ON THE REPRESENTATION OF FIBONACCI AND LUCAS NUMBERS IN AN INTEGER BASES

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Abstract. Résumé. Nous présentons plusieurs théorèmes sur l'écriture des nombres entiers dans deux bases indépendantes. Nous dressons la liste complète des nombres de Fibonacci et des nombres de Lucas qui s'écrivent en binaire avec au plus quatre chiffres 1.
Abstract. We discuss various results on the representation of integers in two unrelated bases. We give the complete list of all the Fibonacci numbers and of all the Lucas numbers which have at most four digits 1 in their binary representation.

1. Introduction

Let a and b be integers at least equal to 2. In 1973, Senge and Straus [12] proved that the number of integers, the sum of whose digits in each of the bases a and b lies below a fixed bound, is finite if, and only if, a and b are multiplicatively independent. Their proof rests on the Thue–Siegel–Roth theorem and, hence, is ineffective. Using Baker's theory of linear forms in logarithms, Stewart [13] succeeded in establishing an effective version of Senge and Straus' theorem. He showed that if a and b are multiplicatively independent, then, for every $c \geq 1$, each integer $m > 25$ whose sum of digits in base a as well as in base b is bounded by c satisfies

$$\frac{\log\log m}{\log\log\log m + c_1} < 2c + 1, \tag{1}$$

where c_1 is a positive constant which is effectively computable in terms of a and b only (see also Mignotte [9]).

Stewart was also able to deal with digital expansions of elements of a linear recurrence sequence of integers. Recall that the general term of such a sequence of integers $(U_n)_{n\geq 1}$ can be written as

$$U_n = P_1(n)\,\alpha_1^n + P_2(n)\,\alpha_2^n + \cdots + P_k(n)\,\alpha_k^n, \qquad \text{with } |\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_k|,$$

where the α_i's are complex numbers and the $P_i(X)$'s are polynomials.

Theorem S (Stewart). *Let $b \geq 2$ be an integer. Keep the above notation and assume that $P_1(X)$ is non-zero and that*

$$|\alpha_1| > |\alpha_2|, \quad |\alpha_1| > 1.$$

There exists a positive constant C_0, which is effectively computable in terms of b and of $(U_n)_{n\geq 1}$, such that, for every $n > 4$, the b-ary representation of U_n has at least

$$\frac{\log n}{\log\log n + C_0} - 1$$

non-zero digits.

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Observe that (1) and Theorem S are of similar strength since, in Stewart's theorem, U_n grows exponentially in n.

Instead of computing an upper bound for the integers having few digits in two unrelated bases, we may as well bound their number from above. This was done by Schlickewei [10], as a particular case of a general result on linear equations in integers with bounded sum of digits (see Theorem 3.2 below). His proof rests on the quantitative subspace theorem and the same ideas can be also used to bound the number of indices n for which the n-th term of the recursion $(U_n)_{n\geq 1}$ as in Stewart's theorem has k non-zero digits in some integer base b.

In Section 2, we consider the two most popular binary sequences, namely the Fibonacci and the Lucas sequences, denoted by $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$, respectively. Recall that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \quad (n \geq 0),$$

where we have set

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

We prove that, for any positive integer k and any integer base $b \geq 2$, the number of Fibonacci (respectively Lucas) numbers having at most k digits in their representation in base b is bounded by some constant depending only on k. The proof rests on the quantitative absolute subspace theorem. Furthermore, we give the complete list of all Fibonacci numbers and all Lucas numbers having at most four non-zero digits in their binary representation.

In Section 3 we discuss various extensions or improvements of Stewart's and Schlickewei's results. Rather than counting the number of non-zero digits, we consider the number of times that a digit different from the previous one is read, a problem investigated by Blecksmith, Filaseta, and Nicol [3] and by Barat, Tichy, and Tijdeman [2]. Sections 4 and 5 contain the proofs of our results.

All these results are illustrations of the same principle "in two unrelated number systems, two miracles cannot happen simultaneously for large integers".

