On the Littlewood conjecture in simultaneous Diophantine approximation

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Abstract. *For any given real number α with bounded partial quotients, we construct explicitly continuum many real numbers β with bounded partial quotients for which the pair (α, β) satisfies a strong form of the Littlewood conjecture. Our proof is elementary and rests on the basic theory of continued fractions.*

1. Introduction

It follows from the theory of continued fractions that, for any real number α, there exist infinitely many positive integers q such that

$$q \cdot \|q\alpha\| < 1, \tag{1.1}$$

where $\| \cdot \|$ denotes the distance to the nearest integer. In particular, for any given pair (α, β) of real numbers, there exist infinitely many positive integers q such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| < 1.$$

A famous open problem in simultaneous Diophantine approximation, called the Littlewood conjecture [8], claims that in fact, for any given pair (α, β) of real numbers, a stronger result holds, namely

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0. \tag{1.2}$$

Throughout the present Note, we denote by $\boldsymbol{Bad}$ the set of badly approximable numbers, that is,

$$\boldsymbol{Bad} = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

It is well-known that a real number lies in $\boldsymbol{Bad}$ if, and only if, it has bounded partial quotients in its continued fraction expansion. It then follows that the Littlewood conjecture holds true for the pair (α, β) if α or β has unbounded partial quotients in its continued

2000 *Mathematics Subject Classification :* 11J13, 11J70.

* Supported by the Austrian Science Fundation FWF, grant M822-N12.

fraction expansion. It also holds when the numbers 1, α, and β are linearly dependent over the rational integers, as follows from (1.1).

The first significant contribution towards the Littlewood conjecture goes back to Cassels & Swinnerton-Dyer [3] who showed that (1.2) holds when α and β belong to the same cubic field. However, since it is still not known whether or not cubic real numbers have bounded partial quotients, their result does not yield examples of pairs of badly approximable real numbers for which the Littlewood conjecture holds.

In view of the above discussion, it is natural to restrict our attention to independent parameters α and β, both lying in $\boldsymbol{Bad}$. The present paper is mainly devoted to the study of the following problem:

Question 1. Given α in $\boldsymbol{Bad}$, is there any independent β in $\boldsymbol{Bad}$ so that the Littlewood conjecture is true for the pair (α, β)?

Apparently, Question 1 remained unsolved until 2000. It has then been answered positively by Pollington & Velani [12], who established the following stronger result.

Theorem PV. *Given α in $\boldsymbol{Bad}$, there exists a subset $A(\alpha)$ of $\boldsymbol{Bad}$ with Hausdorff dimension one, such that, for any β in $A(\alpha)$, there exist infinitely many positive integers q with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{\log q}. \tag{1.3}$$

In particular, the Littlewood conjecture holds for the pair (α, β) for any β in $A(\alpha)$.

The proof of Theorem PV depends on sophisticated tools from metric number theory. At the end of [12], Pollington & Velani gave an alternative proof of a weaker version of Theorem PV, namely with (1.3) replaced by (1.2). However, even for establishing this weaker version, deep tools from metric number theory are still needed, including a result of Davenport, Erdős and LeVeque on uniform distribution [4] and the *Kaufman measure* constructed in [7].

Very recently, Einsiedler, Katok & Lindenstrauss [6] proved the outstanding result that the set of pairs of real numbers for which the Littlewood conjecture does not hold has Hausdorff dimension zero. Obviously, this implies a positive answer to Question 1. Actually, the authors established part of the Margulis conjecture on ergodic actions on the homogeneous space $SL_k(\mathbf{R})/SL_k(\mathbf{Z})$, for $k \geq 3$ (see [9]). It was previously well-known that such a result would have implications to Diophantine questions, including to the Littlewood conjecture. Their sophisticated proof uses, among others, deep tools from algebra and from the theory of dynamical systems, involving in particular the important work developed by Ratner (see for instance [14]).

The main purpose of the present Note is to provide a new, short and elementary, positive answer to a strong form of Question 1. We will only make use of the basic theory of continued fractions. Furthermore, our approach is constructive and allows us to give, for any real number α in $\boldsymbol{Bad}$, continuum many explicit examples of pairs (α, β) of numbers in $\boldsymbol{Bad}$ satisfying the Littlewood conjecture, with 1, α and β linearly independent over the rationals.

