On the quadratic Lagrange spectrum

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Abstract. *We study the quadratic Lagrange spectrum defined by Parkkonen and Paulin by considering the approximation by elements of the orbit of a given real quadratic irrational number for the action by homographies and anti-homographies of $PSL_2(\mathbf{Z})$ on $\mathbf{R} \cup \{\infty\}$. Our approach is based on the theory of continued fractions.*

1. Introduction

We use throughout the superscript $^\sigma$ to denote the Galois conjugate of a quadratic number. For every real quadratic irrational number α, Parkkonen and Paulin [9, 10] introduced the quantity

$$h(\alpha) := \frac{2}{|\alpha - \alpha^\sigma|}, \tag{1.1}$$

which may be viewed as a measure of the complexity of α. As noted in [9], this quantity behaves in a very different way from the naïve height of α (the naïve height of an algebraic number is the maximum of the absolute values of the coefficients of its minimal defining polynomial over the rational integers), a notion which is commonly used in Diophantine approximation; see e.g. [5].

Let α_0 be a fixed real quadratic irrational number and

$$\mathcal{E}_{\alpha_0} = PSL_2(\mathbf{Z}) \cdot \{\alpha_0, \alpha_0^\sigma\}$$

its orbit for the action by homographies and anti-homographies of $PSL_2(\mathbf{Z})$ on $\mathbf{R} \cup \{\infty\}$. In other words, $\mathcal{E}_{\alpha_0}$ is the set of quadratic numbers whose continued fraction expansion is ultimately periodic with the same period as α_0 or as α_0^σ.

For a real number ξ not in $\mathbf{Q} \cup \mathcal{E}_{\alpha_0}$, Parkkonen and Paulin [9, 10] defined by

$$c_{\alpha_0}(\xi) := \liminf_{\alpha \in \mathcal{E}_{\alpha_0} : |\alpha - \alpha^\sigma| \to 0} 2\,\frac{|\xi - \alpha|}{|\alpha - \alpha^\sigma|} = \liminf_{\alpha \in \mathcal{E}_{\alpha_0} : |\alpha - \alpha^\sigma| \to 0} |\xi - \alpha| \cdot h(\alpha) \tag{1.2}$$

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the approximation constant $c_{\alpha_0}(\xi)$ of ξ by elements of $\mathcal{E}_{\alpha_0}$ and they proved that $c_{\alpha_0}(\xi)$ is always finite. Observe that it follows immediately from (1.2) that

$$c_{\alpha_0+k}(\xi + k') = c_{\alpha_0}(\xi), \tag{1.3}$$

for every integers k and k'.

Furthermore, Parkkonen and Paulin [9, 10] defined the *quadratic Lagrange spectrum* of $(\alpha_0, PSL_2(\mathbf{Z}))$ by

$$Sp_{\alpha_0} := \{c_{\alpha_0}(\xi) : \xi \in \mathbf{R} \setminus (\mathbf{Q} \cup \mathcal{E}_{\alpha_0})\}.$$

Among other results, they showed that Sp_{α_0} is a closed subset of $[0, (1+\sqrt{2})\sqrt{3}]$. They also claimed that, in the special case when α_0 is the Golden Ratio $\varphi := (1+\sqrt{5})/2$, the maximum K_{α_0} of Sp_{α_0} is equal to $1 - 1/\sqrt{5}$. There is a slight overlook in the proof they proposed, but, fortunately, it can be fixed [11], and the exact value of $K_{(1+\sqrt{5})/2}$ is $-1 + 3/\sqrt{5} = 0.341\ldots$ Apparently, their method does not give the exact value of K_{α_0} for a general real quadratic number α_0.

The purpose of this note is to present a number theoretical interpretation of the approximation constant $c_{\alpha_0}(\xi)$ by means of the theory of continued fractions. We are then able, in principle, to compute K_{α_0} for every quadratic irrational number α_0, although this computation is in general not an easy task. Our main new result states that the quadratic Lagrange spectrum is always contained in $[0, 1/2]$.

Theorem 1.1. *For any real quadratic irrational number α_0, the maximum K_{α_0} of Sp_{α_0} satisfies*

$$0 < K_{\alpha_0} \leq \frac{1}{2}.$$

If the continued fraction expansion of α_0 terminates in an infinite string of digits 1, then

$$K_{\alpha_0} = K_{(1+\sqrt{5})/2} = \frac{3}{\sqrt{5}} - 1 = 0.341\ldots$$

Furthermore,

$$\lim_{m\to+\infty} K_{(m+\sqrt{m^2+4})/2} = \frac{3}{\sqrt{5}} - 1.$$

Like in [9, 11], our proof also gives that $K_{(1+\sqrt{5})/2}$ is not an isolated point in the spectrum $Sp_{(1+\sqrt{5})/2}$.

We believe that $-1 + 3/\sqrt{5}$ is a common upper bound for all the values K_{φ_0}. This seems to be, however, quite difficult to confirm.

The present note is organized as follows. Various results on continued fraction expansions are stated and proved in Section 2. They are applied to estimate the quantity $h(\alpha)$ defined in (1.1) and, in Section 3, to the proof of Theorem 1.1. We first explain how elementary arguments allow us to show that K_{α_0} is always at most equal to 4 (thus, slightly improving the upper bound $(1+\sqrt{2})\sqrt{3}$ obtained in [9]). Then, we refine our analysis to get the upper bound $1/2$. Several remarks and additional results on the quantity $h(\alpha)$ are gathered in Section 4.

Throughout, we use the notation

$$[a_0; a_1, a_2 \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+s}}] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

to indicate that the block of partial quotients $a_{r+1}, \ldots, a_{r+s}$ is repeated infinitely many times.

