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ON THE EXPANSIONS OF REAL NUMBERS IN TWO MULTIPLICATIVE DEPENDENT BASES

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Abstract

Let $r \geq 2$ and $s \geq 2$ be multiplicatively dependent integers. We establish a lower bound for the sum of the block complexities of the r-ary expansion and of the s-ary expansion of an irrational real number, viewed as infinite words on $\{0, 1, \ldots, r-1\}$ and $\{0, 1, \ldots, s-1\}$, and we show that this bound is best possible.

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1. Introduction

Throughout this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Let $b \geq 2$ be an integer. For a real number ξ, write

$$\xi = \lfloor \xi \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor \xi \rfloor + 0.a_1 a_2 \ldots,$$

where each digit a_k is an integer from $\{0, 1, \ldots, b-1\}$ and infinitely many digits a_k are not equal to $b-1$. The sequence $\mathbf{a} := (a_k)_{k \geq 1}$ is uniquely determined by the fractional part of ξ. With a slight abuse of notation, we call it the b-ary expansion of ξ and we view it also as the infinite word $\mathbf{a} = a_1 a_2 \ldots$ over the alphabet $\{0, 1, \ldots, b-1\}$.

For an infinite word $\mathbf{x} = x_1 x_2 \ldots$ over a finite alphabet and for a positive integer n, set

$$p(n, \mathbf{x}) = \mathrm{Card}\{x_{j+1} \ldots x_{j+n} : j \geq 0\}.$$

This notion from combinatorics on words is now commonly used to measure the complexity of the b-ary expansion of a real number ξ. Indeed, for a positive integer n, we denote by $p(n, \xi, b)$ the total number of distinct blocks of n digits in the b-ary expansion $\mathbf{a}$ of ξ, that is,

$$p(n, \xi, b) := p(n, \mathbf{a}) = \mathrm{Card}\{a_{j+1} \ldots a_{j+n} : j \geq 0\}.$$

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Obviously, we have $1 \le p(n, \xi, b) \le b^n$, and both inequalities are sharp. If ξ is rational, then its b-ary expansion is ultimately periodic and the numbers $p(n, \xi, b)$, $n \ge 1$, are uniformly bounded by a constant depending only on ξ and b. If ξ is irrational, then, by a classical result of Morse and Hedlund [8], we know that $p(n, \xi, b) \ge n + 1$ for every positive integer n, and this inequality is sharp.

Definition 1.1. A Sturmian word $\mathbf{x}$ is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + 1, \quad \text{for } n \ge 1.$$

A quasi-Sturmian word $\mathbf{x}$ is an infinite word which satisfies

$$p(n, \mathbf{x}) = n + k, \quad \text{for } n \ge n_0,$$

for some positive integers k and n_0.

The following rather general problem was investigated in [2]. Recall that two positive integers x and y are called *multiplicatively independent* if the only pair of integers (m, n) such that $x^m y^n = 1$ is the pair $(0, 0)$.

Problem 1.2. *Are there irrational real numbers having a 'simple' expansion in two multiplicatively independent bases?*

We established in [3] that the complexity function of the r-ary expansion of an irrational real number and that of its s-ary expansion cannot both grow too slowly when r and s are multiplicatively independent positive integers.

Theorem 1.3 ([3]). *Let r and s be multiplicatively independent positive integers. Any irrational real number ξ satisfies*

$$\lim_{n \to +\infty} \big(p(n, \xi, r) + p(n, \xi, s) - 2n \big) = +\infty.$$

Said differently, ξ cannot have simultaneously a quasi-Sturmian r-ary expansion and a quasi-Sturmian s-ary expansion.

We complement Theorem 1.3 by the following statement addressing expansions of a real number in two multiplicatively dependent bases.

Theorem 1.4. *Let $r, s \ge 2$ be multiplicatively dependent integers and m, ℓ be the smallest positive integers such that $r^m = s^\ell$. Then, there exist uncountably many real numbers ξ satisfying*

$$\lim_{n \to +\infty} \big(p(n, \xi, r) + p(n, \xi, s) - 2n \big) = m + \ell$$

and every irrational real number ξ satisfies

$$\lim_{n \to +\infty} \big(p(n, \xi, r) + p(n, \xi, s) - 2n \big) \ge m + \ell.$$

The next result, used in the proof of Theorem 1.4, has its own interest.

Theorem 1.5. *Let $b \geq 2$ be an integer and ρ, σ be positive integers. If σ divides ρ, then every real number whose b^ρ-ary expansion is quasi-Sturmian has a quasi-Sturmian b^σ-ary expansion. Moreover, every real number whose b^ρ-ary and b^σ-ary expansions are both quasi-Sturmian has a quasi-Sturmian b^μ-ary expansion, where μ is the least common multiple of ρ and σ.*

We conclude by an immediate consequence of Theorems 1.3 and 1.4.

Corollary 1.6. *Let $r, s \geq 2$ be distinct integers. No real number can have simultaneously a Sturmian r-ary expansion and a Sturmian s-ary expansion.*

Our paper is organized as follows. Section 2 gathers auxiliary results on Sturmian and quasi-Sturmian words. Theorems 1.4 and 1.5 are established in Section 4.

