(n+1+\varepsilon)}$, then $\mathrm{H}(\alpha) \leq g_j^{2/(n+1+\varepsilon)}$ and it follows from (V_j), (16) and (17) that

$$\begin{aligned} |z - \alpha| &\geq |\xi_j - \alpha| - |\xi_j - z| \\ &\geq g_j^{-1} - g_j^{-\nu} \geq 2\sqrt{2} g_j^{-\nu} \geq 2\sqrt{2}\lambda \mathrm{H}(\alpha)^{-\chi}. \end{aligned} \tag{18}$$

Otherwise, we have

$$g_j^{\nu/(\chi-(n+1)/2-\varepsilon)} > g_j^{2/(n+1+\varepsilon)}, \tag{19}$$

and, by (11), we get

$$\begin{aligned} |z-\alpha| &\ge |\xi_j-\alpha| - |\xi_j - z| \\ &\ge c_6(n)\, g_j^{-n(n+s)}\, \mathrm{H}(\alpha)^{-n} - g_j^{-\nu} \\ &\ge c_6(n)\, g_j^{-n(n+s)}\, \mathrm{H}(\alpha)^{-n}/2, \end{aligned} \tag{20}$$

To check the last inequality, we have to verify that

$$2g_j^{-\nu} \le c_6(n)\, g_j^{-n(n+s)}\, \mathrm{H}(\alpha)^{-n}. \tag{21}$$

In view of (16), inequality (21) is true as soon as

$$2g_j^{n\nu/(\chi-(n+1)/2-\varepsilon)} \le c_6(n)\, g_j^{\nu}\, g_j^{-n(n+s)},$$

which, by (19), holds for g_j large enough when

$$\frac{n}{\chi-(n+1)/2-\varepsilon} < 1 - \frac{n(n+1+\varepsilon)(n+s)}{2\chi-n-1-2\varepsilon}, \tag{22}$$

in particular when χ satisfies (6).

Moreover, we have

$$c_6(n)\, g_j^{-n(n+s)}\, \mathrm{H}(\alpha)^{-n} \ge 4\sqrt{2}\lambda \mathrm{H}(\alpha)^{-\chi}. \tag{23}$$

Indeed, by (16), $\lambda < 1$ and (19), we get

$$\begin{aligned} \mathrm{H}(\alpha)^{\chi-n} &\ge (\lambda g_j^{\nu})^{(\chi-n)/\chi} \\ &\ge \lambda\, g_j^{(\chi-n)(2\chi-n-1-2\varepsilon)/(\chi n+\chi+\chi\varepsilon)} \\ &\ge 4\sqrt{2}\lambda\, c_6(n)^{-1}\, g_j^{n(n+s)}, \end{aligned}$$

since we infer from (6) that

$$(\chi-n)(2\chi-n-1-2\varepsilon) > \chi\, n\, (n+1+\varepsilon)\, (n+s). \tag{24}$$

Combining (20) and (23), we have checked that we have

$$|z-\alpha| \ge 2\sqrt{2}\lambda \mathrm{H}(\alpha)^{-\chi},$$

when (19) holds; hence, by (18), (15) is true if $\alpha \ne \xi_j$ satisfies (16). Consequently, if g_j is large enough, then the complement J_j^c of J_j in I_j is contained in the union of the rectangles

$$\big(\Re e\, \alpha - 2\lambda \mathrm{H}(\alpha)^{-\chi}, \Re e\, \alpha + 2\lambda \mathrm{H}(\alpha)^{-\chi}\big) \times \big(\Im m\, \alpha - 2\lambda \mathrm{H}(\alpha)^{-\chi}, \Im m\, \alpha + 2\lambda \mathrm{H}(\alpha)^{-\chi}\big),$$

where α runs over the set of algebraic numbers of degree $\le n$ and with height greater than $g_j^{\nu/(\chi-(n+1)/2-\varepsilon)}$. The Lebesgue measure of J_j^c is then

$$\ll \sum_{H > g_j^{\nu/(\chi-(n+1)/2-\varepsilon)}} H^{n-2\chi} = o(g_j^{-2\nu}) = o(\mathrm{Leb}(I_j)).$$

Thus, we conclude that we can find g_j large enough such that $\mathrm{Leb}(J_j) \ge \mathrm{Leb}(I_j)/2$. This completes step (B_j) as well as the proof of Theorem 5. □

Remark. Observe that the size of the function $n \mapsto H(n)$ occurring in the statement of Theorem 5 is implied by the conditions (9), (22), and (24), the most constraining one being (22).

