

SOME PÓLYA-TYPE IRREDUCIBILITY CRITERIA FOR MULTIVARIATE POLYNOMIALS

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ABSTRACT. We provide irreducibility criteria for multivariate polynomials with coefficients in an arbitrary field, that extend a classical result of Pólya for polynomials with integer coefficients. In particular, we provide irreducibility conditions for polynomials of the form $f(X)(Y - f_1(X)) \cdots (Y - f_n(X)) + g(X)$, with f, f_1, \dots, f_n, g univariate polynomials over an arbitrary field.

1. INTRODUCTION

There are many irreducibility criteria for multivariate polynomials in the literature, obtained by using various methods. Some of the classical techniques used in this respect require the study of Riemann surfaces and Puiseux expansions [17], the study of convex polyhedra [22], or the study of nonarchimedean absolute values over arbitrary fields [14] and [15].

Some more recent irreducibility results have been obtained for various classes of multivariate polynomials, such as difference polynomials [19], [16], [2] and [3], linear combinations of relatively prime polynomials [9], compositions of polynomials [1] and [6], multiplicative convolutions [5], polynomials having one coefficient of dominant degree [7], lacunary polynomials [4], and polynomials obtained from irreducible polynomials in fewer variables [8]. For an excellent account on the techniques used in the study of reducibility of polynomials over arbitrary fields, the reader is referred to Schinzel's book [20].

In [18] Pólya proved the following elegant irreducibility criterion for polynomials with integer coefficients.

Theorem (Pólya, 1919) *If for n integral values of x , the integral polynomial $f(x)$ of degree n has values which are different from zero, and in absolute value less than*

$$\frac{[n/2]!}{2^{\lceil n/2 \rceil}},$$

then $f(x)$ is irreducible over \mathbb{Q} .

The proof of this irreducibility criterion relies on the following lemma.

Lemma (Pólya, 1919) *Let $f(x)$ be an integral polynomial of degree n . Then among any pairwise distinct integers x_0, \dots, x_n there exists at least one, say x_i , such that $|f(x_i)| \geq n!/2^n$.*

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The aim of this paper is to provide several irreducibility criteria for multivariate polynomials, that extend in a natural way Pólya's Theorem. The first result we will prove is the following irreducibility criterion for bivariate polynomials over an arbitrary field.

Theorem 1.1. *Let K be a field and $F(X, Y) \in K[X, Y]$ with $\deg_Y F(X, Y) = n$. If for n polynomials $c_1, \dots, c_n \in K[X]$ with pairwise distinct degrees one has*

$$0 \leq \deg F(X, c_i(X)) < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2}, \quad i = 1, \dots, n, \quad (1)$$

then $F(X, Y)$ is irreducible over $K(X)$. The same conclusion holds if

$$0 \leq \deg F(X, c_i(X)) < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2}, \quad i = 1, \dots, n, \quad (2)$$

provided $c_1, \dots, c_n \in K[X]$ are non-zero polynomials with pairwise distinct degrees.

Let us note here that (1) makes sense only for $n \geq 3$, while (2) makes sense for $n \geq 2$. By Theorem 1.1, one may immediately deduce the following result.

Corollary 1.2. *Let K be a field. For any polynomials $f_1, \dots, f_n \in K[X]$ with pairwise distinct degrees, and any non-zero polynomials $f, g \in K[X]$ with*

$$\deg g < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2}, \quad (3)$$

the polynomial $F(X, Y) = f(X)(Y - f_1(X)) \cdots (Y - f_n(X)) + g(X)$ is irreducible over $K(X)$. The same conclusion holds if

$$\deg g < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2}, \quad (4)$$

provided $f_1, \dots, f_n \in K[X]$ are non-zero polynomials with pairwise distinct degrees.