2. Auxiliary results

We will make use of the following characterisation of quasi-Sturmian words.

Lemma 2.1. *An infinite word $\mathbf{x}$ written over a finite alphabet $\mathcal{A}$ is quasi-Sturmian if and only if there are a finite word W, a Sturmian word $\mathbf{s}$ defined over $\{0,1\}$ and a morphism ϕ from $\{0,1\}^*$ into $\mathcal{A}^*$ such that $\phi(01) \neq \phi(10)$ and*

$$\mathbf{x} = W\phi(\mathbf{s}).$$

Proof. See [4]. □

Throughout this paper, for a finite word W and an integer t, we write W^t for the concatenation of t copies of W and W^∞ for the concatenation of infinitely many copies of W. We denote by $|W|$ the length of W, that is, the number of letters composing W. A word U is called periodic if $U = W^t$ for some finite word W and an integer $t \geq 2$. If U is periodic, then the period of U is defined as the length of the shortest word W for which there exists an integer $t \geq 2$ such that $U = W^t$.

Lemma 2.2. *Let U be a finite word. Assume that there exist words U_1, U_2, V, W such that $U = U_1U_2$ and $UU = VU_2U_1W$, with $|U_1| \neq |V|$ and $0 < |V| < |U|$. Then, the word U is periodic.*

Proof. Since V is a prefix of U and W is a suffix of U, we get

$$U = U_1U_2 = VW,$$

thus, $VU_2U_1W = UU = VWVW$. This implies

$$U_2U_1 = WV.$$

If $|U_1| < |V|$, then we can write $V = V'U_1$ for a nonempty word V', thus $U_2 = WV'$. Therefore,

$$U_1WV' = U_1U_2 = VW = V'U_1W.$$

Our assumption $0 < |V| < |U|$ implies that the word $Z := U_1W$ is nonempty. Since $ZV' = V'Z$, it follows from Theorem 1.5.3 of [1] that $U = ZV'$ is periodic. The proof of the case $|U_1| > |V|$ is similar. □

Lemma 2.3. *Let $\mathcal{A}$ be a finite set,* $\mathbf{s}$ *a Sturmian word over* $\{0,1\}$*, and* ϕ *a morphism from* $\{0,1\}^*$ *into* $\mathcal{A}^*$ *satisfying* $\phi(01) \neq \phi(10)$*. Then there exists an integer* n_0 *such that, for any factor* A *of* $\mathbf{s}$ *of length greater than* n_0*, if one can write* $\phi(A)$ *as* $V_1\phi(b_2b_3\dots b_{m-1})V_2$*, where* $B = b_1b_2\dots b_{m-1}b_m$ *is a factor of* $\mathbf{s}$*, the word* V_1 *is a nonempty suffix of* $\phi(b_1)$*, and* V_2 *is a nonempty prefix of* $\phi(b_m)$*, then* $V_1 = \phi(b_1)$*,* $V_2 = \phi(b_m)$ *and* $A = B$*.*

Proof. We may assume that 1 is the isolated letter in $\mathbf{s}$, i.e., that 11 is not a factor of $\mathbf{s}$. Since $\mathbf{s}$ is balanced, there exists a positive integer k such that 10^t1 is a factor of $\mathbf{s}$ if and only if $t = k$ or $k+1$.

We first consider the case where $V_1 = \phi(b_1)$. Suppose that $A \neq B$. Then, by deleting the maximal common prefix of A and B, we may assume that A and B have no common prefix. Thus, the prefixes of A and B are 00 and 10.

If $\phi(00) = \phi(10)V_2$, then $\phi(0) = \phi(1)V_2 = V_2\phi(1)$ and there exist a word U and positive integers s,t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

If $\phi(10) = \phi(0^h)V_2$ for some integer $h \geq 2$ and a nonempty prefix V_2 of $\phi(0)$, then, writing $\phi(0) = V_2V'$, we get $\phi(0) = V_2V' = V'V_2$, thus there exist a word U and positive integers s,t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

If $\phi(10) = \phi(0^h)V_2$ for some integer $h \geq 2$ and a nonempty prefix V_2 of $\phi(1)$, then there exists a positive integer ℓ and a prefix V' of $\phi(0)$ such that $\phi(1) = \phi(0)^\ell V'$. Write $\phi(0) = V'V''$. Then, $\phi(10) = \phi(0)^\ell V'\phi(0) = \phi(0)^{\ell+1}V'$ and we get $V'\phi(0) = \phi(0)V'$. Thus, there exist a word U and positive integers s,t such that $\phi(1) = U^s$ and $\phi(0) = U^t$. This gives a contradiction to $\phi(01) = \phi(10)$.

Similarly, we show that, if $V_2 = \phi(b_m)$, then $A = B$.

It only remains for us to treat the case where $V_1 \neq \phi(b_1)$ and $V_2 \neq \phi(b_m)$. There exists an integer n_0 such that any factor A of $\mathbf{s}$ of length greater than n_0 contains $10^k10^{k+1}10$. It is sufficient to consider the case where $\phi(10^k10^{k+1}10) = V_1\phi(b_2b_3\dots b_{m-1})V_2$, for a factor $b_1b_2\dots b_m$ of $\mathbf{s}$ and with V_1 a proper nonempty suffix of $\phi(b_1)$ and V_2 a proper nonempty prefix of $\phi(b_m)$.