Likewise, we can rework the proof of Theorem 3 from [2] using the upper bound $\mathrm{H}(\gamma_j) \le c_3(n)\, g_j^s$. Then, modifying accordingly the inequalities displayed on p. 97, l.-5, on p.99, l.-4, and on p.100, l.5 from [2], we see that χ has to satisfy simultaneously

$$\begin{aligned} \chi - n &> n(n+1)(n+s), \\ n &< \chi - n - 1 - \varepsilon - n(n+s)(n+1+\varepsilon), \\ (\chi-n)(\chi-n-1-\varepsilon) &> \chi n(n+s)(n+1+\varepsilon). \end{aligned}$$

The most constraining inequality is the second one, which gives

$$\chi > n(n+s)(n+1) + 2n + 1 \tag{25}$$

when we omit ε. This observation will be used in Section 6.

5. Completion of the proof of Theorem 2

For $n = 5$ or 7, let Δ be in $[7/16, (n+5)/16]$ and set

$$\mu = \frac{n + 16\Delta - 7}{n - 16\Delta + 7}.$$

Observe that μ is in $[1, n-1]$. For $n \ge 9$, let Δ be in $[1 - 9/(2n), (n+5)/16]$. If $\Delta \in [1 - 9/(2n), (n-1)/16]$, then we set

$$\mu = \frac{4n\Delta - 3n + 9}{n - 9},$$

and we observe that μ is in $[1, (n-4)/4]$. If $\Delta \in [(n-1)/16, (n+5)/16]$, then we set

$$\mu = \frac{n + 16\Delta - 7}{n - 16\Delta + 7},$$

and we observe that μ is in $[(n-4)/4, n-1]$. Let $w_n > (5n^3 + 5n^2 + 5n - 3)/2$ and set $w_n^* = w_n - \Delta$. Set $\nu = n(w_n^* + 1)$ and $\chi = w_n - n + 2$, in such a way that $\chi >$

$(5n^3+5n^2+3n+1)/2$. The sequence $(\xi_j)_{j\geq 1}$ obtained by applying Theorem 5 with these parameters is a Cauchy sequence, thus it converges towards a complex number denoted by ξ. Our choice for the γ_j's implies that ξ is non-real.

We write $A \ll B$ if there exists a constant $c(n)$, depending only on n, such that $|A| < c(n)B$, and we write $A \asymp B$ if we have both $A \ll B$ and $B \ll A$.

By the definition of γ_j, the minimal defining polynomial of ξ_j is

$$Q_j(X) := (g_jX - c_j)\big((g_jX-c_j)^2 - 2(g_jX-c_j) + (g_j+d_j)^2 + 1\big)^{(n-1)/2} +$$
$$+2\big([g_j^\mu]^2(g_jX-c_j)^2 - 2[g_j^\mu]([g_j^\mu]+1)(g_jX-c_j) + 2[g_j^\mu] + 1 + [g_j^\mu]^2 + [g_j^\mu]^2(g_j+d_j)^2\big)^2.$$

This polynomial is indeed irreducible and primitive by (I_j) and the first statement of Lemma 1.

Furthermore, for any $j \geq 1$ we have

$$g_j^{-\nu}/2 \leq |\xi - \xi_j| \leq 2\, g_j^{-\nu},$$

and we deduce that

$$|\xi - \xi_j| \asymp \mathrm{H}(\xi_j)^{-\nu/n} \asymp \mathrm{H}(\xi_j)^{-w_n^*-1}. \tag{26}$$

Further, if α is of degree $\leq n$ and is not equal to one of the ξ_j's, then we have

$$|\xi - \alpha| \geq \lambda \mathrm{H}(\alpha)^{-\chi},$$

whence

$$|\xi - \alpha| \geq \mathrm{H}(\alpha)^{-w_n^*-1}, \tag{27}$$

since $\chi \leq w_n^* + 1$, by (1). It follows from (26) and (27) that

$$w_n^*(\xi) = w_n^*.$$

It now remains for us to prove that $w_n(\xi) = w_n$. Denote by $\xi_j = \beta_{j1}, \beta_{j2}, \ldots, \beta_{jn}$ the roots of the polynomial $Q_j(X)$. Observe that, for any $k \geq 1$, we have

$$\beta_{jk} = \frac{c_j + \delta_k}{g_j}$$

for a suitable root δ_k of the polynomial $P_{n,[g_j^\mu],g_j+d_j}(X)$. By (26) and Lemma 1, we get