2. Fibonacci and Lucas numbers with few non-zero digits

For a given integer base $b \geq 2$ and a positive integer k, Theorem S implies that there are only finitely many Fibonacci and Lucas numbers having k non-zero digits in their b-ary representation, a result already established in [12] (but not stated explicitly there). The next theorem gives an upper estimate for their number.

Theorem 2.1. *Let $b \geq 2$ and $k \geq 1$ be integers. There are at most*

$$b^k\,(2k)^{17(k+3)^5} \tag{2}$$

Fibonacci (respectively Lucas) numbers having at most k non-zero digits in their representation in base b.

The dependence on b cannot be removed in the upper bound given in Theorem 2.1. Indeed, since F_n is at most 2^n for any positive integer n, there are, regardless of the positive integer N, clearly at least N Fibonacci numbers having one single digit in any base b greater than 2^N. The fact that the bound (2) is polynomial in b, and not exponential in a power of b, illustrates the power of the absolute quantitative subspace theorem, which is one of the main ingredients for the proof.

The next theorem is specific for the base $b = 2$. It lists all the Fibonacci and Lucas numbers having no more than four non-zero digits in their binary representation.

Theorem 2.2. *The Fibonacci numbers with at most two binary digits are*

$$F_0 = 0,\ F_1 = F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_9 = 34,\ F_{12} = 144,$$

those with three binary digits are

$$F_7 = 13,\ F_8 = 21,$$

and those with four binary digits are

$$F_{11} = 89,\ F_{15} = 610,\ F_{18} = 2584.$$

The Lucas numbers with at most two binary digits are

$$L_0 = 2,\ L_1 = 1,\ L_2 = 3,\ L_3 = 4,\ L_6 = 18,$$

those with three binary digits are

$$L_4 = 7,\ L_5 = 11,\ L_9 = 76,\ L_{12} = 322,\ L_{13} = 521,$$

and $L_7 = 29$ *is the only Lucas number with four binary digits.*

The proof of Theorem 2.2 combines elementary properties of the Fibonacci and Lucas numbers, sharp estimates for linear forms in two and three logarithms, and repeated use of the classical Baker–Davenport Lemma (see Proposition 5.3 below). In principle, the same method can be employed to list all the Fibonacci and Lucas numbers having an arbitrarily given number of binary digits.

3. Further results on the representation of integers in distinct bases

Besides the number of non-zero digits in the representation of an integer m, we may also estimate the number of blocks composed of the same digit in this representation. This was first considered by Blecksmith, Filaseta, and Nicol [3], who proved that, for multiplicatively independent positive integers a and b, we have

$$\lim_{n\to\infty} BC(a^n, b) = +\infty,$$

where $BC(m, b)$ stands for the number of times that a digit different from the previous one is read in the b-ary representation of the positive integer m. Their result was subsequently quantified by Barat, Tichy, and Tijdeman [2], who, under the same assumption, showed that there are effectively computable numbers c_0, n_0, depending only on a and b, such that

$$BC(a^n, b) \geq c_0 \frac{\log n}{\log\log n}, \quad \text{for } n > n_0. \tag{3}$$

This result and its proof are closely related to the theorem of Stewart quoted in Introduction. Again, we have the same order of magnitude as in 1) and in Theorem S.

We also note that if m has at most k non-zero digits in its b-ary representation, then $BC(m, b)$ does not exceed $2k$. However, the converse is not true, since a number m such that $BC(m, b)$ is small may have many non-zero digits in its b-ary representation.

Following Stewart's arguments, we improve and extend 3) as follows.

Theorem 3.1. *Let a and b be multiplicatively independent integers. Then we have*

$$BC(m, a) + BC(m, b) \geq \frac{\log\log m}{\log\log\log m + C} - 1,$$

for $m > 25$, where C is a positive number which is effectively computable in terms of a and b only.

The next theorem is an extension of Schlickewei's result.