If $b_2b_3\dots b_{m-1} = 0^{k+1}10^k1$, then $b_1 = 1$ and $b_m = 0$. Thus $|V_1| < |\phi(1)|$ and $|V_2| < |\phi(0)|$, which contradicts

$$|V_1| + |V_2| < |\phi(1)| + |\phi(0)| = |\phi(10^k10^{k+1}10)| - |\phi(0^{k+1}10^k1)|.$$

Therefore, since any subword of $\mathbf{s}$ in which 10^k10 and $10^{k+1}1$ do not occur is a factor of $0^{k+1}10^k1$, we deduce that if $\phi(10^k10^{k+1}10) = V_1\phi(b_2\dots b_{m-1})V_2$ as above, then $b_2\dots b_{m-1}$ contains 10^k10 or $10^{k+1}1$.

We distinguish three cases:

Case (i) : $\phi(10^k10^{k+1}10) = W_1\phi(10^k10)W_2$, where $0 < |W_1| < |\phi(10^k)|$.
Then

$$\phi(10^k10^k) = W_1\phi(10^k)W_2', \qquad \phi(0^k100^k10) = W_1'\phi(0^k10)W_2,$$

where $|W_2'| = |W_2| - |\phi(0)|$ and $|W_1'| = |W_1|$.

Case (ii) : $\phi(10^k10^{k+1}10) = W_1\phi(10^k10)W_2$, where $|\phi(10^k)| < |W_1| < |\phi(10^{k+1})|$. Then

$$\phi(10^k10^k) = W_1'\phi(0^k1)W_2', \qquad \phi(0^k100^k10) = W_1''\phi(0^k10)W_2,$$

where $|W_1'| = |W_1| - |\phi(0^k)|$, $|W_2'| = |W_2| + |\phi(0^{k-1})|$ and $|W_1''| = |W_1|$.

Case (iii) : $\phi(10^k10^{k+1}10) = W_1\phi(10^{k+1}1)W_2$, where $0 < |W_1| < |\phi(10^{k+1})|$. Then

$$\phi(10^k10^k) = W_1\phi(10^k)W_2', \qquad \phi(0^k100^k10) = W_1'\phi(0^{k+1}1)W_2,$$

where $|W_2'| = |W_2| - |\phi(0)|$ and $|W_1'| = |W_1|$.

By Lemma 2.2, in each Case (i), (ii), (iii), the factors $\phi(10^k)$ and $\phi(0^k10)$ are periodic. Denoting by λ_1, λ_2 the periods of $\phi(10^k)$, $\phi(0^k10)$, we get

$$\lambda_1 \le \frac{|\phi(10^k)|}{2} = \frac{k|\phi(0)| + |\phi(1)|}{2}, \quad \lambda_2 \le \frac{|\phi(0^k10)|}{2} = \frac{(k+1)|\phi(0)| + |\phi(1)|}{2}.$$

Write $\phi(10^k) = U^t$ for a word U with $|U| = \lambda_1$ and integer $t \ge 2$. Then $\phi(1) = U^{t_1}U_1$, $\phi(0^k) = U_2U^{t_2}$ for some words U_1, U_2 with $U = U_1U_2$ and some nonnegative integers t_1, t_2 satisfying $t_1 + t_2 = t - 1$. Thus, we get

$$\phi(0^k1) = U_2(U_1U_2)^{t_2}(U_1U_2)^{t_1}U_1 = (U_2U_1)^t, \qquad |U_2U_1| = \lambda_1.$$

Since $\phi(0)$ is a prefix of $(U_2U_1)^t$, we deduce that $\phi(0^k10) = (U_2U_1)\cdots(U_2U_1)U'$ for a prefix U' of U_2U_1, It then follows from [5, Lemma 3 (v)] that $\lambda_1 = \lambda_2$ or

$$|\phi(0^k10)| < \lambda_1 + \lambda_2 \le (k + \frac{1}{2})|\phi(0)| + |\phi(1)| < |\phi(0^k10)|,$$

in which case we have a contradiction. If $\lambda_1 = \lambda_2$, then λ_1 divides $|\phi(0^k10)|$ and $|\phi(10^k)|$, thus λ_1 divides $|\phi(0)|$ and $|\phi(1)|$. This implies that $\phi(01) = \phi(10) = UU\cdots U$, giving again a contradiction. □

We end this section with an easy result on the convergents of irrational numbers.

