Dynamics for β-shifts and Diophantine approximation

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Abstract. *We investigate the β-expansion of an algebraic number in an algebraic base β. Using tools from Diophantine approximation, we prove several results that may suggest a strong difference between the asymptotic behaviour of eventually periodic expansions and that of non-eventually periodic ones.*

1. Introduction

Among the different possible representations of real numbers in a real (or complex) base β, the β-expansion introduced by Rényi [32] takes a very special place. This expansion and related objects have been studied for many reasons and in many areas, including ergodic theory and symbolic dynamics [32, 30, 22, 10, 33], tilings [36, 9], quasi-crystals and mathematical physics [28, 29], number theory [34, 10, 16, 5, 37], and formal languages and theoretical computer science (see Chap. 7 of [26]). For more details on these topics, the reader is referred to the nice book of Dajani and Kraaikamp [18], Chap. 7 of [26], or to the PhD Thesis of Bernat [8] and the references quoted therein. The present paper mainly focuses on the Rényi β-expansion (or, for short, β-expansion) of an algebraic number in an algebraic base β, with a point of view from Diophantine approximation.

Let $\beta > 1$ be a real number. The β-transformation T_β is defined on $[0,1]$ by $T_\beta : x \longmapsto \beta x \bmod 1$. The β-expansion of a real x in $[0,1]$ is denoted by $d_\beta(x)$, and defined as follows:

$$d_\beta(x) = 0.x_1 x_2 \ldots x_n \ldots,$$

where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$. For $x < 1$, this expansion coincides with the representation of x computed by the 'greedy algorithm'. If β is an integer, the digits x_i of x lie in the set $\{0, 1, \ldots, \beta - 1\}$ and $d_\beta(x)$ corresponds, for $x \neq 1$, to the usual β-adic expansion of x. When β is not an integer, the digits x_i lie in the set $\{0, 1, \ldots, \lfloor \beta \rfloor\}$. Here and throughout the present paper, $\lfloor \cdot \rfloor$ is the integer part function. As proved by Rényi [32], the map T_β has a unique (up to proportionality) invariant measure μ_β that is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. Furthermore, μ_β is ergodic [32] and it is the unique measure of maximal entropy [22]. The symbolic counterpart of this geometric dynamical system is the β-shift S_β, defined as the set, endowed with the shift, of all bi-infinite sequences for which every factor appears in the β-expansion of some x in $[0,1]$.

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An important feature of monotone one-dimensional expanding maps such as T_β is that their dynamics are ruled by the orbits of the critical points. Here, this implies that all the information of the β-shift is already contained in the expansion of 1. This explains why $d_\beta(1)$ plays a crucial role.

Unlike the case of the expansion in an integer base, it generally remains open to decide whether an algebraic number has or does not have an eventually periodic β-expansion in an algebraic base β. Moreover, it seems to be very difficult to describe the β-expansion of an algebraic number when this expansion is neither finite, nor eventually periodic. For instance, if we look at the first digits of the expansion of 1 in the base 3/2 (we know that such a sequence cannot be eventually periodic), there is no evidence of a particular structure. It is then tempting to provide a framework in which such a chaotic-looking expansion would find a more natural explanation.

Dynamical systems sometimes offer such a suitable framework thanks to the ergodic theorem. In particular, we mention the classical 'Borel conjecture' on the normality of algebraic irrational numbers (see for instance [13, 7, 24, 2]), and the speculations of Lang [25] on the continued fraction expansion of algebraic numbers of degree at least 3 (see also [1]). In view of this, we are led to introduce the two Hypotheses below. Of course, we do not claim to confirm them, but we believe that it can be fruitful to keep them in mind as a guideline. This is precisely what we will do along this paper.

Hypothesis $\mathcal{H}_1$. Let $\beta > 1$ and α be two real algebraic numbers. Then, α is either a periodic point or a generic point for the dynamical system $(T_\beta, [0,1], \mu_\beta)$.

When $\beta \geq 2$ is an integer, the measure μ_β is nothing else than the Lebesgue measure, and Hypothesis $\mathcal{H}_1$ thus reduces to the Borel conjecture evoked above.

We can also formulate the following weaker conjecture.

Hypothesis $\mathcal{H}_2$. Let $\beta > 1$ and α be two real algebraic numbers. Then, α is either a periodic point for the map T_β or the orbit of α is dense in $[0,1]$.

Unsurprisingly, both Hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ appear, at this point, to be untractable. However, they seem to be supported by numerical experiments [4].