$$\begin{aligned}
|Q_j(\xi)| &= g_j^n \cdot |\xi - \xi_j| \cdot \prod_{2\leq k\leq n} |\xi - \beta_{jk}| \\
&\asymp g_j^n \cdot |\xi - \xi_j| \cdot \prod_{2\leq k\leq n} |\xi_j - \beta_{jk}| \\
&\asymp g_j \cdot \mathrm{H}(\xi_j)^{-w_n^*-1} \cdot \big|P'_{n,[g_j^\mu],g_j+d_j}\big(\delta^+(n,[g_j^\mu], g_j+d_j)\big)\big| \\
&\asymp \mathrm{H}(\xi_j)^{-w_n^*-1} \cdot g_j^{(n+9)/4-\mu(n-9)/4}.
\end{aligned}$$

Assume first that $1 \leq \mu \leq (n-4)/4$. Then, we have $\mathrm{H}(\xi_j) = \mathrm{H}(Q_j) \asymp g_j^n$, which yields

$$|Q_j(\xi)| \asymp \mathrm{H}(Q_j)^{-w_n^* - 1 + (n+9)/(4n) - \mu(n-9)/(4n)},$$

hence

$$w_n(\xi) \geq w_n^* + \frac{3}{4} - \frac{9}{4n} + \frac{\mu}{4} - \frac{9\mu}{4n} = w_n^* + \Delta, \tag{28}$$

by the definition of Δ. Assume now that $(n-4)/4 \leq \mu \leq n-1$. Then, we have $\mathrm{H}(\xi_j) = \mathrm{H}(Q_j) \asymp g_j^{4(\mu+1)}$, which yields

$$w_n(\xi) \geq w_n^* + \frac{7}{16} + \frac{n}{16} \cdot \frac{\mu - 1}{\mu + 1} = w_n^* + \Delta, \tag{29}$$

by the definition of Δ. In order to show that the inequalities in (28) and (29) are indeed equalities, we argue exactly as Baker [1] did (see also [2]). We omit the details. This completes the proof of the theorem. □

6. Proof of Theorem 3

The proof of Theorem 3 follows step by step that of Theorem 1 from [2]. Instead of working with the integer polynomials given in Lemma 3 from [2], we use Theorem A stated in Section 2. Let $n \geq 6$ be an even integer. Let Δ be in $[(n-1)/n, n/2)$ and set

$$\mu = \frac{n(\Delta - 1) + 1}{n - 2\Delta}.$$

We observe that μ is in $[0, +\infty)$. Keeping the notation from [2, Section 6], we take for γ_j the root of

$$\tilde{P}_{n,a}(X) := (X^{n/2} - aX + 1)^2 - 2X^{n-2}\,(aX - 1)^2$$

nearest to $a^{-1} + a^{-1-n/2} + 2\sqrt{2}\,a^{-n}$, where $a = [g_j^{\mu}]$. Then, the minimal defining polynomial of ξ_j is

$$Q_j(X) = \big((g_jX - c_j)^{n/2} - a(g_jX - c_j) + 1\big)^2 - 2(g_jX - c_j)^{n-2}\,\big(a(g_jX - c_j) - 1\big)^2.$$

Observe further that

$$\mathrm{H}(\xi_j) = \mathrm{H}(Q_j) \asymp g_j^{n+2\mu}. \tag{30}$$

The inequality $\mathrm{H}(\gamma_j) \leq 2g_j^{n-2}$ used in [2, page 98, line 3] does not hold anymore, thus we have to modify accordingly inequality (3) from [2] (i.e., we have to assume that χ is sufficiently large) in order to be able to argue as in [2]. The only consequence is that the fonction $n \mapsto F(n)$ defined in Theorem 1 from [2] must be replaced by a larger one. By the Remark at the end of Section 4, we have to take $s = n + 2\mu$ in (25), thus, we have to assume $\chi > 2n(n+\mu)(n+1) + 2n + 1$.

We argue as in Section 6 from [2] and as in the above Section 5. Since

$$|\tilde{P}'_{n,[g_j^{\mu}]}(\gamma_j)| \asymp a^{-n+2},$$

we get

$$|Q_j(\xi)| \asymp g_j \, \mathrm{H}(Q_j)^{-w_n^*-1} \, g_j^{-\mu(n-2)}.$$

By (30), this implies the lower estimate

$$w_n(\xi) \geq w_n^* + 1 + \frac{\mu(n-2)-1}{n+2\mu}.$$

By definition of μ, we obtain

$$w_n(\xi) \geq w_n^* + \Delta. \tag{31}$$

We prove that there is equality in (31) exactly as in [1] or [2]. □

7. Proof of Theorem 4

Let $n \geq 6$ be an even integer. As in Theorem 2 of [1], the number ξ is obtained as the limit of a sequence of algebraic numbers of the form