Lemma 2.4. *Let $(\frac{p_k}{q_k})_{k\ge0}$ be the sequence of convergents of an irrational number $[0; a_1, a_2, \ldots]$ in $(0, 1)$ and $d \ge 2$ be an integer. Let c_1, c_2 be integers not both multiple of d. Then, for any positive integer k, we have $c_1p_k + c_2q_k \not\equiv 0 \pmod d$ or $c_1p_{k+1} + c_2q_{k+1} \not\equiv 0 \pmod d$.*

Proof. Since

$$\begin{bmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix}\cdots\begin{bmatrix} 0 & 1 \\ 1 & a_{k+1} \end{bmatrix},$$

we have

$$\begin{bmatrix} c_1p_k + c_2q_k & c_1p_{k+1} + c_2q_{k+1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix}\cdots\begin{bmatrix} 0 & 1 \\ 1 & a_{k+1} \end{bmatrix},$$

thus

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} c_1p_k + c_2q_k & c_1p_{k+1} + c_2q_{k+1} \end{bmatrix}\begin{bmatrix} -a_{k+1} & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} -a_2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -a_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, if $\begin{bmatrix} c_1p_k + c_2q_k & c_1p_{k+1} + c_2q_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ modulo d, then c_1 and c_2 are multiple of d. □

3. Proofs of Theorems 1.4 and 1.5

We begin with the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $b \geq 2$ be an integer and ρ, σ be positive integers. Assume that $\rho = d\sigma$ for some integer $d \geq 2$. Let ξ be a real number and assume that there are integers $a_1, a_2, \ldots$ in $\{0, 1, \ldots, b^\rho - 1\}$ and k, n_0 such that

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{a_i}{b^{\rho i}} \quad \text{and} \quad p(n, \xi, b^\rho) = n + k \text{ for } n \geq n_0.$$

Then, by Lemma 2.1, there are a finite word W, a Sturmian word $\mathbf{s}$ defined over $\{0, 1\}$ and a morphism ϕ from $\{0, 1\}^*$ into $\{0, 1, \ldots, b^\rho - 1\}^*$ such that $\phi(01) \neq \phi(10)$ and

$$\mathbf{a} = a_1 a_2 \ldots = W\phi(\mathbf{s}).$$

Let a be in $\{0, 1, \ldots, b^\rho - 1\}$ and consider its representation in base b^σ given by $a = c_1 b^{(d-1)\sigma} + c_2 b^{(d-2)\sigma} + \ldots + c_d b^{0 \cdot \sigma}$, where $c_1, \ldots, c_d$ are in $\{0, 1, \ldots, b^\sigma - 1\}$. Define the function $\phi_{\rho,\sigma}$ on $\{0, 1, \ldots, b^\rho - 1\}$ by setting $\phi_{\rho,\sigma}(a) = c_1 c_2 \ldots c_d$. It extends to a morphism from $\{0, 1, \ldots, b^\rho - 1\}^*$ to $\{0, 1, \ldots, b^\sigma - 1\}^*$, which we also denote by $\phi_{\rho,\sigma}$. Then, we have

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{d_i}{b^{\sigma i}}, \quad \text{where } \mathbf{d} = d_1 d_2 \ldots = \phi_{\rho,\sigma}(W)\, (\phi_{\rho,\sigma} \circ \phi)(\mathbf{s}).$$

We deduce from Lemma 2.1 that the b^σ-ary expansion of ξ is quasi-Sturmian. Thus we have established the first assertion of the theorem.

For the second assertion of the theorem, we may assume that ρ and σ are relatively prime (otherwise, we replace b by b^g where g is the greatest common divisor of ρ and σ).

Let ξ be a real number and write

$$\xi = \lfloor \xi \rfloor + \sum_{i \geq 1} \frac{a_i}{b^{\rho i}} = \lfloor \xi \rfloor + \sum_{j \geq 1} \frac{b_j}{b^{\sigma j}},$$

where $a_1, a_2, \ldots$ are in $\{0, 1, \ldots, b^\rho - 1\}$ and $b_1, b_2, \ldots$ are in $\{0, 1, \ldots, b^\sigma - 1\}$. Assume that $\mathbf{a} = a_1 a_2 \ldots$ and $\mathbf{b} = b_1 b_2 \ldots$ are both quasi-Sturmian. By Lemma 2.1, there are a finite word W, a Sturmian word $\mathbf{s}$ defined over $\{0, 1\}$ and a morphism ϕ from $\{0, 1\}^*$ into $\{0, 1, \ldots, b^\rho - 1\}^*$ such that $\phi(01) \neq \phi(10)$ and

$$\mathbf{a} = a_1 a_2 \ldots = W\phi(\mathbf{s}).$$

We claim that $|\phi(0)| =: l_0$ and $|\phi(1)| =: l_1$ are both multiple of σ.

In order to deduce a contradiction, we suppose that σ does not divide at least one of l_0 and l_1.

Let $\phi_{\rho,1}$ be the morphism $\phi_{\rho,\sigma}$ defined above in the case $\sigma = 1$. For each factor U of $\mathbf{s}$, let

$$\Lambda(U) := \{0 \leq j \leq \sigma - 1 : \phi_{\rho,1}(\mathbf{a}) = V\phi_{\rho,1} \circ \phi(U) \text{ for some } V \text{ with } |V| \equiv j \pmod{\sigma}\}$$

denote the nonempty set of positions modulo σ where $\phi_{\rho,1} \circ \phi(U)$ occurs in $\phi_{\rho,1}(\mathbf{a})$. If U' is a prefix of U, then $\Lambda(U)$ is a subset of $\Lambda(U')$. Consequently, there exists N such that $\Lambda(s_1 \ldots s_n) = \Lambda(s_1 \ldots s_N)$ for each $n \geq N$.

