Transcendence criteria for pairs of continued fractions

Boris Adamczewski (Lyon) & Yann Bugeaud (Strasbourg)

Abstract. *The aim of the present note is to establish two extensions of some transcendence criteria for real numbers given by their continued fraction expansions. We adopt the following point of view: rather than giving sufficient conditions ensuring the transcendence of a given number α, we take a pair (α, α') of real numbers, and we prove that, under some condition, at least one of them is transcendental.*

1. Introduction and results

Very little is known on the continued fraction expansion of any algebraic real number of degree at least three. It is likely that the sequence of its partial quotients is unbounded, but we seem to be still very far away from a proof. Recently, a small step was made in this direction by means of several new transcendence criteria for continued fractions [1,2,3]. They illustrate the fact that if the sequence of partial quotients of a real irrational number α has some special combinatorial property, for example if long blocks of partial quotients repeat unusually close to the beginning, then α must be either transcendental, or quadratic.

The purpose of the present note is to establish two extensions of some of our criteria. We adopt a slightly different point of view: rather than giving sufficient conditions ensuring the transcendence of a given number α, we take a pair (α, α') of real numbers, and we aim at proving that, under some condition, at least one of them is transcendental. Clearly, if one knows in advance that one of them is algebraic, or if we consider the pair (α, α), this plainly gives a transcendence criterion. Like in [1,2,3], the proofs rest on the Schmidt Subspace Theorem.

To state our results, it is convenient to use the terminology from combinatorics on words.

Let $\mathcal{A}$ be a countable set. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. The mirror image or the reversal of $W := a_1 \dots a_m$ is the word $\overline{W} := a_m \dots a_1$. In particular, W is a palindrome if and only if $W = \overline{W}$.

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Let $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ and $\mathbf{a}' = (a'_\ell)_{\ell \ge 1}$ be sequences of elements from $\mathcal{A}$, that we identify with the infinite words $a_1 a_2 \ldots$ and $a'_1 a'_2 \ldots$, respectively. We say that the pair $(\mathbf{a}, \mathbf{a}')$ satisfies Condition $(*)$ if there exists a sequence of finite words $(V_n)_{n \ge 1}$ such that:

(i) For every $n \ge 1$, the word V_n is a prefix of the word $\mathbf{a}$;
(ii) For every $n \ge 1$, the word $\overline{V}_n$ is a prefix of the word $\mathbf{a}'$;
(iii) The sequence $(|V_n|)_{n \ge 1}$ is increasing.

Theorem 1. *Let* $\mathbf{a}$ *and* $\mathbf{a}'$ *be sequences of positive integers satisfying Condition* $(*)$*. Set*

$$\alpha = [0; a_1, a_2, \ldots], \qquad \alpha' = [0; a'_1, a'_2, \ldots].$$

Then, either one (at least) of α *and* α' *is transcendental, or both are in the same real quadratic field.*

We stress that there is no assumption on the growth of the sequences $\mathbf{a}$ and $\mathbf{a}'$.

We point out two immediate consequences of Theorem 1.

Corollary 1. *Let* $(W_j)_{j \ge 0}$ *be an arbitrary sequence of finite words on the alphabet* $\mathbf{Z}_{\ge 1}$*. Set* $X_0 = W_0$ *and* $X_j = X_{j-1} W_j X_{j-1}$ *for any* $j \ge 1$*. Then, the sequences* $(X_j)_{j \ge 0}$ *and* $(\overline{X}_j)_{j \ge 0}$ *converge. Denote their limits by* $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ *and* $\mathbf{a}' = (a'_\ell)_{\ell \ge 1}$*, respectively, and set*

$$\alpha = [0; a_1, a_2, \ldots], \qquad \alpha' = [0; a'_1, a'_2, \ldots].$$

Then at least one among α *and* α' *is transcendental, or both are in the same real quadratic field.*

Applying Theorem 1 with $\mathbf{a} = \mathbf{a}'$, we recover Theorem 1 from [3], stated below.

Corollary 2. *Let* $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ *be a sequence of positive integers. If the word* $\mathbf{a}$ *begins in arbitrarily long palindromes, then the real number* $\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots]$ *is either quadratic or transcendental.*

Our next statement deals with a wider class of continued fractions. Keep the above notation. We say that the pair $(\mathbf{a}, \mathbf{a}')$ satisfies Condition $(**)$ if there exist two sequences of finite words $(U_n)_{n \ge 1}$ and $(V_n)_{n \ge 1}$ such that:

(i) For every $n \ge 1$, the word V_n is a prefix of the word $\mathbf{a}$;
(ii) For every $n \ge 1$, the word $U_n \overline{V}_n$ is a prefix of the word $\mathbf{a}'$;
(iii) The sequence $(|U_n|/|V_n|)_{n \ge 1}$ is bounded from above;
(iv) The sequence $(|V_n|)_{n \ge 1}$ is increasing.