$$\xi_j = \frac{c_j + id_j + \gamma_j}{g_j},$$

where the γ_j's are suitable real algebraic numbers of degree $n/2$ and the c_j's, d_j's and g_j's satisfy $g_j < c_j < 2g_j$ and $5g_j < d_j < 6g_j$. We omit the details of the construction of the ξ_j's, since it is very similar to that in Theorem 5 above. Set $m = n/2$. Let μ, μ' and μ'' be real numbers in $[0, (m-2)/2]$, in $[0, 1]$ and in $[0, +\infty)$, respectively. Set $a = [g_j^{\mu}]$, $a' = [g_j^{\mu'}]$ and $a'' = [g_j^{\mu''}]$. We choose for γ_j roots of the polynomials

$$\hat{P}_{m,a}(X) = X^m - 2(aX-1)^2, \qquad \check{P}_{m,a'}(X) = X^m - 2a'^m$$

or, when n is divisble by 4 (that is, when m is even),

$$\tilde{P}_{m,a''}(X) := (X^{m/2} - a''X + 1)^2 - 2X^{m-2}\,(a''X-1)^2.$$

We observe that the ξ_j's are of degree n and roots of polynomials of the form either $\hat{P}_{m,a}(g_jX - c_j - id_j) \times \hat{P}_{m,a}(g_jX - c_j + id_j)$, or $\check{P}_{m,a'}(g_jX - c_j - id_j) \times \check{P}_{m,a'}(g_jX - c_j + id_j)$, or $\tilde{P}_{m,a''}(g_jX - c_j - id_j) \times \tilde{P}_{m,a''}(g_jX - c_j + id_j)$, whose heights are $\asymp g_j^{2m}$, $\asymp g_j^{2m}$, and $\asymp g_j^{2(m+2\mu)}$, respectively. Furthermore, we have

$$\left| \frac{d}{dx} \big(\hat{P}_{m,a}(X - id_j) \times \hat{P}_{m,a}(X + id_j)\big)(\gamma_j + id_j) \right| \ll g_j^{m-\mu(m-2)/2},$$

$$\left| \frac{d}{dx} \left(\check{P}_{m,a'}(X - id_j) \times \check{P}_{m,a'}(X + id_j) \right)(\gamma_j + id_j) \right| \ll g_j^{m+\mu'(m-1)},$$

and

$$\left| \frac{d}{dx} \left(\tilde{P}_{m,a''}(X - id_j) \times \tilde{P}_{m,a''}(X + id_j) \right)(\gamma_j + id_j) \right| \ll g_j^{m-\mu''(m-4)}.$$

These estimates imply that, firstly, working with polynomials of the form $\hat{P}_{m,a}(g_jX - c_j - id_j) \times \hat{P}_{m,a}(g_jX - c_j + id_j)$, we construct a non-real complex number ξ such that

$$w_n(\xi) = w_n^*(\xi) + 1 - \frac{m+1}{2m} + \mu \cdot \frac{m-2}{4m}.$$

Secondly, working with polynomials of the form $\check{P}_{m,a'}(g_jX - c_j - id_j) \times \check{P}_{m,a'}(g_jX - c_j + id_j)$, we construct a non-real complex number ξ such that

$$w_n(\xi) = w_n^*(\xi) + 1 - \frac{m+1}{2m} - \mu' \cdot \frac{m-1}{2m}.$$

Thirdly, working with polynomials of the form $\tilde{P}_{m,a''}(g_jX - c_j - id_j) \times \tilde{P}_{m,a''}(g_jX - c_j + id_j)$, we construct a non-real complex number ξ such that

$$w_n(\xi) = w_n^*(\xi) + 1 - \frac{m+1}{2(m+2\mu'')} + \frac{\mu''(m-4)}{2(m+2\mu'')}.$$

Recalling that $n = 2m$ and letting μ, μ' and μ'' vary in their respectives ranges of values, this proves that the set of values taken by the function $w_n - w_n^*$ contains the intervals

$$\left[\frac{1}{2} - \frac{1}{n}, \frac{n}{16}\right] \qquad \text{and} \qquad \left[0, \frac{1}{2} - \frac{1}{n}\right],$$

and, if n is divisible by 4, the interval

$$\left[\frac{1}{2} - \frac{1}{n}, \frac{n}{8}\right].$$

This completes the proof of Theorem 4. □

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