Let $[0; a_1, a_2, \ldots]$ denote the continued fraction expansion of the slope of $\mathbf{s}$ and, for $k \geq 1$, let q_k be the denominator of the convergent $[0; a_1, \ldots, a_k]$ to this slope. Define the sequence $(M_k)_{k\geq 0}$ of finite words over $\{0, 1\}$ by

$$M_0 = 0, \quad M_1 = 0^{a_1-1}1, \quad \text{and} \quad M_{k+1} = (M_k)^{a_k} M_{k-1}, \quad (k \geq 1).$$

For $k \geq 1$, the word M_k is a factor of length q_k of $\mathbf{s}$ (see e.g. [7]). Since there are p_k occurrences of the digit 1 in M_k, we get

$$|\phi(M_k)| = l_0(q_k - p_k) + l_1 p_k = (l_1 - l_0)p_k + l_0 q_k.$$

By Lemma 2.4 and the assumption that σ does not divide at least one of l_0 and l_1, we conclude that at least one of $|\phi(M_k)|$ and $|\phi(M_{k+1})|$ is not a multiple of σ.

Let U be a factor of $\mathbf{s}$. Then U is a factor of M_k for some integer k. Since $M_k M_k$ is a factor of $M_{k+2}M_{k+1} = (M_{k+1})^{a_{k+2}} M_k (M_k)^{a_{k+1}} M_{k-1}$, which is a factor of $\mathbf{s}$, there are two positions of $\phi(U)$ which differ by $|\phi(M_k)|$. Thus, there exist two occurrences of $\phi(U)$ in $\phi(\mathbf{s})$ separated by exactly $\rho|\phi(M_k)|$ letters. Replacing k by $k+1$ is necessary, we can assume that $\rho|\phi(M_k)|$ is not a multiple of σ and we deduce that $|\Lambda(U)| \geq 2$ for any factor U of $\mathbf{s}$.

A finite word U is called right special if U is a prefix of two different factors of $\mathbf{s}$ of the same length. If the initial word $s_1 \ldots s_n$ of $\mathbf{s}$ is not a prefix of a right special word, then either $s_{j+1} \ldots s_{j+n} \neq s_1 \ldots s_n$ for all $j \geq 1$, or $\mathbf{s}$ is periodic. Since a Sturmian word is recurrent and not periodic (see, e.g., [6, page 158]), there are infinitely many prefixes $s_1 \ldots s_n$ of $\mathbf{s}$ which are right special. Let $n \geq N$ be such that $s_1 \ldots s_n$ is right special. Then, there exists a letter c such that $c \neq s_{n+1}$ and $s_1 \ldots s_n c$ is a factor of $\mathbf{s}$. Thus, we get

$$\Lambda(s_1 \ldots s_n s_{n+1}) = \Lambda(s_1 \ldots s_n) \supset \Lambda(s_1 \ldots s_n c).$$

Choose i, j in $\Lambda(s_1 \ldots s_n c)$ with $0 \leq i < j \leq \sigma - 1$. Then we can write

$$\phi_{\rho,1}(\mathbf{a}) = UU_1\phi_{\rho,1} \circ \phi(s_1 \ldots s_n c)U_1' \ldots = U'U_2\phi_{\rho,1} \circ \phi(s_1 \ldots s_n s_{n+1})U_2' \ldots$$

and

$$\phi_{\rho,1}(\mathbf{a}) = VV_1\phi_{\rho,1} \circ \phi(s_1 \ldots s_n c)V_1' \ldots = V'V_2\phi_{\rho,1} \circ \phi(s_1 \ldots s_n s_{n+1})V_2' \ldots,$$

for some words $U, U', V, V', U_1, U_2, V_1, V_2, U_1', U_2', V_1', V_2'$ written over $\{0, \ldots, b-1\}$ and satisfying

$$|U_1| = |U_2| = i, \ |V_1| = |V_2| = j, \ |U| \equiv |U'| \equiv |V| \equiv |V'| \equiv 0 \pmod{\sigma},$$
$$0 \leq |U_1'| = |U_2'| \leq \sigma - 1, \quad 0 \leq |V_1'| = |V_2'| \leq \sigma - 1,$$

and σ divides $i + (n+1)\rho + |U_1'|$ and $j + (n+1)\rho + |V_1'|$. Thus, there exist u_1, u_2, v_1, v_2 in $\{0, 1, \ldots, b^\sigma - 1\}$ and words X, Y, A_1, A_2, B_1, B_2 written over $\{0, 1, \ldots, b^\sigma - 1\}$ with

$$|X| = \left\lfloor \frac{i + n\rho}{\sigma} \right\rfloor - 1, \quad |Y| = \left\lfloor \frac{j + n\rho}{\sigma} \right\rfloor - 1$$

and

$$A_1 \neq A_2, \quad B_1 \neq B_2, \quad |A_1| = |A_2| < \frac{\rho}{\sigma} + 2, \quad |B_1| = |B_2| < \frac{\rho}{\sigma} + 2,$$

such that

$$\begin{aligned} U_1\phi_{\rho,1} \circ \phi(s_1 \dots s_n c)U_1' &= \phi_{\sigma,1}(u_1 X A_1), \\ U_2\phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1})U_2' &= \phi_{\sigma,1}(u_2 X A_2), \\ V_1\phi_{\rho,1} \circ \phi(s_1 \dots s_n c)V_1' &= \phi_{\sigma,1}(v_1 Y B_1), \\ V_2\phi_{\rho,1} \circ \phi(s_1 \dots s_n s_{n+1})V_2' &= \phi_{\sigma,1}(v_2 Y B_2). \end{aligned}$$