We recall that an irrational real number α is quadratic if, and only if, its continued fraction expansion is ultimately periodic, that is, is of the form

$$\alpha = [a_0; a_1, \ldots, a_r, \overline{b_1, \ldots, b_s}].$$

When we express α as in (1.4) we tacitly assume that s is minimal and that $a_r \neq b_s$. We call $b_1, \ldots, b_s$ the shortest periodic part in the continued fraction expansion of α.

2. Auxiliary lemmas on continued fraction expansions

We assume that the reader is familiar with the theory of continued fractions. Good references include [12, 8] and the first chapters of [14] and [5].

Let us simply recall that two irrational real numbers α, β are called *equivalent* if there exist rational integers a, b, c, d with $ad - bc = \pm 1$ such that

$$\alpha = \frac{a\beta + b}{c\beta + d}.$$

It is easily shown (see [12], page 65) that α and β are equivalent if, and only if, the tails of their continued fraction expansions coincide.

Our first lemma establishes a link between the quantity $h(\alpha)$ defined in (1.1) and the preperiod of the continued fraction expansion of the real quadratic number α.

Lemma 2.1. *Let α be a quadratic real number with ultimately periodic continued fraction expansion*

$$\alpha = [a_0; a_1, \ldots, a_r, \overline{b_1, \ldots, b_s}],$$

with $r \geq 3$ and $s \geq 1$, and denote by α^σ its Galois conjugate. Let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to α. Set $B := \max\{b_1, \ldots, b_s, 2\}$. If $a_r \neq b_s$, then we have

$$\frac{1}{2q_r^2} \leq |\alpha - \alpha^\sigma| \leq \frac{8B^3}{q_r^2},$$

thus,

$$q_r^2/(4B^3) \leq h(\alpha) \leq 4q_r^2. \tag{2.1}$$

This is essentially Lemma 6.1 from [6]. For the sake of completeness, we reproduce its proof.

Proof. A theorem of Galois (see [12], page 83) states that the Galois conjugate of

$$\tau := [b_1; \overline{b_2, \ldots, b_s, b_1}]$$

is the quadratic number

$$\tau^\sigma = -[0; \overline{b_s, \ldots, b_2, b_1}].$$

Since, by Theorem 1.7 of [5], we have

$$\alpha = \frac{p_r \tau + p_{r-1}}{q_r \tau + q_{r-1}} \quad \text{and} \quad \alpha^\sigma = \frac{p_r \tau^\sigma + p_{r-1}}{q_r \tau^\sigma + q_{r-1}},$$

we get

$$\begin{aligned} |\alpha - \alpha^\sigma| &= \frac{\tau - \tau^\sigma}{(q_r \tau + q_{r-1}) \cdot |q_r \tau^\sigma + q_{r-1}|} \\ &\leq \frac{\tau + 1}{\tau q_r \cdot |q_r \tau^\sigma + q_{r-1}|} \leq \frac{2}{q_r \cdot |q_r \tau^\sigma + q_{r-1}|}. \end{aligned} \tag{2.2}$$

Likewise, using $|q_r \tau^\sigma + q_{r-1}| \leq q_r$, we obtain

$$|\alpha - \alpha^\sigma| \geq \frac{\tau - \tau^\sigma}{(\tau + 1) q_r^2} \geq \frac{1}{2 q_r^2}. \tag{2.3}$$

Since $a_r \neq b_s$ the mirror formula (see Lemma 3F of [14])

$$q_{r-1}/q_r = [0; a_r, a_{r-1}, \ldots, a_1]$$

implies that

$$|q_r \tau^\sigma + q_{r-1}| = |[0; a_r, a_{r-1}, \ldots, a_1] - [0; \overline{b_s, \ldots, b_2, b_1}]| \cdot q_r.$$

If $a_r \geq 2B$, then one gets

$$|q_r \tau^\sigma + q_{r-1}| \geq \Big(\frac{1}{B+1} - \frac{1}{2B} \Big) q_r \geq \frac{q_r}{6B}.$$

Otherwise, if $a_r < b_s$, then $a_r \leq B - 1$ and an easy computation shows that

$$\begin{aligned} |q_r \tau^\sigma + q_{r-1}| &\geq \Big(\frac{1}{a_r + 1} - \frac{1}{b_s + 1/(b_{s-1} + 1)} \Big) \cdot q_r \\ &\geq \frac{q_r}{(a_r + 1)(b_s(b_{s-1} + 1) + 1)} \geq \frac{q_r}{3B^3}, \end{aligned}$$

while, if $b_s < a_r \leq 2B - 1$, then the similar estimate

$$|q_r \tau^\sigma + q_{r-1}| \geq \frac{q_r}{a_r b_s (2B)} \geq \frac{q_r}{4B^3}$$

holds. By (2.2) and (2.3), this completes the proof of the lemma. □

Our second auxiliary lemma is very close to Lemma 5 from [1] (see also Lemma 5.5 from [3]).

Lemma 2.2. *Let B be a positive integer. Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers. Assume that there exists a positive integer n such that $a_i = b_i$ for any $i = 1, \ldots, n$ and $a_{n+1} \neq b_{n+1}$. Then, we have*

$$|\alpha - \beta| \geq \frac{1}{12 q_{n+1}^2 \max\{b_{n+2} + 1, b_{n+3} + 2\}}, \tag{2.4}$$

where $(p_\ell / q_\ell)_{\ell \geq 1}$ is the sequence of convergents to β.