We note here that there are many classical irreducibility criteria for polynomials of the form $a_0(X - a_1) \cdots (X - a_n) + a_{n+1}$, where the a_i are rational integers. Many of these criteria depend upon the arithmetical properties of the values taken by such a polynomial at some integral arguments. The first criterion of this kind was suggested by Schur [21], who raised the question of the irreducibility of the polynomials of the form $(X - a_1) \cdots (X - a_n) \pm 1$. For a unifying approach of the irreducibility criteria for polynomials of this type, we refer the interested reader to [10], [11] and [12].

Another immediate consequence of Theorem 1.1 is a similar result for polynomials in $r \geq 3$ variables X_1, X_2, \dots, X_r over K . For any polynomial $f \in K[X_1, \dots, X_r]$ we denote by $\deg_r f$ the degree of f as a polynomial in X_r with coefficients in $K[X_1, \dots, X_{r-1}]$. The next result follows from Theorem 1.1 by writing Y for X_r , X for X_{r-1} and by replacing K with $K(X_1, \dots, X_{r-2})$.

Theorem 1.3. *Let K be a field, $r \geq 3$, and let $F(X_1, \dots, X_r) \in K[X_1, \dots, X_r]$ with $\deg_r F = n$. If for n polynomials $c_1, \dots, c_n \in K[X_1, \dots, X_{r-1}]$ with $\deg_{r-1} c_1, \dots, \deg_{r-1} c_n$ pairwise distinct, one has*

$$0 \leq \deg_{r-1} F(X_1, \dots, X_{r-1}, c_i) < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2}, \quad i = 1, \dots, n, \quad (5)$$

then F is irreducible over $K(X_1, \dots, X_{r-1})$. The same conclusion holds if

$$0 \leq \deg_{r-1} F(X_1, \dots, X_{r-1}, c_i) < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2}, \quad i = 1, \dots, n, \quad (6)$$

provided c_1, \dots, c_n are non-zero polynomials with $\deg_{r-1} c_1, \dots, \deg_{r-1} c_n$ pairwise distinct.

For the proof of Theorem 1.1 we will need the following analogue of Pólya's Lemma.

Lemma 1.4. *Let K be a field and $F(X, Y) \in K[X, Y]$ with $\deg_Y F(X, Y) = n$. Then among any $n + 1$ polynomials $c_0, \dots, c_n \in K[X]$ with pairwise distinct degrees there exists at least one, say c_i , such that $\deg F(X, c_i(X)) \geq n(n - 1)/2$. Moreover, among any $n + 1$ non-zero polynomials $c_0, \dots, c_n \in K[X]$ with pairwise distinct degrees, there exists at least one, say c_i , such that $\deg F(X, c_i(X)) \geq n(n + 1)/2$.*

In fact, a result similar to Lemma 1.4 holds with less restrictive conditions on the degrees of the polynomials c_0, \dots, c_n , as follows.

Lemma 1.5. *Let K be a field and let $F(X, Y) \in K[X, Y]$ with $\deg_Y F(X, Y) = n$. Then among any $n + 1$ polynomials $c_0, \dots, c_n \in K[X]$ with pairwise distinct leading coefficients and $\deg c_0 \leq \deg c_1 \leq \dots \leq \deg c_n$, there exists at least one, say c_i , such that $\deg F(X, c_i(X)) \geq \deg c_1 + \dots + \deg c_n$.*

An immediate application of Lemma 1.5 gives the following related irreducibility criterion.

Theorem 1.6. *Let K be a field and let $F(X, Y) \in K[X, Y]$ with $\deg_Y F(X, Y) = n$. If for n polynomials $c_1, \dots, c_n \in K[X]$ with $\deg c_1 \leq \deg c_2 \leq \dots \leq \deg c_n$ and pairwise distinct leading coefficients one has*

$$0 \leq \deg F(X, c_i(X)) < \deg c_2 + \deg c_3 + \dots + \deg c_{\lceil n/2 \rceil + 1}, \quad i = 1, \dots, n, \quad (7)$$

then F is irreducible over $K(X)$.