Theorem 3.2. *Let k be a positive integer. Let $\ell \geq 2$ be an integer. Let $b_1, \ldots, b_\ell$ be integers ≥ 2 such that b_{ℓ_1} and b_{ℓ_2} are multiplicatively independent for every integers ℓ_1 and ℓ_2 with $1 \leq \ell_1 < \ell_2 \leq \ell$. Let $n_1, \ldots, n_\ell$ be positive integers whose sum of digits in base b_i is bounded from above by k for $i = 1, \ldots, \ell$. Then there exists an effectively computable constant c such that the equation*

$$\pm n_1 \pm n_2 \pm \cdots \pm n_\ell = 0$$

has at most

$$(\ell k)^{c(\ell k)^5}$$

solutions.

The proof of Theorem 3.2 is similar to Schlickewei's proof of its Theorems 1 and 2 from [10], in which he used his own version of the quantitative subspace theorem. The tools available at that time did not enable one to get rid of the dependence on the number of distinct prime factors of the product $b_1 \cdots b_k$. Some years later, an important progress was made by Schlickewei, who established the absolute subspace theorem, see his paper with Evertse [5] for a quantitative statement. The deep Theorem ESSAV quoted in the next Section is a (not immediate!) consequence of the absolute quantitative subspace theorem.

4. Proofs of Theorems 1, 3 and 4

Let K be a field of characteristic 0 and write K^* for its multiplicative group of non-zero elements. Let $(K^*)^n$ be the direct product of n copies of K^*, so for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ in $(K^*)^n$ we write $\mathbf{x} * \mathbf{y} = (x_1 y_1, \ldots, x_n y_n)$.

We quote below Theorem 6.2 of Amoroso and Viada [1], which strengthens Theorem 1.1 of Evertse, Schlickewei, and Schmidt [6]. The improvement comes from a better estimate for the number of solutions of 'very small height', a step of the proof which is independent of the quantitative subspace theorem.

Theorem ESSAV. *Keep the above notation. Let $(a_1, \ldots, a_n)$ be in $(K^*)^n$ and Γ be a subgroup of rank r of $(K^*)^n$. Then, the number of solutions $\mathbf{x} = (x_1, \ldots, x_n)$ in Γ to*

$$a_1 x_1 + \cdots + a_n x_n = 1, \tag{4}$$

such that no proper subsum of the left-hand side of (4) vanishes, is bounded above by

$$(8n)^{4n^4(n+r+1)}.$$

Theorem ESSAV is the key tool for the proof of Theorems 2.1 and 3.2.

Proof of Theorem 2.1.

Let $b \geq 2$ be an integer. Let $a_1, \ldots, a_k$ be integers in $\{0, 1, \ldots, b-1\}$. Let α_1 and β_1 be non-zero elements of $\mathbb{Q}(\sqrt{5})$. Let k be a non-zero integer. We consider the Diophantine equation

$$\alpha_1 \alpha^n + \beta_1 \beta^n = a_1 b^{n_1} + \cdots + a_k b^{n_k}, \tag{5}$$

in non-negative integers $n, n_1, \ldots, n_k$ satisfying $n_1 > \ldots > n_k$. Dividing both sides of (5) by $a_k b^{n_k}$, we get

$$\alpha_1 \alpha^n a_k^{-1} b^{-n_k} + \beta_1 \beta^n a_k^{-1} b^{-n_k} - a_1 a_k^{-1} b^{n_1 - n_k} - \cdots - a_k^{-1} b^{n_{k-1} - n_k} = 1. \tag{6}$$

We apply Theorem ESSAV to the field $\mathbb{Q}(\sqrt{5})$ and the multiplicative group Γ of rank $k+3$ generated by $(\alpha, 1, 1, \ldots, 1)$, $(1, \beta, 1, \ldots, 1)$, $(b, 1, 1, 1, \ldots, 1), \ldots, (1, \ldots, 1, b)$. It gives that (6), and, consequently, (5), have no more than $\exp\{17(k+3)^5 \log(2k)\}$ solutions. Since there are b^k possible choices for the k-tuple $(a_1, \ldots, a_k)$, this proves the theorem. $\square$