Proof. Set $\alpha' = [a_{n+1}; a_{n+2}, \ldots]$ and $\beta' = [b_{n+1}; b_{n+2}, \ldots]$. Since $a_{n+1} \neq b_{n+1}$, a rapid calculation shows that

$$|\alpha' - \beta'| \geq \min\Big\{ \frac{1}{b_{n+2} + 1}, \frac{1}{b_{n+3} + 2} \Big\}. \tag{2.5}$$

Using that the first n partial quotients of α and β are assumed to be the same, we get

$$\alpha = \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} \quad \text{and} \quad \beta = \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}},$$

thus,

$$|\alpha - \beta| = \left| \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} - \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}} \right| = \left| \frac{\alpha' - \beta'}{(q_n \alpha' + q_{n-1})(q_n \beta' + q_{n-1})} \right|. \tag{2.6}$$

Note that $q_n \beta' + q_{n-1} \leq 2 q_{n+1}$.

If $\alpha' \leq 2\beta' + 1 \leq 2 b_{n+1} + 3$, then $q_n \alpha' + q_{n-1} \leq 2(b_{n+1} + 2) q_n$ and, by (2.5) and (2.6), we get

$$|\alpha - \beta| \geq \frac{1}{4(b_{n+1} + 2) q_n q_{n+1} \max\{b_{n+2} + 1, b_{n+3} + 2\}}. \tag{2.7}$$

If $\alpha' \geq 2\beta' + 1$, then $|\alpha' - \beta'| \geq (\alpha' + 1)/2$ and

$$|\alpha - \beta| \geq \frac{1}{4 q_n q_{n+1}}. \tag{2.8}$$

Since $(b_{n+1} + 2) q_n \leq 3 q_{n+1}$, the estimate (2.4) follows from (2.7) and (2.8). □

We display an easy consequence of Lemma 2.2.

Corollary 2.3. *Let*

$$\tau = [0; \overline{b_1, b_2 \ldots, b_s}]$$

be a quadratic number whose shortest periodic part is $b_1, \ldots, b_s$. Let

$$\xi = [a_0; a_1, a_2, \ldots]$$

be an irrational real number not in $\mathcal{E}_\tau$. For positive integers r and t, let $a'_1, \ldots, a'_t$ be positive integers with $a'_1 \neq a_{r+1}$ and $a'_t \neq b_s$, and set

$$\alpha := [a_0; a_1, a_2, \ldots, a_r, a'_1, \ldots, a'_t, \overline{b_1, \ldots, b_{s-1}, b_s}].$$

Putting $B := \max\{b_1, \dots, b_s, 2\}$, we have

$$|\xi - \alpha| \cdot h(\alpha) \geq \frac{1}{96B^3(B+1)^2}. \tag{2.9}$$

Furthermore, $|\xi - \alpha| \cdot h(\alpha) \geq 1$ if $t \geq 9.6 + 4.4 \log B$.

Proof. Let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to α. We deduce from Lemma 2.2 that

$$|\xi - \alpha| \geq \frac{1}{24 q_{r+1} q_{r+3}} \tag{2.10}$$

and from Lemma 2.1 that

$$h(\alpha) \geq q_{r+t}^2/(4B^3). \tag{2.11}$$

This gives

$$|\xi - \alpha| \cdot h(\alpha) \geq 1/(96B^3)$$

if $t \geq 3$. If $t = 1$, then $q_{r+3} \leq (B+1)^2 q_{r+1}$ and we obtain the upper bound

$$|\xi - \alpha| \cdot h(\alpha) \geq \frac{1}{96B^3(B+1)^2}. \tag{2.12}$$

If $t = 2$, then, using $q_{r+3} \leq (B+1) q_{r+2}$, we see that (2.12) holds. This proves (2.9).

An easy induction shows that $q_\ell \geq \sqrt{2}^{\ell-h-1} q_h$ for $\ell > h \geq 1$. Consequently, for $t \geq 4$, we deduce from (2.10) and (2.11) that

$$|\xi - \alpha| \cdot h(\alpha) \geq \frac{2^t}{768B^3}.$$

This implies that $|\xi - \alpha| \cdot h(\alpha) \geq 1$ if $t \geq 9.6 + 4.4 \log B$. $\square$

For positive integers $a_1, \dots, a_\ell$, denote by $K_\ell(a_1, \dots, a_\ell)$ the denominator of the rational number $[0, a_1, \dots, a_\ell]$. It is commonly called the *continuant* of $a_1, \dots, a_\ell$. A proof of the next lemma can be found on page 15 of [12].

Lemma 2.4. *For any positive integers $a_1, \dots, a_\ell$ and any integer k with $1 \leq k \leq \ell - 1$, we have*

$$K_\ell(a_1, \dots, a_\ell) = K_\ell(a_\ell, \dots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \dots, a_k) \cdot K_{\ell-k}(a_{k+1}, \dots, a_\ell) &\leq K_\ell(a_1, \dots, a_\ell) \\ &\leq 2\, K_k(a_1, \dots, a_k) \cdot K_{\ell-k}(a_{k+1}, \dots, a_\ell). \end{aligned}$$

3. Proof of Theorem 1.1

3.1. Preliminaries.

Let $b_1, \ldots, b_s$ be the (shortest) periodic part in the continued fraction expansion of α_0 and set

$$\tau = [b_1; \overline{b_2, \ldots, b_s, b_1}].$$

For $j = 1, \ldots, s$, set

$$\tau_j := [b_j; \overline{b_{j+1}, \ldots, b_{j-1}, b_j}]$$

and

$$\tau'_j = [b_{j-1}; \overline{b_{j-2}, b_{j-3}, \ldots, b_j, b_{j-1}}],$$

where the indices are understood modulo s. Observe that $\tau = \tau_1$ and $\mathcal{E}_\tau = \mathcal{E}_{\tau_1} = \ldots = \mathcal{E}_{\tau_s} = \mathcal{E}_{\tau'_1} = \ldots = \mathcal{E}_{\tau'_s} = \mathcal{E}_{\alpha_0}$. Furthermore, by the theorem of Galois already mentioned at the beginning of the proof of Lemma 2.1, we have

$$\tau_j^\sigma = -[0; \overline{b_{j-1}, b_{j-2}, \ldots, b_{j+1}, b_j}] = -1/\tau'_j,$$

for $j = 1, \ldots, s$.