Note that in some cases the price paid for using less restrictive conditions on the degrees of c_1, \dots, c_n may consist of stronger conditions on the degrees of $F(X, c_1(X)), \dots, F(X, c_n(X))$.

Corollary 1.7. *Let K be a field. For any polynomials $f_1, \dots, f_n \in K[X]$ with pairwise distinct leading coefficients and $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_n$, and any non-zero polynomials $f, g \in K[X]$ with*

$$\deg g < \deg f_2 + \deg f_3 + \dots + \deg f_{\lceil n/2 \rceil + 1}, \quad (8)$$

the polynomial $F(X, Y) = f(X)(Y - f_1(X)) \cdots (Y - f_n(X)) + g(X)$ is irreducible over $K(X)$.

By Theorem 1.6 we immediately obtain a similar result for polynomials in $r \geq 3$ variables X_1, X_2, \dots, X_r over K .

Theorem 1.8. *Let K be a field, $r \geq 3$, and let $F(X_1, \dots, X_r) \in K[X_1, \dots, X_r]$ with $\deg_r F = n$. If for n polynomials $c_1, \dots, c_n \in K[X_1, \dots, X_{r-1}]$ with pairwise distinct leading coefficients with respect to X_{r-1} and $\deg_{r-1} c_1 \leq \deg_{r-1} c_2 \leq \dots \leq \deg_{r-1} c_n$ one has*

$$0 \leq \deg_{r-1} F(X_1, \dots, X_{r-1}, c_i) < \deg_{r-1} c_2 + \deg_{r-1} c_3 + \dots + \deg_{r-1} c_{\lceil n/2 \rceil + 1} \quad (9)$$

for each $i = 1, \dots, n$, then F is irreducible over $K(X_1, \dots, X_{r-1})$.

The reader may naturally wonder how sharp the above results are. For instance, conditions (3) and (4) in Corollary 1.2 and condition (8) in Corollary 1.7 are best possible in the sense that there exist polynomials for which equality in (3) or (4), or (8) holds, and which are reducible over $K(X)$.

To see that the inequality sign in condition (3) can not be replaced by \leq when the c_i have pairwise distinct degrees, take for instance $K = \mathbb{Q}$ and $F_1(X, Y) = Y(Y-1)(Y-X) + 2X + 2$. In this case we have $n = \deg_Y F_1 = 3$, $\frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2} = 1$, $f(X) = 1$, $f_1(X) = 0$, $f_2(X) = 1$, $f_3(X) = X$ and $g(X) = 2X + 2$. Since $\deg g = 1$, we have equality in (3), while F_1 is obviously reducible, since $F_1(X, Y) = (Y + 1) \cdot (Y^2 - (X + 2)Y + 2X + 2)$.

Now, to see that the inequality sign in condition (4) can not be replaced by \leq when the c_i are non-zero polynomials with pairwise distinct degrees, let $K = \mathbb{Q}$ and consider the polynomial $F_2(X, Y) = (Y - 1)(Y - X) - 2X - 2$. Here $n = \deg_Y F_2 = 2$, $\frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2} = 1$, $f(X) = 1$, $f_1(X) = 1$, $f_2(X) = X$ and $g(X) = -2X - 2$. Since $\deg g = 1$, we have equality in (4), while F_2 is reducible too, since $F_2(X, Y) = (Y + 1) \cdot (Y - X - 2)$.

For an example related to Corollary 1.7, let $K = \mathbb{Q}$ and $F_3(X, Y) = (Y - X)(Y - 2X) - 2X^2 + 3X - 1$. Here $n = 2$, $\lceil n/2 \rceil + 1 = 2$, $f(X) = 1$, $f_1(X) = X$, $f_2(X) = 2X$, $\deg f_1 = \deg f_2$ and $g(X) = -2X^2 + 3X - 1$. We have equality in (8), while F_3 is obviously reducible, since $F_3(X, Y) = (Y - 1) \cdot (Y - 3X + 1)$.