2. Main results

Before stating our main result, we recall the obvious fact that, for any given α and β in $\boldsymbol{Bad}$, there exists a positive constant $c(\alpha, \beta)$ such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \geq \frac{c(\alpha, \beta)}{q}, \tag{2.1}$$

for any positive integer q. Our Theorem 1 gives a positive answer to a strong form of Question 1 and solves a question posed by de Mathan at the end of [10].

Theorem 1. *Let φ be a positive, non-increasing function defined on the set of positive integers and satisfying $\varphi(1) = 1$, $\lim_{q \to +\infty} \varphi(q) = 0$ and $\lim_{q \to +\infty} q\varphi(q) = +\infty$. Given α in $\boldsymbol{Bad}$, there exists an uncountable subset $B_\varphi(\alpha)$ of $\boldsymbol{Bad}$ such that, for any β in $B_\varphi(\alpha)$, there exist infinitely many positive integers q with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{q \cdot \varphi(q)}. \tag{2.2}$$

In particular, the Littlewood conjecture holds for the pair (α, β) for any β in $B_\varphi(\alpha)$. Furthermore, the set $B_\varphi(\alpha)$ can be effectively constructed.

To the best of our knowledge, the first explicit examples of independent pairs of real numbers (α, β) satisfying the Littlewood conjecture, with α and β both lying in $\boldsymbol{Bad}$, have been recently given by de Mathan in [10]. In particular, for any quadratic real number α, the method introduced by de Mathan allows him to construct an independent β in $\boldsymbol{Bad}$ such that the pair (α, β) satisfies the Littlewood conjecture. However, his results yield a positive answer to Question 1 only for a very restricted class of real numbers α.

The proof of Theorem 1 is elementary, in the sense that it rests only on the theory of continued fractions. For given α and φ, we construct inductively the sequence of partial quotients of a suitable real number β such that (2.2) holds for the pair (α, β). This sequence can easily be explicited, as we show now.

Throughout this Note, we identify any finite or infinite word $W = w_1 w_2 \ldots$ on the alphabet $\{1, 2, \ldots\} = \mathbf{Z}_{\geq 1}$ with the sequence of partial quotients $w_1, w_2, \ldots$ Further, if $U = u_1 \ldots u_m$ and $V = v_1 v_2 \ldots$ are words on $\mathbf{Z}_{\geq 1}$, with V finite or infinite, then $[0; U, V]$ denotes the continued fraction $[0; u_1, \ldots, u_m, v_1, v_2, \ldots]$. The mirror image of any finite word $W = w_1 \ldots w_m$ is denoted by $\overline{W} := w_m \ldots w_1$.

Theorem 2. *Let $M \geq 2$ be an integer and ε be a positive real number with $\varepsilon < 1$. Let $\alpha := [0; a_1, a_2, \ldots]$ be in $\boldsymbol{Bad}$ with partial quotients bounded from above by M. For any positive integer n, denote by A_n the finite word $a_1 a_2 \ldots a_n$. Let $(t_i)_{i \geq 1}$ be any sequence with values in the set $\{M+1, M+2\}$, and let $(n_i)_{i \geq 1}$ be any sequence of positive integers satisfying*

$$\liminf_{i \to +\infty} \frac{n_{i+1}}{n_i} > \frac{4 \log(M+3)}{\varepsilon \log 2}. \tag{2.3}$$

Set

$$\beta = [0; \overline{A}_{n_1}, t_1, \overline{A}_{n_2}, t_2, \overline{A}_{n_3}, t_3, \ldots]. \tag{2.4}$$

Then, 1, α and β are linearly independent over the rationals, and there exist infinitely many positive integers q such that

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{q^{1-\varepsilon}}. \tag{2.5}$$

In particular, the Littlewood conjecture holds for the pair (α, β).

With a slight change in their construction, we may ensure that the real numbers β satisfying the conclusion of Theorem 2 are transcendental. Indeed, keep the notation of that theorem and set $B_1 = \overline{A}_{n_1}$ and $B_j := \overline{A}_{n_1} t_1 \overline{B}_1 \overline{A}_{n_2} t_2 \overline{B}_2 \ldots \overline{B}_{j-1} \overline{A}_{n_j}$ for any $j \geq 2$. Then, the real number

$$\beta = [0; \overline{A}_{n_1}, t_1, \overline{B}_1, \overline{A}_{n_2}, t_2, \overline{B}_2 \overline{A}_{n_3}, t_3, \overline{B}_3 \ldots]$$

begins in infinitely many palindromes, hence, by Theorem 1 from [1], it is transcendental. To reach the full conclusion of Theorem 2 with these β, it is then sufficient to slightly weaken (2.3).