Here, $\phi_{\sigma,1}$ is defined analogously as $\phi_{\rho,1}$. Therefore, u_1XA_1, u_2XA_2 and v_1YB_1, v_2YB_2 are all factors of $\phi_{\sigma,1}^{-1}(\phi_{\rho,1}(\phi(\mathbf{s})))$. Denoting by A (resp., by B) the longest common prefix (it could be the empty word) of A_1 and A_2 (resp., of B_1 and B_2), we deduce that XA and YB are both right special.

Let W_0 be the longest common prefix of $\phi_{\rho,1}\circ\phi(s_1 \dots s_n s_{n+1})$ and $\phi_{\rho,1}\circ\phi(s_1 \dots s_n c)$. Then, there exist finite words W_1, W_2, W_1', W_2' over $\{0, \dots, b-1\}$ satisfying $|W_1| = \sigma - i$, $|W_2| = \sigma - j$, $|W_1'| < \sigma$, $|W_2'| < \sigma$, and

$$W_0 = W_1\phi_{\sigma,1}(XA)W_1' = W_2\phi_{\sigma,1}(YB)W_2',$$

Thus, we get $|XA| \le |YB| \le |XA| + 1$.

Suppose that XA is a suffix of YB. Then, there exists a nonempty finite word W' of length less than σ such that

$$\begin{aligned} W_0 &= W_2W'\phi_{\sigma,1}(XA)W_1' = W_2\phi_{\sigma,1}(XA)W_2', && \text{if } |XA| = |YB|, \\ W_0 &= W_1\phi_{\sigma,1}(XA)W_1' = W_1W'\phi_{\sigma,1}(XA)W_2', && \text{if } |XA| + 1 = |YB|. \end{aligned}$$

It then follows from Theorem 1.5.2 of [1] that we have $W_0 = W_2(W')^tW''W_1'$ or $W_1(W')^tW''W_2'$, respectively, for some integer t and a prefix W'' of W'. Since ρ, σ are fixed and $\mathbf{s}$ is Sturmian, we deduce from Lemma 2.3 of [3] that $(W')^t$ cannot be a factor of $\phi_{\rho,1} \circ \phi(s_1 \dots s_n)$ when n is sufficiently large. This shows that the lengths of XA and YB are bounded independently of n.

Consequently, the right special words XA and YB are not suffixes of each others if n is sufficiently large. Hence, there are arbitrarily large integers m such that $\phi_{\sigma,1}^{-1} \circ \phi_{\rho,1} \circ \phi(\mathbf{s})$ has two distinct right special words of length m. This implies that $\mathbf{b} = \phi_{\sigma,1}^{-1} \circ \phi_{\rho,1}(\mathbf{a})$ is not quasi-Sturmian, which gives a contradiction. Therefore, we have established that $|\phi(0)|$ and $|\phi(1)|$ are both multiple of σ.

Write

$$\xi = \lfloor \xi \rfloor + \sum_{i \ge 1} \frac{c_i}{b^{\rho\sigma i}}, \qquad \mathbf{c} = c_1c_2 \dots = \phi_{\rho\sigma,\rho}^{-1}(\mathbf{a}) = \phi_{\rho\sigma,\rho}^{-1}(W\phi(\mathbf{s})).$$

Put $|W| = h\sigma + d$ for integers $h \ge 0$ and d with $0 \le d < \sigma$. Let $\phi(0) = X_1X_2$, $\phi(1) = Y_1Y_2$, where $|X_1| = |Y_1| = \sigma - d$. Assume that 11 is not a factor of $\mathbf{s}$. Then there exists a positive integer k such that 10^m1 is a factor of $\mathbf{s}$ if and only if $m = k$ or $k + 1$. Thus, we can represent $\mathbf{s}$ as

$$\mathbf{s} = 0^w t_0t_1t_2t_3 \dots, \qquad t_0 = 10^k, \ t_i \in \{10^k, 0\}, \ 0 \le w \le k + 1.$$

It is not difficult to check that $\mathbf{t} := t_0t_1t_2\dots$ is Sturmian. Define ϕ' by

$$\phi'(10^k) = X_2Y_1Y_2(X_1X_2)^{k-1}X_1, \quad \phi'(0) = X_2X_1.$$

Then we get

$$\phi(\mathbf{s}) = (X_1X_2)^w Y_1Y_2(X_1X_2)^{k-1}X_1\phi'(t_1t_2t_3\dots),$$

thus

$$\mathbf{c} = \phi^{-1}_{\rho\sigma,\rho}(W\phi(\mathbf{s})) = \phi^{-1}_{\rho\sigma,\rho}(W(X_1X_2)^w Y_1Y_2(X_1X_2)^{k-1}X_1)(\phi^{-1}_{\rho\sigma,\rho}\circ\phi')(t_1t_2t_3\dots).$$