Let

$$\xi = [0; a_1, a_2, \ldots]$$

be an irrational real number not in $\mathcal{E}_\tau$, which we wish to approximate by numbers from $\mathcal{E}_\tau$. Let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to ξ. By (1.3), our assumption that ξ is in $(0, 1)$ is not restrictive.

3.2. An observation.

For $r \geq 1$, the quadratic number

$$\alpha_r := [0; a_1, a_2, \ldots, a_r, \overline{b_1, \ldots, b_{s-1}, b_s}] = [0; a_1, a_2, \ldots, a_r, \tau]$$

is a quite good approximation to ξ in $\mathcal{E}_\tau$ and

$$\ell_\tau(\xi) := \liminf_{r \to +\infty} |\xi - \alpha_r| \cdot h(\alpha_r)$$

is greater than or equal to $c_\tau(\xi)$.

By taking also into account the circular permutations of the periodic part of the continued fraction expansion of τ and of that of τ^σ, we see that

$$c_\tau(\xi) \leq \min_{1 \leq j \leq s} \min\{\ell_{\tau_j}(\xi), \ell_{\tau'_j}(\xi)\}. \tag{3.1}$$

At first sight, we may expect that equality holds in (3.1). In order to confirm this, we would need to estimate the quantities $|\xi - \alpha| \cdot h(\alpha)$ for quadratic numbers α of the form

$$[0; a_1, a_2, \ldots, a_r, a'_1, \ldots, a'_t, \tau_j] \quad \text{and} \quad [0; a_1, a_2, \ldots, a_r, a'_1, \ldots, a'_t, \tau'_j],$$

where $1 \le j \le s$, $t \ge 1$ and $a'_1, \ldots, a'_t$ are positive integers. By Corollary 2.3, we know already that $|\xi - \alpha| \cdot h(\alpha) \ge 1$ if t is sufficiently large in terms of $b_1, \ldots, b_s$. However, even for small values of t, it is difficult to estimate precisely $|\xi - \alpha| \cdot h(\alpha)$ for a general quadratic number τ.

Actually, we will see below that, unlike what would be expected, there exist quadratic numbers τ for which we do not have equality in (3.1).

3.3. A first upper bound.

Since ξ and α_r have the same first r partial quotients, we deduce from Theorem 1.7 of [5] that

$$|\xi - \alpha_r| \le \frac{1}{q_r^2}.$$

It then follows from (2.1) that

$$|\xi - \alpha_r| \cdot h(\alpha_r) \le 4,$$

which gives that

$$c_{\alpha_0}(\xi) \le 4.$$

We recover, with a fairly simple proof, that the quantities K_{α_0} are always finite and, even, absolutely bounded. The upper bound 4 is slightly smaller than the value $(1+\sqrt{2})\sqrt{3} = 4.18\ldots$ obtained in [9]. However, our method using continued fraction expansions allows us to improve further this upper bound.

3.4. Preliminary calculation.

Since, by Theorem 1.7 of [5], we have

$$\alpha_r = \frac{p_r\tau + p_{r-1}}{q_r\tau + q_{r-1}} \quad \text{and} \quad \alpha_r^\sigma = \frac{p_r\tau^\sigma + p_{r-1}}{q_r\tau^\sigma + q_{r-1}},$$

we deduce that

$$|\alpha_r - \alpha_r^\sigma| = \frac{\tau - \tau^\sigma}{(q_r\tau + q_{r-1}) \cdot |q_r\tau^\sigma + q_{r-1}|}. \tag{3.2}$$

Observe now that

$$\begin{aligned} |\xi - \alpha_r| &= \left| \frac{p_r[a_{r+1}; a_{r+2}, \ldots] + p_{r-1}}{q_r[a_{r+1}; a_{r+2}, \ldots] + q_{r-1}} - \frac{p_r\tau + p_{r-1}}{q_r\tau + q_{r-1}} \right| \\ &= \frac{|\tau - [a_{r+1}; a_{r+2}, \ldots]|}{(q_r[a_{r+1}; a_{r+2}, \ldots] + q_{r-1}) \cdot (q_r\tau + q_{r-1})} \end{aligned}$$

and, using (3.2),

$$2\,\frac{|\xi - \alpha_r|}{|\alpha_r - \alpha_r^\sigma|} = \frac{2\,|\tau - [a_{r+1}; a_{r+2}, \ldots]| \cdot |q_r\tau^\sigma + q_{r-1}|}{(\tau - \tau^\sigma) \cdot (q_r[a_{r+1}; a_{r+2}, \ldots] + q_{r-1})}. \tag{3.3}$$

Set

$$d_r := \frac{q_r}{q_{r-1}} = [a_r; a_{r-1}, \ldots, a_1]$$

and

$$D_r := [a_{r+1}; a_{r+2}, \ldots].$$

The right hand side of (3.3) then becomes

$$\frac{2}{\tau - \tau^\sigma} \cdot \frac{|\tau - D_r| \cdot |\tau^\sigma d_r + 1|}{1 + d_r D_r}. \tag{3.4}$$

We stress that, to establish (3.4) we do not have used that the continued fraction expansion of τ is purely periodic.