We point out that a slight modification of the proof of Theorem 1 yields the following result.

Theorem 3. *Let φ be as in Theorem 1. Let M be a positive real number. Let $\mathcal{A}$ be a countable subset of **Bad** such that the partial quotients of every element of $\mathcal{A}$ are bounded by M. Then, there exists an uncountable subset $B_\varphi(\mathcal{A})$ of **Bad** such that, for any α in $\mathcal{A}$ and for any β in $B_\varphi(\mathcal{A})$, there exist infinitely many positive integers q with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{q \cdot \varphi(q)}.$$

Furthermore, the set $B_\varphi(\mathcal{A})$ can be effectively constructed.

To establish Theorem 3, it is sufficient to follow the proof of Theorem 1, but, instead of working with the same α at each step, to work alternatively with each element of $\mathcal{A}$. We omit the details.

Actually, the method for proving Theorem 1 gives us much freedom, and allows us to get various results in the same spirit as Theorem 2. Some of them will be stated in Section 4, with a particular focus on the case when α and β are equivalent real numbers. Section 5 is devoted to additional remarks and comments.

3. Proofs of Theorems 1 and 2

For the reader convenience, we recall some well-known results from the theory of continued fractions, whose proofs can be found e.g. in the book of Perron [11].

Lemma 1. *Let $\alpha = [0; a_1, a_2, \ldots]$ be a real number with convergents $(p_j/q_j)_{j\geq 1}$. Then, for any $j \geq 2$, we have*

$$\frac{q_{j-1}}{q_j} = [0; a_j, a_{j-1}, \ldots, a_1].$$

Lemma 2. *Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers. Assume that there exists a positive integer n such that $a_i = b_i$ for any $i = 1, \ldots, n$. We then have $|\alpha - \beta| \leq q_n^{-2}$, where q_n denotes the denominator of the n-th convergent to α.*

For positive integers $a_1, \ldots, a_m$, denote by $K_m(a_1, \ldots, a_m)$ the denominator of the rational number $[0; a_1, \ldots, a_m]$. It is commonly called a *continuant.*

Lemma 3. *For any positive integers $a_1, \ldots, a_m$ and any integer k with $1 \leq k \leq m-1$, we have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) &\leq K_m(a_1, \ldots, a_m) \\ &\leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m). \end{aligned}$$

Lemma 4. *Let $(a_i)_{i\geq 1}$ be a sequence of positive integers at most equal to M. For any positive integer n, we have*

$$2^{(n-1)/2} \leq K_n(a_1, \ldots, a_n) \leq (M+1)^n.$$

We further need the following auxiliary result.

Lemma 5 . *Let M be a positive real number. Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers whose partial quotients are at most equal to M. Assume that there exists a positive integer n such that $a_i = b_i$ for any $i = 1, \ldots, n$ and $a_{n+1} \neq b_{n+1}$. Then, we have*

$$|\alpha - \beta| \geq \frac{1}{(M+2)^3 q_n^2},$$

where q_n denotes the denominator of the n-th convergent to α.

Proof. Set $\alpha' = [a_{n+1}; a_{n+2}, \ldots]$ and $\beta' = [b_{n+1}; b_{n+2}, \ldots]$. Since $a_{n+1} \neq b_{n+1}$, we have

$$|\alpha' - \beta'| \geq 1 - [0; 1, M+1] = \frac{1}{M+2}. \tag{3.1}$$

Furthermore, since the partial quotients of both α and β are bounded by M, we immediately obtain

$$\alpha' \leq M+1 \quad \text{and} \quad \beta' \leq M+1. \tag{3.2}$$

Denote by $(p_j/q_j)_{j\geq 1}$ the sequence of convergents to α. Then, the theory of continued fractions gives that

$$\alpha = \frac{p_n\alpha' + p_{n-1}}{q_n\alpha' + q_{n-1}} \quad \text{and} \quad \beta = \frac{p_n\beta' + p_{n-1}}{q_n\beta' + q_{n-1}},$$

since the first n-th partial quotients of α and β are assumed to be the same. We thus obtain

$$|\alpha - \beta| = \left| \frac{p_n\alpha' + p_{n-1}}{q_n\alpha' + q_{n-1}} - \frac{p_n\beta' + p_{n-1}}{q_n\beta' + q_{n-1}} \right| = \left| \frac{\alpha' - \beta'}{(q_n\alpha' + q_{n-1})(q_n\beta' + q_{n-1})} \right|.$$