Our results are quite flexible and may be used in various applications where other irreducibility criteria fail. The proofs of the main results are given in Section 2 below. We will also provide some examples of infinite families of irreducible polynomials in the last section of the paper.

2. PROOFS OF THE MAIN RESULTS

2.1. Proof of Lemma 1.4. Let $c_0, \dots, c_n \in K[X]$ be $n + 1$ arbitrary polynomials with pairwise distinct degrees. By Lagrange's interpolation formula, we may write F as

$$F(X, Y) = \sum_{i=0}^n F(X, c_i) \cdot \frac{(Y - c_0) \cdots (Y - c_{i-1})(Y - c_{i+1}) \cdots (Y - c_n)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)}. \quad (10)$$

Note that (10) makes sense, since the c_i are supposed to be pairwise distinct. By (10) we deduce that the leading coefficient of F , regarded as a polynomial in Y with coefficients

$a_i \in K[X]$, may be expressed as

$$a_n(X) = \sum_{i=0}^n \frac{F(X, c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)}. \quad (11)$$

We now introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, as follows. We fix an arbitrary real number $\rho > 1$, and for any polynomial $u(X) \in K[X]$ we define $|u(X)|$ by the equality

$$|u(X)| = \rho^{\deg u(X)}.$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity. Thus for any $w(X) \in K(X)$, $w(X) = \frac{u(X)}{v(X)}$, with $u(X), v(X) \in K[X]$, $v(X) \neq 0$, we let $|w(X)| = \frac{|u(X)|}{|v(X)|}$.

Let us note that for any non-zero element $u(X)$ of $K[X]$ one has $|u(X)| \geq 1$. In particular, since $a_n \neq 0$, we have

$$|a_n(X)| \geq 1. \quad (12)$$

The proof proceeds by reduction to absurd. Let us assume that $\deg F(X, c_i(X)) < n(n-1)/2$ for each $i = 0, \dots, n$. Then, by (11) and the fact that our absolute value is nonarchimedean, we deduce that

$$\begin{aligned} |a_n(X)| &= \left| \sum_{i=0}^n \frac{F(X, c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} \right| \\ &\leq \max_{0 \leq i \leq n} \left| \frac{F(X, c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} \right| \\ &\leq \frac{\max_{0 \leq i \leq n} |F(X, c_i)|}{\min_{0 \leq i \leq n} |(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)|} \\ &< \frac{\rho^{\frac{n(n-1)}{2}}}{\min_{0 \leq i \leq n} |(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)|}. \end{aligned}$$

Recall now that the degrees of c_0, \dots, c_n are pairwise distinct, so we may assume without loss of generality that $\deg c_0 < \deg c_1 < \cdots < \deg c_n$. In particular, this shows that $\deg c_i \geq i - 1$ for $i = 1, \dots, n$. We then deduce that

$$\begin{aligned} |a_n(X)| &< \frac{\rho^{\frac{n(n-1)}{2}}}{\min_{1 \leq i \leq n} |c_i|^i \cdot |c_{i+1}| \cdots |c_n|} = \frac{\rho^{\frac{n(n-1)}{2}}}{|c_1| \cdot |c_2| \cdots |c_n|} \\ &= \frac{\rho^{\frac{n(n-1)}{2}}}{\rho^{\deg c_1 + \deg c_2 + \cdots + \deg c_n}} \leq \frac{\rho^{\frac{n(n-1)}{2}}}{\rho^{1+2+\cdots+(n-1)}} = 1, \end{aligned}$$

which contradicts (12). This completes the proof of the first part of the lemma.

The proof of the second part of the lemma follows in a similar way, except that here the fact that $c_0, \dots, c_n \in K[X]$ are non-zero polynomials with pairwise distinct degrees implies $\deg c_i \geq i$ for $i = 0, \dots, n$, which allows one to replace the bound $n(n-1)/2$ by $n(n+1)/2$.