Theorem 2. *Let* $\mathbf{a}$ *and* $\mathbf{a}'$ *be sequences of positive integers satisfying Condition* $(**)$*. Set*

$$\alpha = [0; a_1, a_2, \ldots], \qquad \alpha' = [0; a'_1, a'_2, \ldots].$$

Denote by $(p_\ell / q_\ell)_{\ell \ge 1}$ *the sequence of convergents to* α'*. If the sequence* $(q_\ell^{1/\ell})_{\ell \ge 1}$ *is bounded, then either one (at least) of* α *and* α' *is transcendental, or both are in the same real quadratic field.*

Applying Theorem 2 with $\mathbf{a} = \mathbf{a}'$, we recover Theorem 2 from [3]. Applying Theorem 2 with a purely periodic sequence $\mathbf{a}$, we can derive Theorem 3.2 from [2], a particular case of which is stated below.

Corollary 3. *Let $(a_\ell)_{\ell\ge 1}$ be a bounded sequence of positive integers. Assume that there are positive integers $b_1,\ldots,b_m$ and sequences $(n_k)_{k\ge1}$ and $(\lambda_k)_{k\ge1}$ of positive integers with*

$$a_{n_k+j+hm} = b_j \quad \text{for } 1 \le j \le m \text{ and } 0 \le h \le \lambda_k - 1,$$

and $n_{k+1} > n_k + \lambda_k m$ for every $k \ge 1$. If

$$\limsup_{k\to+\infty} \frac{\lambda_k}{n_k} > 0,$$

then the real number $[0; a_1, a_2, \ldots]$ is either quadratic, or transcendental.

To get Corollary 3, we apply Theorem 2 with $\mathbf{a}$ being the purely periodic sequence of period $b_m,\ldots,b_1$.

Theorem 2 describes a way to perturb the continued fraction expansion of an algebraic number to get a transcendental number.

2. Proofs

The proofs of our theorems rest on the Schmidt Subspace Theorem, recalled below.

Theorem A (W. M. Schmidt). *Let $m \ge 2$ be an integer. Let $L_1,\ldots,L_m$ be linearly independent linear forms in $\mathbf{x} = (x_1,\ldots,x_m)$ with real algebraic coefficients. Let ε be a positive real number. Then, the set of solutions $\mathbf{x} = (x_1,\ldots,x_m)$ in $\mathbf{Z}^m$ to the inequality*

$$|L_1(\mathbf{x}) \ldots L_m(\mathbf{x})| \le (\max\{|x_1|,\ldots,|x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbf{Q}^m$.

Proof. See e.g. [5] or [6]. □

We also need three lemmas. Except for the last assertion of Lemma 2, we omit the proofs, since they can be found in Perron's book [4].

Lemma 1. *Let $\alpha = [0; a_1, a_2, \ldots]$ be a real number with convergents $(p_\ell/q_\ell)_{\ell\ge1}$. Then, for any $\ell \ge 2$, we have*

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1].$$

Lemma 2. *Let $\alpha = [0; a_1, a_2, \ldots]$ and $\beta = [0; b_1, b_2, \ldots]$ be real numbers. Assume that there exists a positive integer n such that $a_i = b_i$ for any $i = 1,\ldots,n$. We then have $|\alpha - \beta| \le q_n^{-2}$, where q_n denotes the denominator of the n-th convergent to α. Furthermore, if $a_{n+1} \ne b_{n+1}$, then we have*

$$|\alpha - \beta| \ge \frac{1}{6(b_{n+1}+2)^2 \max\{b_{n+2}, b_{n+3}\} q_n^2}.$$

Proof. We content ourselves to establish the last assertion. Set $\alpha' = [a_{n+1}; a_{n+2}, \ldots]$ and $\beta' = [b_{n+1}; b_{n+2}, \ldots]$. If $a_{n+1} > b_{n+1}$, then we have

$$\alpha' - \beta' \geq 1 - [0; 1, b_{n+3} + 1] = \frac{1}{b_{n+3} + 2}. \tag{1}$$