Together with (3.1) and (3.2), this yields

$$|\alpha - \beta| \geq \frac{1}{(M+2)^3 q_n^2},$$

concluding the proof of the lemma. □

We can now proceed with the proofs of Theorems 1 and 2.

Proof of Theorem 1. Write $\alpha = [0; a_1, a_2, \ldots, a_k, \ldots]$. We first construct inductively a rapidly increasing sequence $(n_j)_{j\geq 1}$ of positive integers. We set $n_1 = 1$ and we proceed with the inductive step. Assume that $j \geq 2$ is such that $n_1, \ldots, n_{j-1}$ have been constructed. Then, we choose n_j sufficiently large in order that

$$\varphi(2^{(m_j-1)/2}) \leq \frac{1}{4} \cdot \left(\frac{1}{(M+3)^{m_{j-1}+1}} \right)^2, \tag{3.3}$$

where $m_j = n_1 + n_2 + \ldots + n_j + (j-1)$. Such a choice is always possible since φ tends to zero at infinity and since the right hand side of (3.3) depends only on $n_1, n_2, \ldots, n_{j-1}$.

Our sequence $(n_j)_{j\geq 1}$ being now constructed, for an arbitrary integer sequence $\mathbf{t} = (t_k)_{k\geq 1}$ with values in $\{M+1, M+2\}$, we set

$$\begin{aligned} \beta_{\mathbf{t}} =& [0; b_1, b_2, \ldots] \\ =& [0; a_{n_1}, \ldots, a_1, t_1, a_{n_2}, \ldots, a_1, t_2, a_{n_3}, \ldots, a_1, \ldots, a_1, t_{j-1}, a_{n_j}, \ldots]. \end{aligned}$$

Then, we introduce the set

$$B_\varphi(\alpha) = \left\{ \beta_{\mathbf{t}},\ \mathbf{t} \in \{M+1, M+2\}^{\mathbf{Z}_{\geq 1}} \right\}.$$

Clearly, the set $B_\varphi(\alpha)$ is uncountable.

Let β be in $B_\varphi(\alpha)$. It remains for us to prove that (2.2) with this pair (α, β) holds for infinitely many integers q. Denote by $(p_j/q_j)_{j\geq 1}$ (resp. by $(r_j/s_j)_{j\geq 1}$) the sequence of convergents to α (resp. to β). We infer from Lemma 1 that

$$\frac{s_{m_j-1}}{s_{m_j}} = [0; a_1, \ldots, a_{n_j}, t_{j-1}, a_1, \ldots, a_{n_{j-1}}, t_{j-2}, \ldots, t_1, a_1, \ldots, a_{n_1}], \tag{3.4}$$

which, by Lemma 2, yields

$$\|s_{m_j}\alpha\| \le s_{m_j}\, q_{n_j}^{-2}. \tag{3.5}$$

Now, we proceed to bound $s_{m_j}\, q_{n_j}^{-2}$ from above.

By Lemma 4, we have

$$s_{m_j} \ge 2^{(m_j-1)/2}$$

and, since the partial quotients of β are bounded by $M+2$, we also get

$$K_{m_{j-1}+1}(b_1, \dots, b_{m_{j-1}+1}) < (M+3)^{m_{j-1}+1}.$$

Thus, using that φ is non-increasing, inequality (3.3) implies that

$$4\,\varphi(s_{m_j}) \le K_{m_{j-1}+1}(b_1, \dots, b_{m_{j-1}+1})^{-2} = K_{m_j-n_j}(b_1, \dots, b_{m_j-n_j})^{-2} \tag{3.6}$$