Proof of Theorem 3.1.
Let the a-ary representation of m be given by

$$a_h a_{h-1} \dots a_1,$$

with $a_h \neq 0$, and define the integers $n_1 < n_2 < \dots$ by $a_1 = \dots = a_{n_1}$, $a_{n_1} \neq a_{n_1+1}$ and $a_{n_j+1} = \dots = a_{n_{j+1}}$, $a_{n_{j+1}} \neq a_{n_{j+1}+1}$ for $j \geq 1$, until we reach a_h. Define r by $a_{n_r} = a_h$. Then, observe that

$$\begin{aligned} m &= a_{n_1} \frac{a^{n_1} - 1}{a - 1} + a_{n_2} a^{n_1} \frac{a^{n_2 - n_1} - 1}{a - 1} + a_{n_3} a^{n_2} \frac{a^{n_3 - n_2} - 1}{a - 1} + \cdots \\ &= -\frac{a_{n_1}}{a-1} + \frac{a_{n_1} - a_{n_2}}{a-1} a^{n_1} + \frac{a_{n_2} - a_{n_3}}{a-1} a^{n_2} + \cdots + \frac{a_{n_r}}{a-1} a^{n_r}. \end{aligned}$$

Likewise, if

$$b_k b_{k-1} \dots b_1,$$

where $b_k \neq 0$, denotes the b-ary representation of m, we define the integers $\ell_1 < \ell_2 < \dots$ by $b_1 = \dots = b_{\ell_1}$, $b_{\ell_1} \neq b_{\ell_1+1}$ and $b_{\ell_j+1} = \dots = b_{\ell_{j+1}}$, $b_{\ell_{j+1}} \neq b_{\ell_{j+1}+1}$ for $j \geq 1$, until we reach b_k. Defining t by $b_{\ell_t} = b_k$, we have

$$m = -\frac{b_{\ell_1}}{b-1} + \frac{b_{\ell_1} - b_{\ell_2}}{b-1} b^{\ell_1} + \frac{b_{\ell_2} - b_{\ell_3}}{b-1} b^{\ell_2} + \cdots + \frac{b_{\ell_t}}{b-1} b^{\ell_t}.$$

This is of the same shape as in the proof of Theorem 1 of [13]. So, it is sufficient to follow this proof to establish Theorem 3.1. □

Proof of Theorem 3.2.
We follow step by step the argument of Schlickewei [10], using however Theorem ESSAV in place of the result from [11] recalled in [10]. □

5. Proof of Theorem 2

About 30 minutes of computation are sufficient to list all the Fibonacci and Lucas numbers having at most four binary digits and small index.

Lemma 5.1. *The only Fibonacci and Lucas numbers having at most four binary digits and index at most 10^4 are those listed in Theorem 2.2.*

Therefore from now on we assume $n > 10^4$. Moreover, to treat simultaneously the Fibonacci and the Lucas cases, we put

$$U_n = \begin{cases} F_n, & \text{in the Fibonacci case,} \\ L_n, & \text{in the Lucas case,} \end{cases} \qquad \rho = \begin{cases} \sqrt{5}, & \text{in the Fibonacci case,} \\ 1, & \text{in the Lucas case,} \end{cases}$$

so that

$$|\rho U_n - \alpha^n| = \alpha^{-n}$$

in both cases.

The proof of Theorem 2.2 requires sharp estimates for linear forms in two and three logarithms. For linear forms in two logarithms, we apply the bounds established in [7], while, for linear forms in three logarithms, we use a special case of a general estimate due to Matveev [8].