2.2. *Proof of Lemma 1.5.* Since $\deg c_0 \leq \deg c_1 \leq \dots \leq \deg c_n$, our assumption that the c_i have pairwise distinct leading coefficients shows that

$$\deg(c_i - c_j) = \deg c_{\max\{i,j\}} \quad (13)$$

for each pair of distinct indices i, j . On the other hand, the fact that the c_i have pairwise distinct leading coefficients shows that the c_i are pairwise distinct. In particular, at most one of the c_i , namely c_0 , may be the zero polynomial.

Now, assuming that $\deg F(X, c_i(X)) < \deg c_1 + \dots + \deg c_n$ for each $i = 0, \dots, n$ and using the method employed in the proof of Lemma 1.4, we deduce that

$$|a_n(X)| < \frac{\rho^{\deg c_1 + \dots + \deg c_n}}{\min_{0 \leq i \leq n} |(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)|}.$$

Using (13), we further obtain

$$|a_n(X)| < \frac{\rho^{\deg c_1 + \dots + \deg c_n}}{\min_{1 \leq i \leq n} |c_i|^i \cdot |c_{i+1}| \cdots |c_n|} = \frac{\rho^{\deg c_1 + \dots + \deg c_n}}{|c_1| \cdot |c_2| \cdots |c_n|} = 1,$$

again a contradiction, and this completes the proof.

2.3. *Proof of Theorem 1.1.* First of all let us note that condition $0 \leq \deg F(X, c_i(X))$ allows us to avoid the trivial case when F is divisible by $Y - c_i(X)$. Now let us assume, for the sake of a contradiction, that F as a polynomial in Y with coefficients in $K[X]$ is reducible over $K(X)$. By the celebrated Lemma of Gauss we may further assume that $F = G \cdot H$, with $G, H \in K[X, Y]$, $\deg_Y G \geq 1$, $\deg_Y H \geq 1$. Let now $m = \deg_Y G$. Without loss of generality we may assume that $\deg_Y G \geq \deg_Y H$, hence

$$\lceil n/2 \rceil \leq m < n.$$

Next, since $F(X, c_i(X)) = G(X, c_i(X))H(X, c_i(X))$ for each $i = 1, \dots, n$, we obtain

$$\begin{aligned} \deg G(X, c_i(X)) &\leq \deg F(X, c_i(X)) \\ &< \lceil n/2 \rceil (\lceil n/2 \rceil - 1)/2 \\ &\leq m(m-1)/2, \quad i = 1, \dots, n. \end{aligned}$$

In particular, since $n \geq m + 1$, we have

$$\deg G(X, c_i(X)) < m(m-1)/2, \quad i = 1, \dots, m+1. \quad (14)$$

On the other hand, according to Lemma 1.4, there must be an index $i \in \{1, \dots, m+1\}$ such that $\deg G(X, c_i(X)) \geq m(m-1)/2$, which contradicts (14), and this completes the proof of the first part of the theorem.

The second part of the proof follows in a similar way, except that instead of (14) one obtains

$$\deg G(X, c_i(X)) < m(m+1)/2, \quad i = 1, \dots, m+1,$$

which cannot hold, according to Lemma 1.4. This completes the proof of the theorem.