holds. However, we infer from Lemma 3 that

$$s_{m_j} \le 2\, K_{m_j-n_j}(b_1, \dots, b_{m_j-n_j})\, K_{n_j}(b_{m_j-n_j+1}, \dots, b_{m_j}) \tag{3.7}$$

and

$$K_{n_j}(b_{m_j-n_j+1}, \dots, b_{m_j}) = K_{n_j}(a_1, \dots, a_{n_j}) = q_{n_j}. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we obtain that

$$q_{n_j}^{-2} \le \frac{1}{s_{m_j}^2\, \varphi(s_{m_j})},$$

which, together with (3.5), yields

$$s_{m_j} \cdot \|s_{m_j}\alpha\| \cdot \|s_{m_j}\beta\| \le \|s_{m_j}\alpha\| \le s_{m_j}\, q_{n_j}^{-2} \le \frac{1}{s_{m_j}\, \varphi(s_{m_j})}.$$

This shows that (2.2) holds for infinitely many positive integers q and completes the proof of Theorem 1. □

We now turn to the proof of Theorem 2.

Proof of Theorem 2. With the notation of Theorem 1, we have $\varphi(q) = q^{-\varepsilon}$ for any positive integer q, thus, inequality (3.6) becomes

$$s_{m_j} \ge \left(2\, K_{m_{j-1}+1}(b_1, \dots, b_{m_{j-1}+1})\right)^{2/\varepsilon}. \tag{3.9}$$

To satisfy (3.9), it follows from (3.7), (3.8) and the equality $m_j = m_{j-1} + n_j + 1$ that it is sufficient to choose n_j such that

$$2^{(m_{j-1}+n_j)/2} \ge \left(2(M+3)^{m_{j-1}+1}\right)^{2/\varepsilon},$$

and thus, such that

$$n_j \geq m_{j-1} \left(\frac{4 \log(M+3)}{\varepsilon \log 2} - 1 \right) + \frac{4 \log(2(M+3))}{\varepsilon \log 2}. \tag{3.10}$$

Our assumption (2.3) implies that (3.10) is satisfied for any sufficiently large j. Consequently, (2.5) holds with β given by (2.4) for any integer $q = s_{m_j}$ large enough.

It thus only remains to prove that 1, α and β are independent over the rationals. Therefore, we assume that they are dependent and we aim at deriving a contradiction. Let (A, B, C) be a non-zero integer triple satisfying

$$A\alpha + B\beta + C = 0.$$

Then, for any positive integer q, we have

$$\|qA\alpha\| = \|qB\beta\|.$$

In particular, we get

$$\|s_{m_j} A\alpha\| = \|s_{m_j} B\beta\| \leq |B| \cdot \|s_{m_j}\beta\| \ll \frac{1}{s_{m_j}}, \tag{3.11}$$

for any $j \geq 2$. Here and below, the constant implied by $\ll$ does not depend on j.

On the other hand, we have constructed the sequence $(n_j)_{j\geq 1}$ in order to guarantee that

$$|s_{m_j}\alpha - s_{m_j-1}| \leq \frac{1}{s_{m_j}\varphi(s_{m_j})}. \tag{3.12}$$

Since by assumption $b_{m_{j-1}+1} = t_{j-1}$ lies in the set $\{M+1, M+2\}$, we have $b_{m_{j-1}+1} \neq a_{n_j+1}$. Then, (3.4) and Lemma 5 imply that

$$|s_{m_j}\alpha - s_{m_j-1}| \geq \frac{s_{m_j}}{(M+5)^3 q_{n_j}^2}.$$

Moreover, by Lemmas 3 and 4, we obtain

$$\begin{aligned} s_{m_j} &= K_{m_j}(a_1, \ldots, a_{n_j}, b_{m_{j-1}+1}, a_1, \ldots, a_{n_{j-1}}, b_{m_{j-2}+1}, a_1, \ldots, a_{n_1}) \\ &\geq K_{n_j}(a_1, \ldots, a_{n_j}) \cdot K_{m_j - n_j}(b_{m_{j-1}+1}, a_1, \ldots, a_{n_{j-1}}, b_{m_{j-2}+1}, a_1, \ldots, a_{n_1}) \\ &\geq q_{n_j} 2^{m_{j-1}/2}, \end{aligned}$$

hence, we get

$$|s_{m_j}\alpha - s_{m_j-1}| \gg \frac{2^{m_{j-1}}}{s_{m_j}}. \tag{3.13}$$

For j large enough, we deduce from (3.12) that

$$|s_{m_j}A\alpha - s_{m_j-1}A| < \frac{1}{2},$$

thus,

$$\|s_{m_j} A\alpha\| = |s_{m_j} A\alpha - s_{m_j - 1} A| = |A| \cdot |s_{m_j}\alpha - s_{m_j - 1}|.$$