Define

$$\tau' := -1/\tau^\sigma = 1/|\tau^\sigma| = [b_s; \overline{b_{s-1}, \ldots, b_1, b_s}].$$

Then, (3.4) becomes

$$\frac{2}{\tau\tau' + 1} \cdot \frac{|\tau - D_r| \cdot |\tau' - d_r|}{1 + d_r D_r}.$$

Keeping the notation of Subsection 3.1, this proves that

$$\ell_{\tau_j}(\xi) = \liminf_{r \to +\infty} \frac{2}{\tau_j \tau_j' + 1} \cdot \frac{|\tau_j - D_r| \cdot |\tau_j' - d_r|}{1 + d_r D_r},$$

for $j = 1, \ldots, s$. Similarly, we obtain

$$\ell_{\tau_j'}(\xi) = \liminf_{r \to +\infty} \frac{2}{\tau_j \tau_j' + 1} \cdot \frac{|\tau_j' - D_r| \cdot |\tau_j - d_r|}{1 + d_r D_r}.$$

Consequently, the quantity

$$c_\tau'(\xi) := \min_{1 \le j \le s} \frac{2}{\tau_j \tau_j' + 1} \cdot \liminf_{r \to +\infty} \frac{\min\{|\tau_j - D_r| \cdot |\tau_j' - d_r|, |\tau_j' - D_r| \cdot |\tau_j - d_r|\}}{1 + d_r D_r} \tag{3.5}$$

is greater than or equal to $c_\tau(\xi)$. As explained below (3.1), it could be strictly greater than $c_\tau(\xi)$, since the quadratic numbers we have considered are special elements of the orbit of α_0 and of that of α_0^σ.

3.5. A refined upper bound.

Our goal is to establish that the quantity $c_\tau'(\xi)$ defined in (3.5) is always at most equal to 1/2.

First, we assume that neither τ, nor ξ is equivalent to the Golden Ratio φ. This means that both have infinitely many partial quotients at least equal to 2.

In particular, either infinitely many partial quotients of ξ are at least equal to 3, in which case $d_r D_r \geq 3$ for infinitely many r, or

$$d_r D_r \geq \Big(1 + \frac{1}{3}\Big) \cdot \Big(2 + \frac{1}{3}\Big) = \frac{28}{9}, \tag{3.6}$$

for infinitely many r.

Since τ and ξ are not equivalent to φ, we can suppose that $\tau' > 2$ and $D_r > 2$. Furthermore, arguing as for (3.6), we can suppose that

$$\tau\tau' \geq 3. \tag{3.7}$$

If τ, τ', d_r, D_r are all greater than 2, then it follows from (3.5) that

$$c'_\tau(\xi) \leq \frac{2\max\{\tau, D_r\} \cdot \max\{\tau', d_r\}}{\tau\tau' d_r D_r} \leq \frac{1}{2}.$$

Assume that $\tau > 2$ and $1 < d_r < 2$.

If $\tau \geq D_r/2$, then $|\tau - D_r| \leq \tau$ and, since $|\tau' - d_r| \leq \tau'$, we get from (3.6) that

$$c'_\tau(\xi) \leq \frac{2}{1 + d_r D_r} \leq \frac{1}{2}.$$

The same upper bound holds if $\tau' \geq D_r/2$, since then we have $|\tau - d_r| \leq \tau$. Consequently, we can assume that $D_r \geq 2\tau$ and $D_r \geq 2\tau'$, thus,

$$c'_\tau(\xi) \leq \min\Big\{\frac{2|\tau' - d_r|}{d_r(1 + \tau\tau')}, \frac{2|\tau - d_r|}{d_r(1 + \tau\tau')}\Big\}.$$

We get $c'_\tau(\xi) \leq 1/2$ if $\tau \geq 4$ or $\tau' \geq 4$. So, we can assume that $2 < \tau, \tau' < 4$. If $\tau \leq \tau'$, then

$$c'_\tau(\xi) \leq \frac{2(\tau - 1)}{1 + \tau\tau'} \leq \frac{2(\tau - 1)}{1 + \tau\tau^2} \leq \frac{1}{2},$$

and, by symmetry, the estimate $c'_\tau(\xi) \leq 1/2$ also holds if $\tau' \leq \tau$.

Assume that $1 < \tau \leq 2$.

We get $|\tau - D_r| \leq D_r$. If $d_r \geq \tau'/2$, it then follows from (3.7) that

$$c'_\tau(\xi) \leq \frac{2}{\tau\tau' + 1} \leq \frac{1}{2}.$$

Using the symmetry, the same upper bound holds when $D_r \geq \tau'/2$. Consequently, we can assume that $\tau' \geq 2D_r$ and $\tau' \geq 2d_r$.

Arguing as above, we see that one can assume that D_r and d_r are both < 4 and, moreover, that $1 < d_r < 2$. In particular, the partial quotients of ξ belong to $\{1, 2, 3\}$. Arguing as in (3.6), this implies that $d_r \geq 5/4$ and $D_r \geq 9/4$. Furthermore, since $5/4 \leq d_r < 2$ and $1 < \tau < 2$, we get that $|\tau - d_r| \leq 3\tau/4$ and, consequently,

$$c'_\tau(\xi) \leq \min\Big\{\frac{2|\tau - d_r|, 2|\tau - D_r|}{\tau(1 + d_r D_r)}\Big\} \leq \frac{2}{5}.$$

To summarize, we have established that, unless τ or ξ is equivalent to the Golden Ratio, we always have

$$c_\tau(\xi) \le c'_\tau(\xi) \le 1/2.$$

3.6. When τ is the Golden Ratio.

Assume that $\tau = \varphi = (1+\sqrt{5})/2$. To determine K_φ, we need to find $\xi = [0; a_1, a_2, \dots]$ for which $c_\tau(\xi)$ is as large as possible. Clearly, all the a_j have to be taken very large and we then derive from (3.5) that

$$c'_\varphi(\xi) \le \frac{2}{1+\varphi^2} \cdot \frac{d_r D_r}{1 + d_r D_r} \le 1 - \frac{1}{\sqrt{5}},$$

and $c'_\varphi(\xi)$ can be arbitrarily close to $1 - 1/\sqrt{5}$. Actually, it turns out that the approximations $[0; a_1, a_2, \dots, a_r, \overline{1}] = [0; a_1, a_2, \dots, a_r, \varphi]$ are not the closest ones to ξ and that we gain by considering the approximations

$$[0; a_1, a_2, \dots, a_r, 2, \overline{1}] = [0; a_1, a_2, \dots, a_r, 1 + \varphi].$$