2.4. *Proof of Theorem 1.6.* Here too, the condition $0 \leq \deg F(X, c_i(X))$ allows us to avoid the trivial case when F is divisible by $Y - c_i(X)$. Let us assume as before that $F = G \cdot H$, with $G, H \in K[X, Y]$, $m = \deg_Y G \geq \deg_Y H \geq 1$, so

$$\lceil n/2 \rceil \leq m < n.$$

Next, since $F(X, c_i(X)) = G(X, c_i(X))H(X, c_i(X))$ for each $i = 1, \dots, n$, we deduce that

$$\begin{aligned} \deg G(X, c_i(X)) &\leq \deg F(X, c_i(X)) \\ &< \deg c_2 + \deg c_3 + \dots + \deg c_{\lceil n/2 \rceil + 1} \\ &\leq \deg c_2 + \deg c_3 + \dots + \deg c_{m+1}, \quad i = 1, \dots, n. \end{aligned}$$

In particular, since $n \geq m + 1$, we have

$$\deg G(X, c_i(X)) < \deg c_2 + \deg c_3 + \dots + \deg c_{m+1}, \quad i = 1, \dots, m + 1. \quad (15)$$

On the other hand, according to Lemma 1.5, there must be an index $i \in \{1, \dots, m + 1\}$ such that $\deg G(X, c_i(X)) \geq \deg c_2 + \deg c_3 + \dots + \deg c_{m+1}$, which contradicts (15), and this completes the proof of the theorem.

We will also prove a p -adic version of Pólya's Lemma, which might be of independent interest. For a rational prime p and an arbitrary nonzero rational number r , we will denote by $\nu_p(r)$ the exponent of p in the prime decomposition of r ($\nu_p(0) = \infty$). With this notation we have the following result.

Lemma 2.1. *Let $F(X) = a_0 + a_1X + \dots + a_nX^n \in \mathbb{Z}[X]$, and let p be a prime number. Then among any $n + 1$ integers c_0, c_1, \dots, c_n with $\nu_p(c_0) > \nu_p(c_1) > \dots > \nu_p(c_n)$ there exists at least one, say c_i , such that $\nu_p(F(c_i)) \leq \nu_p(a_n) + \nu_p(c_1) + \nu_p(c_2) + \dots + \nu_p(c_n)$.*

2.5 *Proof of Lemma 2.1.* We will first introduce a nonarchimedean absolute value $|\cdot|_p$ on \mathbb{Q} , as follows. For an arbitrary rational number r we define $|r|_p = p^{-\nu_p(r)}$. This absolute value satisfies $|ab|_p = |a|_p|b|_p$ and $|a + b|_p \leq \max\{|a|_p, |b|_p\}$, and it can be extended to a number field (see, for instance, [13]), but in our proof we will not need this feature of $|\cdot|_p$.

Let now $c_0, \dots, c_n \in \mathbb{Z}$ be $n + 1$ arbitrary integers with $\nu_p(c_0) > \nu_p(c_1) > \dots > \nu_p(c_n)$, that is

$$|c_0|_p < |c_1|_p < \dots < |c_n|_p. \quad (16)$$

Using again Lagrange's interpolation formula, one may write F as

$$F(X) = \sum_{i=0}^n F(c_i) \cdot \frac{(X - c_0) \cdots (X - c_{i-1})(X - c_{i+1}) \cdots (X - c_n)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)}. \quad (17)$$

Here too (17) makes sense, since the c_i in the statement of Lemma 2.1 must be pairwise distinct, so the leading coefficient of F may be expressed as

$$a_n = \sum_{i=0}^n \frac{F(c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)}. \quad (18)$$

Now, let us assume that $\nu_p(F(c_i)) > \nu_p(a_n) + \nu_p(c_1) + \nu_p(c_2) + \cdots + \nu_p(c_n)$ for each $i = 0, 1, \dots, n$, that is $\max_{0 \leq i \leq n} |F(c_i)|_p < |a_n|_p \cdot |c_1|_p \cdot |c_2|_p \cdots |c_n|_p$. Then, by (18) and the fact that our absolute value is nonarchimedean, we deduce that

$$\begin{aligned} |a_n|_p &= \left| \sum_{i=0}^n \frac{F(c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} \right|_p \\ &\leq \max_{0 \leq i \leq n} \left| \frac{F(c_i)}{(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)} \right|_p \\ &\leq \frac{\max_{0 \leq i \leq n} |F(c_i)|_p}{\min_{0 \leq i \leq n} |(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)|_p} \\ &< \frac{|a_n|_p \cdot |c_1|_p \cdot |c_2|_p \cdots |c_n|_p}{\min_{0 \leq i \leq n} |(c_i - c_0) \cdots (c_i - c_{i-1})(c_i - c_{i+1}) \cdots (c_i - c_n)|_p}. \end{aligned}$$