By (3.13), this yields

$$\|s_{m_j} A\alpha\| \gg \frac{2^{m_{j-1}}}{s_{m_j}},$$

which contradicts (3.11). This concludes the proof of Theorem 2. □

4. Pairs of equivalent numbers

Two real irrational numbers α and β are said to be equivalent (resp. equal up to a rational homography) if there exist integers a, b, c and d with $|ad - bc| = 1$ (resp. with $|ad - bc| \neq 0$) such that

$$\beta = \frac{a\alpha + b}{c\alpha + d}.$$

A classical result (see e.g. [11]) asserts that two real numbers are equivalent if, and only if, their continued fraction expansions coincide, up to finitely many partial quotients. Consequently, if α is in $\boldsymbol{Bad}$, then this is also the case for any real number β equivalent to α. Note that if α is a quadratic real number and if β is a real number equivalent to α, then α and β are dependent over the rational integers. Thus, the Littlewood conjecture holds obviously for any pair of quadratic, equivalent real numbers. Moreover, it is easy to see that if α denotes a non-quadratic irrational number and if β is equivalent to α, then 1, α and β are independent over the rationals, except if there exists an integer m such that $\beta = \pm\alpha + m$. In particular, α and $1/\alpha$ are equivalent and independent, for any non-quadratic irrational number α.

In the present section, we ask wether the Littlewood conjecture is true for any pair of equivalent real numbers. This seemingly innocuous problem is still open, and nothing more is known on it than on the general conjecture, up to the following remark: the Littlewood conjecture is true for the pair $(\alpha, 1/\alpha)$ as soon as α is well approximable by quadratic numbers [10] (this observation originates in the work of M. Queffélec [13] where she proved the transcendence of the Thue–Morse continued fraction). Actually, this result can be slightly refined: under the same assumption on α, the Littlewood conjecture is true for the pair (α, β), where β is any number equivalent to α. We give an explicit related statement in Theorem 4 below and describe in Theorem 5 another class of real numbers α such that the Littlewood conjecture holds for any pair of equivalent parameters (α, β).

In the sequel, we denote by $|W|$ the length of a finite word W. Furthermore, for any positive rational number x, we denote by W^x the word $W^{[x]}W'$, where W' is the prefix of W of length $\lceil (x - [x])|W| \rceil$ and $\lceil y \rceil$ denotes the least integer greater than or equal to y.

Theorem 4. *Let α be in $\boldsymbol{Bad}$ and denote by $(p_n/q_n)_{n \geq 1}$ the sequence of its convergents. Assume that there exist a positive rational number x and a sequence of finite words $(U_k)_{k \geq 1}$*

such that, for every $k \geq 1$, the continued fraction expansion of α begins in $[0; U_k, U_k^x]$ and $|U_{k+1}| > |U_k|$. Set further $M = \limsup_{\ell \to +\infty} q_\ell^{1/\ell}$ and $m = \liminf_{\ell \to +\infty} q_\ell^{1/\ell}$. If we have

$$x \geq 1 \quad \text{or} \quad x > \frac{1}{2} \cdot \frac{\log M}{\log m}, \tag{4.1}$$

then the Littlewood conjecture is true for the pair (α, β), where β is any number equal to α up to a rational homography.