If $a_{n+1} < b_{n+1}$, then we have

$$\beta' - \alpha' \geq [0; b_{n+2} + 1] = \frac{1}{b_{n+2} + 1}. \tag{2}$$

Denote by $(p_\ell/q_\ell)_{\ell \geq 1}$ the sequence of convergents to α. Then, the theory of continued fractions gives that

$$\alpha = \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} \quad \text{and} \quad \beta = \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}},$$

since, by assumption, the first n-th partial quotients of α and β are the same. We thus obtain

$$\begin{aligned} |\alpha - \beta| = \left| \frac{p_n \alpha' + p_{n-1}}{q_n \alpha' + q_{n-1}} - \frac{p_n \beta' + p_{n-1}}{q_n \beta' + q_{n-1}} \right| &= \left| \frac{\alpha' - \beta'}{(q_n \alpha' + q_{n-1})(q_n \beta' + q_{n-1})} \right| \\ &\geq \left| \frac{\alpha' - \beta'}{(a_{n+1} + 2)(b_{n+1} + 2) q_n^2} \right|. \end{aligned}$$

If $a_{n+1} \geq 2(b_{n+1} + 1)$, this yields

$$|\alpha - \beta| \geq \frac{1}{3(b_{n+1} + 2) q_n^2}.$$

Otherwise, we get from (1) and (2) that

$$|\alpha - \beta| \geq \frac{1}{6(b_{n+1} + 2)^2 \max\{b_{n+2}, b_{n+3}\} q_n^2}.$$

This concludes the proof of the lemma. □

For positive integers $a_1, \ldots, a_m$, we denote by $K_m(a_1, \ldots, a_m)$ the denominator of the rational number $[0; a_1, \ldots, a_m]$. It is commonly called a *continuant*.

Lemma 3. *Let $m \geq 2$ be an integer. For any positive integers $a_1, \ldots, a_m$ and any integer k with $1 \leq k \leq m - 1$, we have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1)$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) &\leq K_m(a_1, \ldots, a_m) \\ &\leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m). \end{aligned}$$

Furthermore, we have

$$K_m(a_1,\dots,a_m) \ge K_m(1,\dots,1) \ge 2^{m/2}.$$

Throughout the rest of the paper, if W denotes the finite word $w_1 \dots w_m$ on the alphabet $\mathbf{Z}_{\ge 1}$, then $[0; W]$ denotes the rational number $[0; w_1, \dots, w_m]$ and $K_m(W)$ denotes the denominator of $[0; W]$.

We begin with establishing Theorem 1.

Proof of Theorem 1. We assume that α and α' are algebraic numbers. For $n \ge 1$, set $s_n = |V_n|$. Denote by $(p_\ell/q_\ell)_{\ell\ge1}$ the sequence of convergents to α'. By assumption, we have

$$\frac{p_{s_n}}{q_{s_n}} = [0; \overline{V}_n],$$

and we infer from Lemma 1 that

$$\frac{q_{s_n-1}}{q_{s_n}} = [0; V_n].$$

Consequently, we have

$$|q_{s_n}\alpha - q_{s_n-1}| < q_{s_n}^{-1}, \tag{3}$$

and

$$\lim_{n\to+\infty} \frac{q_{s_n-1}}{q_{s_n}} = \alpha. \tag{4}$$

Furthermore, we clearly have

$$|q_{s_n}\alpha' - p_{s_n}| < q_{s_n}^{-1} \quad \text{and} \quad |q_{s_n-1}\alpha' - p_{s_n-1}| < q_{s_n}^{-1}. \tag{5}$$

Consider now the three linearly independent linear forms with algebraic coefficients:

$$\begin{aligned} L_1(X_1,X_2,X_3) &= \alpha' X_1 - X_3,\\ L_2(X_1,X_2,X_3) &= \alpha X_1 - X_2,\\ L_3(X_1,X_2,X_3) &= X_2. \end{aligned}$$

Evaluating them on the triple $(q_{s_n}, q_{s_n-1}, p_{s_n})$, it follows from (3) and (5) that

$$\prod_{1\le j\le 3} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n})| \le q_{s_n}^{-1}.$$