Since for $i < j$ one has $|c_i - c_j|_p = |c_j|_p$, we then deduce that

$$|a_n|_p < \frac{|a_n|_p \cdot |c_1|_p \cdot |c_2|_p \cdots |c_n|_p}{\min_{1 \leq i \leq n} |c_i|_p^i \cdot |c_{i+1}|_p \cdots |c_n|_p} = \frac{|a_n|_p \cdot |c_1|_p \cdot |c_2|_p \cdots |c_n|_p}{|c_1|_p \cdot |c_2|_p \cdots |c_n|_p} = |a_n|_p,$$

which is a contradiction. This completes the proof of the lemma.

We end this section by noting that no immediate application of Lemma 2.1 seems to be useful to derive similar irreducibility conditions in a p -adic setting, because unlike the usual absolute value and the nonarchimedean absolute value used in Theorem 1.1, with respect to divisibility, the p -adic modulus is a reverse ordering map, that is $|a|_p \geq |b|_p$ if a divides b .

3. EXAMPLES

1) Let K be a field, and let us consider now bivariate polynomials obtained by adding a linear factor in Y to the polynomial $F(X, Y)$ in Corollary 1.2. We will first consider polynomials of the form

$$F(X, Y) = f(X) \prod_{i=1}^n (Y - f_i(X)) + \sum_{i=1}^n g_i(X)(Y - f_i(X)),$$

with $f, f_1, \dots, f_n, g_1, \dots, g_n \in K[X]$, $f \neq 0$, and $\deg f_i \neq \deg f_j$ for $i \neq j$. It is easy to prove that if $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ and

$$\max_{1 \leq i \leq n} \deg f_i + \max_{1 \leq i \leq n} \deg g_i < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2},$$

then F must be irreducible over $K(X)$. If moreover $f_1 \cdots f_n \neq 0$ then the same conclusion will hold if

$$\max_{1 \leq i \leq n} \deg f_i + \max_{1 \leq i \leq n} \deg g_i < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2}.$$

To prove this, we observe that $F(X, f_j(X)) = \sum_{i=1}^n g_i(X)(f_j(X) - f_i(X))$, which shows that $\deg F(X, f_j(X)) \leq \max_{1 \leq i \leq n} \deg f_i + \max_{1 \leq i \leq n} \deg g_i$ for $j = 1, \dots, n$. The conclusion follows now by Theorem 1.1.

We note here that in Corollary 1.2 the degrees of f_1, \dots, f_n are only asked to be pairwise distinct, while in this case the degrees of f_1, \dots, f_n must be also bounded by the expression in the right side of (1) or (2).

2) One may also consider the case when the linear factor in Y added to the polynomials in Corollary 1.2 does not depend explicitly on the polynomials f_1, \dots, f_n . So let now

$$F(X, Y) = f(X) \prod_{i=1}^n (Y - f_i(X)) + h(X)Y + g(X),$$

with $f, g, h, f_1, \dots, f_n \in K[X]$, $f \neq 0$, and $\deg f_i \neq \deg f_j$ for $i \neq j$. Similarly, if one has $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ and

$$\max\{\deg g, \deg h + \max_{1 \leq i \leq n} \deg f_i\} < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil - 1)}{2},$$

then F must be irreducible over $K(X)$. If moreover $f_1 \cdots f_n \neq 0$ then the same conclusion will hold if

$$\max\{\deg g, \deg h + \max_{1 \leq i \leq n} \deg f_i\} < \frac{\lceil n/2 \rceil (\lceil n/2 \rceil + 1)}{2}.$$

In this case $\deg F(X, f_j(X)) \leq \max\{\deg g, \deg h + \max_{1 \leq i \leq n} \deg f_i\}$ for $j = 1, \dots, n$, and the conclusion follows again by Theorem 1.1. Note that we obviously have $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ if, for instance, $g \neq 0$ and g is not divisible by h .