Proposition 5.2. *Let λ_1, λ_2, λ_3 be $\mathbb{Q}$–linearly independent logarithms of non-zero algebraic numbers and let b_1, b_2, b_3 be rational integers with $b_1 \neq 0$. Define $\alpha_j = \exp(\lambda_j)$ for $j = 1,\ 2,\ 3$ and*

$$\Lambda = b_1 \lambda_1 + b_2 \lambda_2 + b_3 \lambda_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$. Put

$$\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

Let A_1, A_2, A_3 be positive real numbers, which satisfy

$$A_j \geq \max\bigl\{D\mathrm{h}(\alpha_j), |\lambda_j|, 0.16\bigr\} \quad (1 \leq j \leq 3).$$

Assume that

$$B \geq \max\bigl\{|b_j| A_j / A_1;\ 1 \leq j \leq 3\bigr\}.$$

Define also

$$C_1 = \frac{5 \times 16^5}{6\chi}\, \mathrm{e}^3\, (7 + 2\chi) \left(\frac{3\mathrm{e}}{2}\right)^{\chi} \Bigl(20.2 + \log\bigl(3^{5.5} D^2 \log(\mathrm{e}D)\bigr)\Bigr).$$

Then

$$\log|\Lambda| > -C_1\, D^2\, A_1\, A_2 A_3\, \log\bigl(1.5\,\mathrm{e}DB \log(\mathrm{e}D)\bigr).$$

We shall also apply the following version of the classical Baker–Davenport Lemma, due to Dujella and Pethő [4].

Proposition 5.3. *Let A, B, θ, μ be positive real numbers and M be a positive integer. Suppose that P/Q is a convergent of the continued fraction expansion of θ such that $Q > 6M$. Put $\varepsilon = \|\mu Q\| - M\|\theta Q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (j, k) to the inequality*

$$0 < j\theta - k + \mu < A \cdot B^{-j}$$

subject to the restriction that

$$\frac{\log(AQ/\varepsilon)}{\log B} \leq j \leq M.$$

For a positive integer x, we denote by $\sigma(x)$ the number of non-zero digits in the binary expansion of x.

I. The case $\sigma(U_n) = 1$

Since no Lucas number is divisible by 8, the assumption $\sigma(L_n) = 1$ implies that $L_n = 1$, 2 or 4, which corresponds to the indices 1, 0 and 3, respectively.

Let us move to Fibonacci numbers. The shortest proof is two lines long: since any Fibonacci number of index > 12 has a primitive divisor, all powers of 2 appear among the first 13 terms.

Since this argument cannot be adapted for $\sigma(F_n) > 1$, we present a different proof, whose ingredients serve later on. A tool we need is the following well-known lemma. Recall that for a non-zero integer a, one puts $\mathrm{v}_2(a) = k$ if $2^k \mid a$ and $2^{k+1} \nmid a$.

Lemma 5.4. *If F_n is even then $n = 3m$ and if $4 \mid F_n$ then $2 \mid m$ and*

$$\mathrm{v}_2(F_n) = 2 + \mathrm{v}_2(n).$$

If $F_n = 2^r$, $r \geq 3$, then $\alpha^{n-2}/\sqrt{5} < 2^r < \alpha^n/\sqrt{5}$ and $n \geq 3 \cdot 2^{r-2}$. From this and Lemma 5.4 it easily follows that $n < 10^4$.

II. The case $\sigma(U_n) = 2$

Write $U_n = 2^r + 2^s$, with $r > s$. By Lemma 5.4, in the Fibonacci case and for $n \neq 3 \cdot 2^{s-2}$, we have

$$s \leq \log n / \log 2,$$

and $s < 1 + \log n / \log 2$ otherwise. Since no Lucas number is divisible by 8, we have $s \leq 2$ in the Lucas case. We consider the linear form in logarithms

$$\Lambda := n \log\alpha - \log\rho - r\log 2.$$

In the Fibonacci case, we apply Proposition 5.2 for $\alpha_1 = \sqrt{5}$, $\alpha_2 = 2$ $\alpha_3 = \alpha$, $b_1 = -1$, $b_2 = -r$, $b_3 = n$. We obtain $h(\alpha_1) = \log\sqrt{5}$, $h(\alpha_2) = \log 2$, $h(\alpha_3) =$

$(\log\alpha)/2$, $D = 2$, $\chi = 1$, $A_1 = \log 5$, $A_2 = \log 4$, $A_3 = \log\alpha$, $B = (r\log 4)/(\log 5)$. Therefore, with the notation of Proposition 5.2, we have

$$C_1 = \frac{5\cdot 16^5}{6}\cdot \mathrm{e}^3\cdot 9\cdot \frac{3\mathrm{e}}{2}\left(20.2+\log\left(3^{5.5}\cdot 4\log(2\mathrm{e})\right)\right) < 1.8134\cdot 10^{10}.$$

It follows that

$$\log|\Lambda| > -C_2\log(8.3\,n), \tag{7}$$

with $C_2 = 7.788\cdot 10^{10}$.