Proof. We content ourselves to outline the proof. We first recall the dual form of the Littlewood conjecture (see Lemma 5 from [3]). Given two real numbers α and β, then (1.2) is equivalent to the following equality

$$\inf_{(A,B) \in \mathbf{Z} \times \mathbf{Z} \setminus \{(0,0)\}} \max\{|A|, 1\} \cdot \max\{|B|, 1\} \cdot \|A\alpha + B\beta\| = 0.$$

Let α be in ***Bad*** and denote by $(p_n/q_n)_{n \geq 1}$ the sequence of its convergents. For any positive integer k, the quadratic number $\alpha_k := [0; U_k, U_k, \ldots]$ is very close to α. Setting $r_k := |U_k|$, we get $q_{r_k - 1}\alpha_k^2 + (q_{r_k} - p_{r_k - 1})\alpha_k - p_{r_k} = 0$ and

$$|q_{r_k-1}\alpha^2 + (q_{r_k} - p_{r_k-1})\alpha - p_{r_k}| \ll q_{r_k}|\alpha_k - \alpha| \ll q_{r_k}\, q_{(1+x)r_k}^{-2},$$

where, as below, the numerical constant implied in $\ll$ depends on α, but is independent from k. Then, by (4.1) and Lemma 3, there exists $\varepsilon > 0$ such that

$$|q_{r_k-1}\alpha^2 + (q_{r_k} - p_{r_k-1})\alpha - p_{r_k}| \ll q_{r_k}|\alpha_k - \alpha| \ll q_{r_k}^{-2-\varepsilon}. \tag{4.2}$$

Let $\beta = (a\alpha + b)/(c\alpha + d)$ be a number equal to α up to a rational homography. Set $\delta = ad - bc$,

$$A_k = q_{r_k-1}\,, \quad B_k = \delta\big(d\big((q_{r_k} - p_{r_k-1})c - q_{r_k-1}d\big) + p_{r_k}c^2\big)$$

and

$$C_k = \delta b\big((q_{r_k} - p_{r_k-1})c - q_{r_k-1}d\big) + \delta p_{r_k} ca.$$

Then, an easy calculation shows that

$$\|A_k\alpha + B_k\beta\| = |A_k\alpha + B_k\beta - C_k| = \frac{c}{c\alpha + d}\,|q_{r_k-1}\alpha^2 + \ (q_{r_k} - p_{r_k-1})\alpha - p_{r_k}|.$$

Since $|A_k| \ll q_{r_k}$ and $|B_k| \ll q_{r_k}$, it thus follows from (4.2) that

$$\max\{|A_k|, 1\} \cdot \max\{|B_k|, 1\} \cdot \|A_k\alpha + B_k\beta\| \ll q_{r_k}^{-\varepsilon}.$$

This proves that the dual form of the Littlewood conjecture, and thus the Littlewood conjecture, holds for the pair (α, β). $\square$

In Theorem 4, we used repetition to construct suitable real numbers α. Another useful combinatorial tool is palindromy. We recall that a palindrome is a finite word W such that $\overline{W} = W$.

Theorem 5. *Let α be in **Bad** and denote by $(p_n/q_n)_{n\geq 1}$ the sequence of its convergents. Assume that there exist a positive rational number x and two sequences of finite words $(U_k)_{k\geq 1}$ and $(V_k)_{k\geq 1}$ such that, for every $k \geq 1$, the continued fraction expansion of α begins in $[0; V_k, U_k, \overline{U}_k]$ and $|U_{k+1}| > |U_k| \geq x|V_k|$. Set further $M = \limsup_{\ell\to+\infty} q_\ell^{1/\ell}$ and $m = \liminf_{\ell\to+\infty} q_\ell^{1/\ell}$. If we have*

$$x > \frac{3}{2} \cdot \frac{\log M}{\log m} - \frac{1}{2}, \tag{4.3}$$

then the Littlewood conjecture is true for the pair (α, β), where β is any number equal to α up to a rational homography.

Proof. Let $\beta = (a\alpha + b)/(c\alpha + d)$ be a number equal to α up to a rational homography. Let $k \geq 1$ be an integer and let P_k/Q_k be the last convergent to the rational number

$$\frac{P'_k}{Q'_k} := [0; V_k, U_k, \overline{U}_k, \overline{V}_k].$$

It follows from Lemma 1 that $P'_k = Q_k$. Setting $r_k = |U_k|$ and $s_k = |V_k|$, we infer from Lemma 2 that

$$\max\{\|Q_k\alpha\|, \|Q'_k\alpha\|\} \ll Q_k\, q_{s_k+2r_k}^{-2}. \tag{4.4}$$

Here and below, the constants implied by $\ll$ may depend on α and β, but not on k. Observe that

$$\left|\beta - \frac{aP'_k + bQ'_k}{cP'_k + dQ'_k}\right| \ll \left|\alpha - \frac{P'_k}{Q'_k}\right|.$$