3) Let us consider now the case when the degrees of f_1, \dots, f_n are not necessarily pairwise distinct. So let again

$$F(X, Y) = f(X) \prod_{i=1}^n (Y - f_i(X)) + \sum_{i=1}^n g_i(X)(Y - f_i(X)),$$

with $f, f_1, \dots, f_n, g_1, \dots, g_n \in K[X]$, $f \neq 0$, and assume that $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_n$, and that the leading coefficients of f_1, \dots, f_n are pairwise distinct. If $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ and

$$\deg f_n + \max_{1 \leq i \leq n} \deg g_i < \deg f_2 + \deg f_3 + \dots + \deg f_{\lceil n/2 \rceil + 1},$$

then F must be irreducible over $K(X)$.

Indeed, since $F(X, f_j(X)) = \sum_{i=1}^n g_i(X)(f_j(X) - f_i(X))$, we deduce that $\deg F(X, f_j(X)) \leq \deg f_n + \max_{1 \leq i \leq n} \deg g_i$ for $j = 1, \dots, n$, and the conclusion follows by Theorem 1.6.

4) Let now

$$F(X, Y) = f(X) \prod_{i=1}^n (Y - f_i(X)) + h(X)Y + g(X),$$

with $f, g, h, f_1, \dots, f_n \in K[X]$, $f \neq 0$, $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_n$, and assume that the leading coefficients of f_1, \dots, f_n are pairwise distinct. If $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ and

$$\max\{\deg h + \deg f_n, \deg g\} < \deg f_2 + \deg f_3 + \dots + \deg f_{\lceil n/2 \rceil + 1},$$

then F must be irreducible over $K(X)$.

Here $\deg F(X, f_j(X)) \leq \max\{\deg h + \deg f_n, \deg g\}$ for $j = 1, \dots, n$, and the conclusion follows again by Theorem 1.6.

5) One may easily formulate similar irreducibility conditions for polynomials obtained by adding terms of higher degree with respect to Y to the polynomials considered in Corollary 1.2 and Corollary 1.7. Consider, for instance, the polynomial

$$F(X, Y) = f(X) \prod_{i=1}^n (Y - f_i(X)) + \sum_{i=0}^k h_i(X) Y^i,$$

with $f, f_1, \dots, f_n, h_0, \dots, h_k \in K[X]$, $f \neq 0$, $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_n$, and assume that the leading coefficients of f_1, \dots, f_n are pairwise distinct. If $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$ and

$$\max_{0 \leq i \leq k} \{\deg h_i + i \deg f_n\} < \deg f_2 + \deg f_3 + \dots + \deg f_{\lceil n/2 \rceil + 1}, \quad (19)$$

then F must be irreducible over $K(X)$.

Here $\deg F(X, f_j(X)) \leq \max_{0 \leq i \leq k} \{\deg h_i + i \deg f_n\}$ for $j = 1, \dots, n$, and the conclusion follows again by Theorem 1.6. Note that since $\deg f_1 \leq \deg f_2 \leq \dots \leq \deg f_n$, (19) can not hold if $k \geq \lceil n/2 \rceil$, so one may obtain irreducibility criteria for polynomials of this type only if k is sufficiently small. We also note that we obviously have $F(X, f_j(X)) \neq 0$ for $j = 1, \dots, n$, if for instance, the polynomial $\sum_{i=0}^k h_i(X) Y^i$ has no linear factors over $K(X)$.

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