By Theorem A, there exist a non-zero integer triple (x_1, x_2, x_3) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1 q_{s_n} + x_2 q_{s_n-1} + x_3 p_{s_n} = 0, \tag{6}$$

for every n in $\mathcal{N}_1$. By dividing (6) by q_{s_n} and letting n tend to infinity along $\mathcal{N}_1$, it follows from (4) that

$$x_1 + x_2\alpha + x_3\alpha' = 0. \tag{7}$$

We further consider the three linearly independent linear forms with algebraic coefficients:

$$\begin{aligned} L_4(X_1, X_2, X_3) &= \alpha' X_2 - X_3, \\ L_5(X_1, X_2, X_3) &= \alpha X_1 - X_2, \\ L_6(X_1, X_2, X_3) &= X_2. \end{aligned}$$

Evaluating them on the triple $(q_{s_n}, q_{s_n-1}, p_{s_n-1})$, it follows from (3) and (5) that

$$\prod_{4 \le j \le 6} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n-1})| \le q_{s_n}^{-1}.$$

By Theorem A, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$y_1 q_{s_n} + y_2 q_{s_n-1} + y_3 p_{s_n-1} = 0, \tag{8}$$

for every n in $\mathcal{N}_2$. By dividing (8) by q_{s_n} and letting n tend to infinity along $\mathcal{N}_2$, it follows from (4) that

$$y_1 + y_2\alpha + y_3\alpha\alpha' = 0. \tag{9}$$

Observe that x_3 is non-zero since α is irrational. Consequently, it follows from (7) and (9) that

$$y_1 + y_2\alpha - y_3\alpha\left(\frac{x_1 + x_2\alpha}{x_3}\right) = 0. \tag{10}$$

Since $x_2 y_3$ is non-zero, (10) implies that α is a quadratic real number, and we infer from (9) that α' lies in the same quadratic field as α. This concludes the proof of Theorem 1. □

We now establish Theorem 2.

Proof of Theorem 2. Keep the notation from the statement of the theorem, and denote by $(U_n)_{n\ge1}$ and $(V_n)_{n\ge1}$ the sequences occurring in Condition $(**)$. Modifying them if necessary, we may assume that, besides (i) to (iv), they also satisfy

(v) The sequence $(|U_n|)_{n\ge1}$ is increasing;

(vi) For any $n \ge 2$, if c_n denotes the last letter of U_n, then $V_n c_n$ is not a prefix of the word **a**.

Indeed, if this would not be possible to modify $(U_n)_{n\ge1}$ and $(V_n)_{n\ge1}$ accordingly, then an application of Theorem 1 would yield that either one (at least) of α and α' is transcendental, or both are in the same real quadratic field.

Assume that α and α' are algebraic numbers. For $n \ge 1$, set $r_n = |U_n|$ and $s_n = |V_n|$. Recall that $(p_\ell/q_\ell)_{\ell\ge1}$ denotes the sequence of convergents to α'.

By assumption, we have

$$\frac{p_{r_n+s_n}}{q_{r_n+s_n}} = [0; U_n \overline{V}_n],$$

and we infer from Lemma 1 that

$$\frac{q_{r_n+s_n-1}}{q_{r_n+s_n}} = [0; V_n \overline{U}_n].$$

It follows from Lemma 2 that

$$|q_{r_n+s_n}\alpha - q_{r_n+s_n-1}| < q_{r_n+s_n} K_{s_n}(V_n)^{-2}. \tag{11}$$

This shows in particular that

$$\lim_{n\to+\infty} \frac{q_{r_n+s_n-1}}{q_{r_n+s_n}} = \alpha. \tag{12}$$

Furthermore, we clearly have

$$|q_{r_n+s_n}\alpha' - p_{r_n+s_n}| < q_{r_n+s_n}^{-1} \quad \text{and} \quad |q_{r_n+s_n-1}\alpha' - p_{r_n+s_n-1}| < q_{r_n+s_n}^{-1}. \tag{13}$$

Consider now the four linearly independent linear forms with algebraic coefficients:

$$\begin{aligned} L_7(X_1, X_2, X_3, X_4) &= \alpha' X_1 - X_3, \\ L_8(X_1, X_2, X_3, X_4) &= \alpha' X_2 - X_4, \\ L_9(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2, \\ L_{10}(X_1, X_2, X_3, X_4) &= X_2. \end{aligned}$$

Evaluating them on the quadruple $(q_{r_n+s_n}, q_{r_n+s_n-1}, p_{r_n+s_n}, p_{r_n+s_n-1})$, it follows from (11) and (13) that

$$\Pi := \prod_{7\le j\le 10} |L_j(q_{r_n+s_n}, q_{r_n+s_n-1}, p_{r_n+s_n}, p_{r_n+s_n-1})| \le K_{s_n}(V_n)^{-2}.$$