Since $|\phi(0)|$ and $|\phi(1)|$ are both multiple of σ, the morphism $\phi^{-1}_{\rho\sigma,\rho}\circ\phi'$ is well-defined. We conclude that $\mathbf{c}$ is quasi-Sturmian and the proof of the theorem is complete. □

Lemma 3.1. *Let $b \ge 2$, $d \ge 2$, ρ, σ be positive integers with $\rho = d\sigma$. Let $x_1x_2\dots$ be a quasi-Sturmian word over $\{0,1,\dots,b^\rho-1\}$. Then, there exists an integer n_0 such that the real number $\xi = \sum_{k\ge1}\frac{x_k}{b^{\rho k}}$ satisfies*

$$p(nd,\xi,b^\sigma) \ge (n+1)d, \quad \text{for } n \ge n_0.$$

Furthermore, if $s_1s_2\dots$ is a Sturmian word written over $\{0,1\}$, then there exists an integer n_0 such that the real number $\xi = \sum_{k\ge1}\frac{s_k}{b^{\rho k}}$ satisfies

$$p(n,\xi,b^\sigma) = n+d, \quad \text{for } n \ge n_0.$$

Proof. Set $\mathcal{A} := \{0,1,\dots,b^\rho-1\}$. There exist a Sturmian word $\mathbf{s}$ written over $\{0,1\}$, a morphism ϕ from $\{0,1\}^*$ into $\mathcal{A}^*$ satisfying $\phi(01)\ne\phi(10)$, and a factor W of $\mathbf{x} := x_1x_2\dots$ such that $\mathbf{x} = W\phi(\mathbf{s})$. Then, the word

$$\mathbf{y} := \phi_{\rho,\sigma}(\mathbf{x}) = \phi_{\rho,\sigma}(W\phi(\mathbf{s})) = \phi_{\rho,\sigma}(W)(\phi_{\rho,\sigma}\circ\phi)(\mathbf{s})$$

is quasi-Sturmian.

Let n be a positive integer larger than the integer n_0 given by Lemma 2.3 applied to the morphism $\phi_{\rho,\sigma}\circ\phi$. We claim that if $U_1\phi_{\rho,\sigma}(A_1)V_1 = U_2\phi_{\rho,\sigma}(A_2)V_2$, where A_1, A_2 are factors of $\phi(\mathbf{s})$ of length n and U_1, U_2 (resp., V_1, V_2) are nonempty suffixes (resp., proper prefixes) of words of the form $\phi_{\rho,\sigma}(a)$ for a in $\mathcal{A}$, then $U_1 = U_2$, $A_1 = A_2$ and $V_1 = V_2$.

Suppose not. Then we may assume that there exist A_1, A_2 and U, V such that

$$\phi_{\rho,\sigma}(A_1)V = U\phi_{\rho,\sigma}(A_2).$$

Thus there exist a_1, a_2 in $\mathcal{A}$, a factor A of $\phi(\mathbf{s})$ of length n, and a factor A' of $\phi(\mathbf{s})$ of length $n-1$ such that $\phi_{\rho,\sigma}(A) = W_1\phi_{\rho,\sigma}(A')W_2$, where W_1 (resp., W_2) is a nonempty proper suffix (resp., prefix) of $\phi_{\rho,\sigma}(a_1)$ (resp., of $\phi_{\rho,\sigma}(a_2)$). Consequently, there exist b, b', c, c' in $\{0,1\}$ and factors B, B' of $\mathbf{s}$ such that $A = U\phi(B)V$, $a_1A'a_2 = U'\phi(B')V'$, where U (resp., U') is a nonempty suffix of $\phi(b)$ (resp., $\phi(b')$) and V (resp., V') is a nonempty prefix of $\phi(c)$ (resp., $\phi(c')$). Then $A' = U''\phi(B')V''$ for words U'', V'' such that $U' = a_1U''$, $V' = V''a_2$. Therefore, we get

$$\phi_{\rho,\sigma}(A) = \phi_{\rho,\sigma}(U)(\phi_{\rho,\sigma}\circ\phi)(B)\phi_{\rho,\sigma}(V) = W_1\phi_{\rho,\sigma}(U'')(\phi_{\rho,\sigma}\circ\phi)(B')\phi_{\rho,\sigma}(V'')W_2.$$

We deduce from Lemma 2.3 that $\phi_{\rho,\sigma}(U) = W_1\phi_{\rho,\sigma}(U'')$, $\phi_{\rho,\sigma}(V) = \phi_{\rho,\sigma}(V'')W_2$ and $B = B'$. This is a contradiction to the fact that W_1 (resp., W_2) is a nonempty proper suffix (resp., prefix) of $\phi_{\rho,\sigma}(a_1)$ (resp., of $\phi_{\rho,\sigma}(a_2)$). Hence, the representation of $X = U\phi_{\rho,\sigma}(A)V$ is unique.