Thus, setting

$$R_k := |cP'_k + dQ'_k| = |cQ_k + dQ'_k|,$$

we get $R_k \ll Q_k$ and

$$\max\{\|R_k\alpha\|, \|R_k\beta\|\} \ll Q_k\, q_{s_k+2r_k}^{-2}, \tag{4.5}$$

using (4.4). Furthermore, Lemma 3 implies that

$$Q_k \ll K_{2(r_k+s_k)}(V_k U_k \overline{U_k}\, \overline{V_k}) \ll q_{s_k}\, q_{s_k+2r_k}.$$

Then, it follows from (4.5) that

$$R_k \cdot \|R_k\alpha\| \cdot \|R_k\beta\| \ll q_{s_k}^3\, q_{s_k+2r_k}^{-1}. \tag{4.6}$$

In virtue of (4.3), this concludes the proof. □

Actually, a sharper conclusion than (1.2) holds for the pairs (α, β) satisfying the conclusion of Theorem 5: there exists a positive real number $\varepsilon < 1$, depending on x, such that (2.5) holds for infinitely many positive integers q. This can be further refined under the strongest assumption that α begins in arbitrarily large palindromes.

Theorem 6. *Let α be in* $\boldsymbol{Bad}$ *such that its continued fraction expansion begins in infinitely many palindromes. Let β be any real number equal to α up to a rational homography. Then, the Littlewood conjecture is true for the pair (α, β) and, moreover, we have*

$$\liminf_{q \to +\infty} q^2 \cdot \|q\alpha\| \cdot \|q\beta\| < +\infty.$$

The proof of Theorem 6 follows the same lines as that of Theorem 5: it essentially amounts to setting $s_k = 0$ in (4.6).

For α being as in Theorem 6, the fact that the Littlewood conjecture is true for the pair $(\alpha, 1/\alpha)$ has previously been noticed by M. Queffélec in her talk held at the I.H.P. in June 2004.

5. Concluding remarks

For the reader convenience, we reformulate inequality (2.1). For any given α and β, both lying in $\boldsymbol{Bad}$, there exists a positive constant $c(\alpha, \beta)$ such that

$$q^2 \cdot \|q\alpha\| \cdot \|q\beta\| > c(\alpha, \beta),$$

for any positive integer q. In view of this and of Theorem 1, we propose the following problem:

Question 2. Given α in $\boldsymbol{Bad}$, is there any independent β in $\boldsymbol{Bad}$ so that

$$\liminf_{q \to +\infty} q^2 \cdot \|q\alpha\| \cdot \|q\beta\| < +\infty \ ? \tag{5.1}$$

We observe that (5.1) holds when α and β are linearly dependent over the rationals, as follows from (1.1). Furthermore, Theorem 6 gives a positive answer to Question 2 for a restricted class of real numbers α. Apart from this partial result, we do not know the answer to Question 2.

Let $\mathbf{K}$ be any field, and let X be an indeterminate. We define a norm $|\cdot|$ on the field $\mathbf{K}((X^{-1}))$ by setting $|0| = 0$ and, for any non-zero formal power series $F(X) = \sum_{h=-m}^{+\infty} f_h X^{-h}$ with $f_m \neq 0$, by setting $|F| = 2^m$. We further write $||F||$ to denote the norm of the fractional part of $F(X)$, that is, of the part of the series which comprises only the negative powers of X. In analogy with the Littlewood conjecture, we may ask whether, given $F(X)$ and $G(X)$ in $\mathbf{K}((X^{-1}))$, we have

$$\inf_{q \in \mathbf{K}[X] \setminus \{0\}} |q| \cdot \|qF\| \cdot \|qG\| = 0.$$

A negative answer to this question has been obtained by Davenport & Lewis [5] (see also Baker [2] for an explicit counter-example) when $\mathbf{K}$ is an infinite field. The question is still unsolved when $\mathbf{K}$ is a finite field. We conclude by pointing out that our construction can also be applied to solve the analogue of Question 1 for formal power series defined over an arbitrary field. This will be part of a subsequent work.

Acknowledgements. We are very grateful to Bernard de Mathan for many useful remarks. In particular, the present version of Theorem 5 and its proof is due to him.

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