Indeed, (3.4) then becomes

$$\frac{2}{\varphi - \varphi^\sigma} \cdot \frac{|1 + \varphi - D_r| \cdot |(1 + \varphi^\sigma) d_r + 1|}{1 + d_r D_r},$$

and, since d_r and D_r are very large, we obtain the limiting value

$$\frac{2(1+\varphi^\sigma)}{\varphi - \varphi^\sigma} = \frac{3}{\sqrt{5}} - 1.$$

This shows that $c_\varphi(\xi) < -1 + 3/\sqrt{5}$ for every irrational number ξ not in $\mathcal{E}_\varphi$. Furthermore, we check that, for $\alpha = [0; a_1, a_2, \dots, a_r, a'_1, \dots, a'_t, \overline{1}]$, with $t \ge 1, a'_t \ne 1, a'_1 \ne a_{r+1}$, and $a'_t \ne 2$ if $t = 1$, the quantity $|\xi - \alpha| \cdot h(\alpha)$ exceeds $-1 + 3/\sqrt{5}$. To see this, setting

$$\frac{p'_t}{q'_t} := [a'_1; a'_2, \dots, a'_t], \quad \frac{p'_{t-1}}{q'_{t-1}} := [a'_1; a'_2, \dots, a'_{t-1}] \quad \text{and} \quad \varphi_t := \frac{p'_t \varphi + p'_{t-1}}{q'_t \varphi + q'_{t-1}},$$

the formula (3.4) becomes

$$\frac{2}{\varphi_t - \varphi_t^\sigma} \cdot \frac{|\varphi_t - D_r| \cdot |\varphi_t^\sigma d_r + 1|}{1 + d_r D_r}.$$

Again, since d_r and D_r are very large, we obtain the limiting value

$$\frac{2|\varphi_t^\sigma|}{\varphi_t - \varphi_t^\sigma}.$$

Noticing that $\varphi_t - \varphi_t^\sigma \le \varphi - \varphi^\sigma$ and $|\varphi_t^\sigma| \ge 1 + \varphi^\sigma$, we see that the minimum of this quantity is attained for $\varphi_t = 1 + \varphi$.

Consequently, we have established that $K_\varphi = -1 + 3/\sqrt{5}$. Moreover, setting

$$\varphi_m := [m; \overline{m}] = (m + \sqrt{m^2+4})/2,$$

for $m \ge 1$, we see that $\varphi = \varphi_1$,

$$c_\varphi(\varphi_m) < 3/\sqrt{5} - 1 \quad \text{and} \quad \lim_{m \to +\infty} c_\varphi(\varphi_m) = 3/\sqrt{5} - 1.$$

3.7. When ξ is the Golden Ratio.

For $\xi = \varphi = (1+\sqrt{5})/2$, we have $D_r = \varphi$ and d_r approaches φ as r tends to infinity. Since we are looking for large values of $c_\tau(\varphi)$ we can assume that $b_1, \ldots, b_s$ are all at least equal to 2. It then follows from (3.5) that

$$c'_\tau(\varphi) \le \frac{2}{1+\varphi^2} \cdot \frac{\tau\tau'}{1+\tau\tau'} < 1 - \frac{1}{\sqrt{5}}.$$

This establishes that $c_\tau(\varphi)$ is always smaller than $1 - 1/\sqrt{5}$. We have considered the approximants $[1; 1, \ldots, 1, \tau]$, but, quite surprisingly, the approximants

$$[1; 1, \ldots, 1, \tau - 1] = [1; 1, \ldots, 1, b_1 - 1, \overline{b_2, \ldots, b_s, b_1}]$$

give a better estimate. Indeed, by (3.4), we then have to consider

$$\frac{2}{\tau - \tau^\sigma} \cdot \frac{|(\tau - 1) - \varphi| \cdot |(\tau^\sigma - 1)\varphi + 1|}{1+\varphi^2},$$

and, when τ and $1/|\tau^\sigma|$ tend to infinity, the above quantity tends to

$$\frac{2 \cdot |1 - \varphi|}{1+\varphi^2} = \frac{3}{\sqrt{5}} - 1.$$

Furthermore, since $|n\varphi + 1| \ge |1 - \varphi|$ for every integer n, the quantity $|\varphi - \alpha| \cdot h(\alpha)$ always exceeds $-1 + 3/\sqrt{5}$.

Consequently, we have established that $c_\tau(\varphi) < -1 + 3/\sqrt{5}$ and, moreover,

$$\lim_{m \to +\infty} c_{\varphi_m}(\varphi) = 3/\sqrt{5} - 1.$$

This shows that

$$\lim_{m \to +\infty} K_{\varphi_m} = K_\varphi,$$

thus completing the proof of Theorem 1.1.