If $\phi(\mathbf{s})$ is written over an alphabet of three letters or more, then

$$p(n-1, \phi(\mathbf{s})) \geq (n-1) + 2 = n+1,$$

which implies that the number of factors X of $(\phi_{\rho,\sigma} \circ \phi)(\mathbf{s})$ of length nd is at least equal to $(n+1)d$. If $\phi(\mathbf{s})$ is written over an alphabet of two letters, say over the alphabet $\mathcal{A} = \{a, b\}$, then we can put $\phi_{\rho,\sigma}(a) = ZX$ and $\phi_{\rho,\sigma}(b) = ZY$, where Z is the longest common prefix of $\phi_{\rho,\sigma}(a), \phi_{\rho,\sigma}(b)$ and the first letters of X, Y are different. If $|V| > |Z|$, then for each right special factor A of $\mathbf{s}$ there are two distinct factors $\phi_{\rho,\sigma}(A)V_1$, $\phi_{\rho,\sigma}(A)V_2$ in $\phi(\mathbf{s})$. If $|V| \leq |Z|$, then $|U| \geq |X| = |Y|$, thus for each left special factor B of $\mathbf{s}$ there are two factors $U_1\phi_{\rho,\sigma}(B)$, $U_2\phi_{\rho,\sigma}(B)$ in $\phi(\mathbf{s})$. For each $c = 0, \dots, d-1$, the number of factors $X = U\phi_{\rho,\sigma}(A)V$ of $(\phi_{\rho,\sigma} \circ \phi)(\mathbf{s})$ of length nd with $|A| = n-1$ and $|U| = d - |V| = c$ is at least equal to $p(n-1, \phi(\mathbf{s})) + 1$. Therefore, we get

$$p(nd, \xi, b^\sigma) \geq p(nd, (\phi_{\rho,\sigma} \circ \phi)(\mathbf{s})) \geq (n+1)d.$$

Since the function $m \mapsto p(m, \xi, b^\sigma)$ is strictly increasing, this implies the first assertion of the theorem.

For the second assertion, let $\mathbf{s} = s_1 s_2 \dots$ be a Sturmian word written over the subset $\{0, 1\}$ of $\{0, 1, \dots, b^\rho - 1\}$ and define

$$\xi = \sum_{i \geq 1} \frac{s_i}{b^{\rho i}}.$$

Since $\phi_{\rho,\sigma}(0) = 0^d$ and $\phi_{\rho,\sigma}(1) = 0^{d-1}1$, for $n \geq 1$, any factor of length dn of $\phi_{\rho,\sigma}(\mathbf{s})$ is a suffix of $\phi_{\rho,\sigma}(A)0^k$, where A is a factor of length n in $\mathbf{s}$ and $0 \leq k \leq d-1$. Since 0^{d-1} is a prefix of $\phi_{\rho,\sigma}(A)0^k$, the number of suffixes of $\phi_{\rho,\sigma}(A)0^k$ of length nd is $d(n+1)$, thus

$$p(dn, \xi, b^\sigma) = d(n+1) = dn + d.$$

Since the function $m \mapsto p(m, \xi, b^\sigma)$ is strictly increasing, this completes the proof of the theorem. □

Proof of Theorem 1.4. Suppose that the two bases $r \geq 2$ and $s \geq 2$ are multiplicatively dependent and let m, ℓ be the coprime positive integers satisfying $r^m = s^\ell$. Then, there exists a positive integer b such that $r = b^\ell$ and $s = b^m$.

Let $\mathbf{s} = s_1 s_2 \dots$ be a Sturmian word over the subset $\{0, 1\}$ of $\{0, 1, \dots, b^{m\ell} - 1\}$ and define

$$\xi = \sum_{i \geq 1} \frac{s_i}{b^{m\ell i}}.$$

By the second assertion of Lemma 3.1, there exists an integer n_0 such that

$$p(n, \xi, b^\ell) = n + m \quad \text{and} \quad p(n, \xi, b^m) = n + \ell, \quad \text{for } n \geq n_0.$$

Thus,

$$\lim_{n\to+\infty} \big(p(n,\xi,r) + p(n,\xi,s) - 2n\big) = m + \ell.$$

This proves the first assertion of the theorem.

For the second assertion of the theorem, it is sufficient to consider a real number ξ whose b^ℓ-ary and b^m-ary expansions are both quasi-Sturmian. By Theorem 1.5, the $b^{\ell m}$-ary expansion of ξ is also quasi-Sturmian and we deduce from the first assertion of Lemma 3.1 that there exists an integer n_0 such that

$$p(mn,\xi,b^\ell) \ge m(n+1) \quad \text{and} \quad p(\ell n,\xi,b^m) \ge \ell(n+1), \quad \text{for } n \ge n_0.$$

Therefore,

$$\lim_{n\to+\infty} \big(p(n,\xi,r) + p(n,\xi,s) - 2n\big) \ge m + \ell.$$

This completes the proof of the theorem. □

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