From the discussion at the beginning of this subsection II, we easily get that

$$\Lambda = \log\left(1+\frac{\beta^n}{2^r\sqrt{5}}+2^{s-r}\right) < \frac{1}{2^r\sqrt{5}}+2^{s-r} < 2^{s-r+1} < \frac{n}{\alpha^{n/2}},$$

thus

$$\log\Lambda < \log n - \frac{n}{2}\,\log\alpha.$$

Combined with (7), this gives

$$n \le \frac{2C_2+1}{\log\alpha}\,\log(8.3n),$$

which implies $n < 1.1\cdot 10^{13}$.

Now we apply Proposition 5.3 with the choice $j = n$, $\theta = (\log\alpha)/(\log 2)$, $k = r$, $\mu = -(\log 5)/(\log 2)$. Since for any real $a > 1$ the function $n \mapsto n/a^n$ is decreasing for $a^n > \mathrm{e}$, we may choose $A = 1.2$ and $B = \alpha^{2/5}$. Simple computations yield $n < 10^3$.

The Lucas case is simpler: we have a linear form in two logarithms

$$\Lambda := n\log\alpha - r\log 2$$

and an application of [7] gives $n < 10^4$.

Thus Theorem 2.1 is completely proved for $\sigma(U_n) = 2$. We employ a similar reasoning to determine the Fibonacci and Lucas number with three or four binary digits, so we shall give less numerical details in the cases $\sigma(U_n) = 3, 4$. The constants $C_3, \dots, C_7$ below are absolute.

III. The case $\sigma(U_n) = 3$

In the present case

$$U_n = 2^r + 2^s + 2^t, \quad \text{with } r > s > t,$$

and we put $h = r - s$. We consider the two linear forms

$$\Lambda := n\log\alpha - \log\rho - r\log 2,$$

as above, which satisfies

$$0 < |\Lambda| \le 2^{-h+1},$$

and

$$\Lambda_1 := n\log\alpha - \log\rho - \log(2^h+1) - s\log 2,$$

for which

$$0 < \Lambda_1 \le 2^{-s+1} = 2^{-r+h+1}.$$

By Proposition 5.2,

$$\log|\Lambda_1| > -C_3(h+1)\log 2\,\log(2\mathrm{e}n),$$

for a certain constant C_3. Hence,

$$r < (C_3+1)(h+1)\,\log(2\mathrm{e}n).$$

We first treat the case of Lucas numbers. If $U_n = L_n$ then $\rho = 1$ and, by [7], we get

$$\log|\Lambda| > -C_4\,\log^2(2n), \quad \text{with } C_4 = 25\times 8\times\log\alpha\times\log 2,$$

and, since $\log|\Lambda| \le -(h-1)\log 2$, we obtain the upper bound

$$h \le 1 + C_4 \log^2(2n)/\log 2.$$

A similar study of Λ_1 gives

$$n\log\alpha \le (r+1)\log 2 \le (C_5+2)(h+1)\log(2en). \tag{8}$$

Hence,

$$n \le (C_5+2)(C_4+2)\log^3(2en),$$

which implies $n < 10^{17}$. But the continued fraction expansion of $(\log\alpha)/(\log 2)$ gives

$$|n\log\alpha - r\log 2| \ge \frac{1}{136n} \quad \text{if } n < 10^{20}.$$

Thus

$$2^{-h+1} \ge \frac{1}{136n} \quad \text{if } n < 10^{17}$$

and we get $h \le 64$. Now (8) implies $n < 10^{14}$ (indeed $n < 5\cdot 10^{13}$) and $h \le 54$. Then one application of Proposition 5.3 gives $n < 10^3$ and Theorem 2.2 holds in this case.