3.8. Some speculation.

It seems that there are two extremal cases to determine the value of K_τ for a given real quadratic number τ. A first one is when all the a_j are large, and a second one when all the a_j are equal to 1, that is, when $\xi = \varphi$. This observation suggests that

$$K_\tau := \max\{c_\tau(\varphi), \lim_{m\to+\infty} c_\tau(\varphi_m)\}. \tag{3.8}$$

Using (3.4) and arguing as in Subsection 3.6, we also believe that

$$\lim_{m\to+\infty} c_\tau(\varphi_m) = \min_{1\le j\le s} \min\Big\{\frac{2\min\{|\tau_j^\sigma|, 1+\tau_j^\sigma\}}{\tau_j - \tau_j^\sigma}, \frac{2\min\{|\tau_j'^\sigma|, 1+\tau_j'^\sigma\}}{\tau_j' - \tau_j'^\sigma}\Big\}.$$

Since $\tau_j' = -1/\tau_j^\sigma$ for $j = 1, \dots, s$, the above formula can be rewritten as

$$\lim_{m\to+\infty} c_\tau(\varphi_m) = \min_{1\le j\le s} \frac{2\min\{|\tau_j^\sigma|, 1+\tau_j^\sigma, |\tau_j^\sigma|(\tau_j - 1)\}}{\tau_j - \tau_j^\sigma}.$$

Likewise, using (3.4) and arguing as in Subsection 3.7, we believe that, putting

$$m_j := \min\{|\tau_j - \varphi|\cdot|1+\varphi\tau_j^\sigma|, |\tau_j - (1+\varphi)|\cdot|1+\varphi\tau_j^\sigma - \varphi|, |\tau_j - \varphi(1+\tau_j)|\cdot|1+\tau_j^\sigma(1+\varphi)|\},$$

for $j = 1, \dots, s$, we have

$$c_\tau(\varphi) = \min_{1\le j\le s} \frac{2m_j}{(\tau_j - \tau_j^\sigma)(1+\varphi^2)}.$$

Establishing (3.8) seems to be a difficult problem.

4. Further results

In this section, we take the point of view of Diophantine approximation to discuss several questions naturally related to the quantity $h(\alpha)$ defined in (1.1).

We begin with a link between the constant $c_\tau(\xi)$ and the continued fraction expansions of ξ and τ.

Lemma 4.1. *Let τ be a quadratic real number. Let $b_1, \dots, b_s$ be the shortest period in its continued fraction expansion. Then, an irrational real number $\xi = [a_0; a_1, a_2, \dots]$ not in $\mathcal{E}_\tau$ satisfies $c_\tau(\xi) > 0$ if, and only if, the infinite word $a_0 a_1 \dots$ does not contain arbitrarily large repetitions neither of $b_1 \dots b_s$, nor of $b_s \dots b_1$.*

Proof. Assume first that the infinite word $a_0 a_1 \dots$ contains arbitrarily large repetitions of $b_1 \dots b_s$ and let K_τ denote the denominator of the rational number $[0; b_1, \dots, b_s]$. Let $m \ge 2$ and $r \ge 1$ be integers such that $a_{r+js+i} = b_i$ for $i = 1, \dots, s$ and $j = 0, \dots, m-1$. Set $\alpha_r := [a_0; a_1, a_2, \dots, a_r, \overline{b_1, \dots, b_s}]$ and denote by $(p_\ell/q_\ell)_{\ell\ge 1}$ the sequence of its convergents.

Since α_r and ξ have (at least) their $r+ms$ first partial quotients in common, it follows from Lemma 2.4 that

$$|\xi - \alpha_r| < q_{r+ms}^{-2} \le q_r^{-2} K_\tau^{-2ms},$$

while Lemma 2.1 asserts that

$$h(\alpha_r) < 4q_r^2.$$

Consequently, we deduce that

$$|\xi - \alpha_r| \cdot h(\alpha_r) < 4K_\tau^{-2ms}.$$

Since m can be taken arbitrarily large, we conclude that $c_\tau(\xi) = 0$.

The converse is slightly more difficult to establish. Assume that $c_\tau(\xi) = 0$ and let ε be a positive real number. Set $B := \max\{b_1, \ldots, b_s, 2\}$. There exists α in $\mathcal{E}_\tau$ such that

$$|\xi - \alpha| \cdot h(\alpha) < \varepsilon. \tag{4.1}$$

It follows from Corollary 2.3 that, if ε is sufficiently small, then there exist a cyclic permutation $c_1, \ldots c_s$ of $b_1, \ldots, b_s$ or of $b_s, \ldots, b_1$ and an integer r such that

$$\alpha = [a_0; a_1, a_2, \ldots, a_r, \overline{c_1, \ldots, c_s}]$$

with $a_r \neq c_s$. Denote by $(p_\ell/q_\ell)_{\ell \ge 1}$ the sequence of convergents of α and observe that, by Lemma 2.1,

$$h(\alpha) \ge q_r^2/(4B^3). \tag{4.2}$$

Let $m(\alpha)$ be the smallest integer for which the $m(\alpha)$-th partial quotients of ξ and of α are different. Then, by Lemma 2.2, we get the lower bound

$$|\xi - \alpha| \ge \frac{1}{12(B+2)q_{m(\alpha)}^2}. \tag{4.3}$$

Since $q_{r+h} \le (B+1)^h q_r$ for $h \ge 1$, it follows from (4.1), (4.2) and (4.3) that

$$q_r^2 < 48\varepsilon(B+2)B^3 q_{m(\alpha)}^2 \le 48\varepsilon(B+2)B^3(B+1)^{m(\alpha)-r} q_r^2.$$

Since ε can be taken arbitrarily small, this shows that $m(\alpha) - r$ can be arbitrarily large, thus, the infinite word $a_0 a_1 a_2 \ldots$ contains arbitrarily large repetitions of the block $b_1 \ldots b_s$ or of the block $b_s \ldots b_1$. This concludes the proof of the lemma. □

In 1966, Schmidt [13, 14] developed his theory of α-β games and winning sets. The next result is an easy consequence of Lemma 4.1.