By assumption, there is $M > 1$ such that $q_\ell \le M^\ell$ and $r_\ell \le M s_\ell$ for every $\ell \ge 1$. This and Lemma 3 imply that, for $n \ge 2$, we have

$$K_{s_n}(V_n) \ge 2^{s_n/2} \ge 2^{s_n/4} \cdot 2^{s_n/(4M)} \ge 2^{(r_n+s_n)/(4M)} \ge q_{r_n+s_n}^{\eta},$$

with $\eta = (\log 2)/(4M \log M)$. Thus, $\Pi \le q_{r_n+s_n}^{-2\eta}$, and, by Theorem A, there exist a non-zero integer quadruple (z_1, z_2, z_3, z_4) and an infinite set of distinct positive integers $\mathcal{N}_3$ such that

$$z_1 q_{r_n+s_n} + z_2 q_{r_n+s_n-1} + z_3 p_{r_n+s_n} + z_4 p_{r_n+s_n-1} = 0, \tag{14}$$

for every n in $\mathcal{N}_3$. By dividing (14) by $q_{r_n+s_n}$ and letting n tend to infinity along $\mathcal{N}_3$, it follows from (12) that

$$z_1 + z_2\alpha + z_3\alpha' + z_4\alpha\alpha' = 0. \tag{15}$$

We get from (15) that

$$\alpha = -\frac{z_1 + z_3\alpha'}{z_2 + z_4\alpha'},$$

and we observe that, for n in $\mathcal{N}_3$, we have

$$\begin{aligned} \left|\alpha - \frac{q_{r_n+s_n-1}}{q_{r_n+s_n}}\right| &= \left|\frac{z_1 + z_3\alpha'}{z_2 + z_4\alpha'} - \frac{z_1 + z_3 p_{r_n+s_n}/q_{r_n+s_n}}{z_2 + z_4 p_{r_n+s_n-1}/q_{r_n+s_n-1}}\right| \\ &\ll \frac{1}{q^2_{r_n+s_n}} \ll (K_{r_n}(U_n)K_{s_n}(V_n))^{-2}, \end{aligned} \tag{16}$$

by (13) and Lemma 3. Here and below, the numerical constant implied by $\ll$ is independent on n. Let c_n, c'_n and c''_n be the last three letters of U_n, and define the word U'_n by $U_n = U'_n c''_n c'_n c_n$. We infer from (vi) and Lemmas 2 and 3 that

$$\left|\alpha - \frac{q_{r_n+s_n-1}}{q_{r_n+s_n}}\right| \gg K_{s_n+3}(V_n c_n c'_n c''_n)^{-2}. \tag{17}$$

Then, Lemma 3 and the combination of (16) and (17) yield that

$$\begin{aligned} K_{s_n}(V_n) \times K_3(c''_n c'_n c_n) &\gg K_{s_n+3}(V_n c_n c'_n c''_n) \\ &\gg K_{s_n}(V_n) \times K_{r_n}(U_n) \\ &\gg K_{s_n}(V_n) \times K_{r_n-3}(U'_n) \times K_3(c''_n c'_n c_n). \end{aligned} \tag{18}$$

However, our assumption (v) implies that $K_{r_n-3}(U'_n)$ tends to infinity with n. This contradicts (18) and finishes the proof of the theorem. □

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References

[1] B. Adamczewski & Y. Bugeaud, *On the complexity of algebraic numbers II. Continued fractions*, Acta Math. 195 (2005), 1–20.

[2] B. Adamczewski & Y. Bugeaud, *On the Maillet–Baker continued fractions*, J. reine angew. Math. To appear.

[3] B. Adamczewski & Y. Bugeaud, *Palindromic continued fractions*. Preprint. `http://arxiv.org/abs/math.NT/0512014`.

[4] O. Perron, Die Lehre von den Ketterbrüchen. Teubner, Leipzig, 1929.

[5] W. M. Schmidt, *Norm form equations*, Ann. of Math. **96** (1972), 526–551.

[6] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics 785, Springer, Berlin, 1980.

Boris Adamczewski
CNRS, Institut Camille Jordan
Université Claude Bernard Lyon 1
Bât. Braconnier, 21 avenue Claude Bernard
69622 VILLEURBANNE Cedex (FRANCE)

Boris.Adamczewski@math.univ-lyon1.fr

Yann Bugeaud
Université Louis Pasteur
U. F. R. de mathématiques
7, rue René Descartes
67084 STRASBOURG Cedex (FRANCE)

bugeaud@math.u-strasbg.fr