In the case of Fibonacci numbers, an application of Proposition 5.2 to Λ gives

$$h \le 1 + C_6\log(2en),$$

and for Λ_1 this result implies

$$s\log 2 \le 1 + C_6(h+1)\log(2en),$$

hence

$$r\log 2 \le (C_6+1)(h+1)\log(2en).$$

Finally,

$$n\log\alpha < \frac{(C_6+1)^2\log^2(2en)}{\log 2},$$

and $n < 3\cdot 10^{24}$. Then, an application of Proposition 5.3 to Λ gives $n < 4.4\times 10^{17}$. Now, using this bound on n and applying Proposition 5.3 to Λ_1 for $1 \le h \le 10^5 =: H$, we obtain

$$1 \le h \le 10^5 \implies n < 150{,}000.$$

Then, applying again Proposition 5.3 to Λ, we see that $n < 4.8\times 10^{16}$ if $h \ge H$, and four applications of Proposition 5.3 then imply $n < 2{,}000$. By applying again Proposition 5.3 to Λ and Λ_1, respectively, we get

$$\begin{cases} h \ge 600 \implies n < 10^4, \\ h < 600 \implies n < 10^3, \end{cases}$$

and the asserted result follows from Lemma 5.1.

IV. The case $\sigma(U_n) = 4$

In this case

$$U_n = 2^r + 2^s + 2^t + 2^u, \quad \text{with } r > s > t > u,$$

and we put $h = r - s$, $k = s - t$. By Lemma 5.4 we know that

$$u \le (\log n)/(\log 2).$$

Now we consider the three linear forms

$$\Lambda := n\log\alpha - \log\rho - r\log 2,$$

which satisfies

$$0 < |\Lambda| \le 2^{-h+1},$$

and

$$\Lambda_1 := n\log\alpha - \log\bigl(\rho(2^h+1)\bigr) - s\log 2,$$

for which

$$0 < |\Lambda_1| \le 2^{-s+t+1} = 2^{-k+1},$$

and

$$\Lambda_2 := n\log\alpha - \log\bigl(\rho(2^{h+k}+2^k+1)\bigr) - t\log 2,$$

which satisfies

$$0 < |\Lambda_2| \le 2^{-t+u}.$$

Applying Proposition 5.2 to each of these linear forms, we get now

$$h \le C_7\,\log(2en), \quad k \le C_7\,(h+1)\log(2en), \quad t-u \le C_7\,(h+k+1)\log(2en).$$

From the trivial relation

$$r = h+k+(t-u)+u$$

we deduce that

$$n \le C_7\,\log(2en) + C_7(C_7+1)\,\log^2(2en) + C_7(C_7+1)^2\,\log^3(2en) + \frac{\log n}{\log 2},$$

which leads to

$$n < 10^{32}.$$

Now we proceed in several steps. First, by repeated use of Proposition 5.3 for Λ, we prove that

$$h \ge 200 \implies n < 10^4.$$

Then, using the above upper bounds, we verify successively that

$$h \le 200 \implies n < 4\cdot 10^{26} \implies h \le 110 \implies n < 2\cdot 10^{26},$$

where the first bound comes from Proposition 5.2 and the second one is obtained by repeated use of Proposition 5.3 for Λ_1. On using Proposition 5.3 for Λ_2, we also verify that

$$n < 2\cdot 10^{26} \implies k \le 190 \implies n < 2\cdot 10^{16},$$

and after a few more steps we conclude that

$$n \le 5\cdot 10^{15}, \quad h \le 75, \quad k \le 80.$$

Then, a last application of Proposition 5.3 shows that $n < 10^3$. In view of Lemma 5.1, Theorem 2.2 is completely proved for $\sigma(U_n) = 4$.

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