Theorem 4.2. *For every quadratic real number τ the set*

$$\{\xi \in \mathbf{R} : c_\tau(\xi) > 0\}$$

has Lebesgue measure zero. It is a winning set, thus it has Hausdorff dimension 1.

The first assertion of Theorem 4.2 follows from the well-known fact that every finite block of positive integers occurs in the continued fraction expansion of almost all real numbers.

Proof. Let $b_1, \ldots, b_s$ denote the shortest period of the continued fraction expansion of τ. By Lemma 4.1, an irrational number ξ satisfies $c_\tau(\xi) > 0$ if, and only if, its continued fraction expansion, viewed as an infinite word, does not contain arbitrarily large repetitions of $b_1 \ldots b_s$ or of $b_s \ldots b_1$. To win the game, the second player has simply to avoid certain intervals, as in Theorem 4 of [13], where it is proved that, under some assumption on α, β and on the integer $b \geq 2$, the set of real numbers having only finitely many digits 0 in their b-ary expansion is (α, β)-winning. We omit the details. □

For an irrational number ξ and an integer $b \geq 2$, let $v_b(\xi)$ denote the supremum of the real numbers v such that

$$||b^n \xi|| < (b^n)^{-v}$$

for infinitely many positive integers n. Here, $|| \cdot ||$ denotes the distance to the nearest integer. The exponents of approximation v_b have been recently introduced and studied in [4]. They detect large repetitions of digits 0 or of digits $b - 1$.

Actually, given an integer $b \geq 2$ and a rational number p/q with $0 \leq p/q < 1$, we can as well define $v_{b,p/q}(\xi)$ to be the supremum of the real numbers v such that

$$||b^n \xi + p/q|| < (b^n)^{-v}$$

for infinitely many positive integers n. This means that we approximate ξ by rational numbers whose b-ary expansion is ultimately periodic, with the same period as the b-ary expansion of p/q. Almost all real numbers ξ satisfy $v_{b,p/q}(\xi) = 0$.

A natural analogue (for restricted quadratic approximation) of the exponents $v_{b,p/q}$ (for restricted rational approximation) in the present setting is given by the exponents v_τ defined as follows.

Definition 4.3. *For a quadratic number τ and an irrational number ξ not in $\mathcal{E}_\tau$, we denote by v_τ the supremum of the real numbers v such that*

$$|\xi - \alpha| < h(\alpha)^{-v},$$

for infinitely many α in $\mathcal{E}_\tau$.

Note that the choice of an integer base b corresponds to that of a continued fraction algorithm and the choice of a rational number p/q corresponds to that of a quadratic number τ. In the present text, we consider only the usual continued fraction algorithm.

It follows from Lemma 2.1 that every real number ξ not in $\mathcal{E}_\tau$ satisfies $v_\tau(\xi) \geq 1$. Furthermore, almost all real numbers ξ satisfy $v_\tau(\xi) = 1$ and, for every real number $v \geq 1$, the Hausdorff dimension of the set

$$\{\xi \in \mathbf{R} : v_\tau(\xi) = v\}$$

is equal to $1/v$; see also Theorem 1.3 of [9].

Our last result is a transcendence statement.

Theorem 4.4. *Let τ be a quadratic real number and ξ an irrational number not in $\mathcal{E}_\tau$. If $v_\tau(\xi) > 1$, then ξ is transcendental.*

Proof. Let $b_1, \ldots, b_s$ be the shortest period of the continued fraction expansion of τ and write $\xi = [a_0; a_1, a_2 \ldots]$. Let $(p_\ell/q_\ell)_{\ell \geq 1}$ be the sequence of partial quotients to ξ. Replacing τ by $-1/\tau^\sigma$ if necessary, there exist, by assumption, a positive real number ε and an infinite set $\mathcal{R}$ of positive integers such that, for any r in $\mathcal{R}$, the quadratic number $\alpha_r = [a_0; a_1, \ldots, a_r, \overline{b_1, \ldots, b_s}]$ in $\mathcal{E}_\tau$ satisfies $a_{r+1} \neq b_1$ and

$$|\xi - \alpha_r| < h(\alpha_r)^{-1-\varepsilon}. \tag{4.4}$$

Note that it follows from Lemma 2.1 that

$$h(\alpha_r) \gg q_r^2, \tag{4.5}$$

where, as below, the numerical constant implied by $\gg$ only depends on $b_1, \ldots, b_s$. Furthermore, denoting by $m(\alpha_r)$ the smallest integer for which the $m(\alpha_r)$-th partial quotients of ξ and of α_r are different, Lemma 2.2 implies that

$$|\xi - \alpha_r| \gg q_{m(\alpha_r)}^{-2}. \tag{4.6}$$

The combination of (4.4), (4.5) and (4.6) gives then

$$q_{m(\alpha_r)} \gg q_r^{1+\varepsilon}, \tag{4.7}$$

and we deduce from Lemma 2.4 that there exists a positive real number ε', depending only on $b_1, \ldots, b_s$, such that

$$m(\alpha_r) \geq r + s\lfloor \varepsilon' \log q_r \rfloor,$$

for any sufficiently large integer r in $\mathcal{R}$.

Then, arguing as in the proof of Theorem 3.2 of [2] (see also Theorem 1.3 of [7]), it follows from the Schmidt Subspace Theorem that ξ is either transcendental or quadratic (we omit the details). But if ξ is quadratic, it must be in $\mathcal{E}_\tau$. This proves the theorem. □

Unlike in Theorem 3.2 of [2] (see also Theorem 1.3 of [7]), we do not need in Theorem 4.4 an assumption on the growth of the denominators of the convergents to ξ. Indeed, here and unlike in those papers, the number of times that the block $b_1, \ldots, b_s$ is repeated is not only at least equal to some absolute constant times r, but also at least equal to some absolute constant times $\log q_r$.

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