The authors of [20] (see also the discussion in [8]) present as a conjecture a statement which can be reformulated as follows: if β is a Perron number, then the origin cannot be an accumulation point for the orbit $(T_\beta^n(1))_{n\geq 1}$. Note that their conjecture is in contradiction with both Hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$.

Our paper is organized as follows. In Section 2, we present our two main results that reveal a (weak) combinatorial dichotomy for the representations of algebraic numbers in an algebraic base β. Theorem 1 applies to general representations in an algebraic (real or complex) base, while Theorem 2 and the rest of the paper are only devoted to β-expansions. In Section 3, we investigate some consequences of Theorem 2 on the complexity of the β-expansion of an algebraic number in an algebraic base β. Section 4 is devoted to Blanchard's classification and related results, while in Section 5 we discuss the gaps (occurrences of strings of 0's) in $d_\beta(1)$. The proofs of Theorems 1 and 2 are based on the approach started in [2, 3] and rest on the Schmidt Subspace Theorem for number fields. They are postponed to Section 6.

2. Main results

We first associate with an infinite sequence $\mathbf{a}$ defined over a finite alphabet a combinatorial exponent, denoted by $\mathrm{Dio}(\mathbf{a})$, the so-called *Diophantine exponent* of $\mathbf{a}$. As we will see, this exponent is a measure of the periodicity of the sequence, and appears to be useful for deriving Diophantine properties of the real (or complex) number whose expansion is given by the sequence $\mathbf{a}$. In particular, we will show that if $\beta > 1$ is an algebraic number, algebraic numbers whose β-expansion is not eventually periodic all have a Diophantine exponent uniformly bounded in terms of the Mahler measure of β.

Let $\mathcal{A}$ be a finite set. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. For any positive integer ℓ, we write W^ℓ for the word $W \ldots W$ (the concatenation of the word W repeated ℓ times). More generally, for any positive real number x, we denote by W^x the word $W^{\lfloor x \rfloor} W'$, where W' is the prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$. Here, and in all what follows, $\lceil y \rceil$ denotes the ceiling of the real number y, that is, the smallest integer greater than, or equal to y. Let $\mathbf{a} = (a_k)_{k \geq 1}$ be a sequence of elements from $\mathcal{A}$, that we identify with the infinite word $a_1 a_2 \ldots$ Let $\rho \geq 1$ be a real number. We say that $\mathbf{a}$ satisfies Condition $(*)_\rho$ if there exist two sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n \geq 1}$, and a sequence of positive real numbers $(w_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $U_n V_n^{w_n}$ is a prefix of the word $\mathbf{a}$;
(ii) For any $n \geq 1$, $|U_n V_n^{w_n}| / |U_n V_n| \geq \rho$;
(iii) The sequence $(|V_n^{w_n}|)_{n \geq 1}$ is strictly increasing.

We then define $\mathrm{Dio}(\mathbf{a})$ to be the supremum of the real numbers ρ for which $\mathbf{a}$ satisfies Condition $(*)_\rho$. It follows from this definition that for any sequence $\mathbf{a}$, we have

$$1 \leq \mathrm{Dio}(\mathbf{a}) \leq +\infty.$$

Clearly all eventually periodic sequences have an infinite Diophantine exponent. Note that many of the classical sequences studied in symbolic dynamics, number theory and combinatorics on words have a Diophantine exponent greater than 1 (in [2], sequences with this property are called stammering sequences). This is for instance the case of the Sturmian sequences, the sequences of sublinear complexity, the automatic sequences and most of the substitutive sequences (see [2]).

With the above notation, the main result of [3] can be reformulated as follows.

Theorem ABL. *Let $b > 1$ be a positive integer and let $\mathbf{a} = (a_k)_{k \geq 1}$ be a sequence of integers with values in $\{0, 1, \ldots, b-1\}$. Assume that $Dio(\mathbf{a}) > 1$. Then, the real number*

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{b^k}$$

is either rational or transcendental.

Theorem ABL was extended in [2] in the following way. Recall that a Pisot (resp. Salem) number is a real algebraic integer > 1, whose complex conjugates lie inside the open unit disc (resp. inside the closed unit disc, with at least one of them on the unit circle). In particular, every integer $b > 1$ is a Pisot number.

Theorem AB. *Let β be a Pisot or a Salem number. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be a bounded sequence of rational integers. Assume that Dio($\mathbf{a}$) > 1. Then, the real number*

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$$

either lies in $\mathbf{Q}(\beta)$, or is transcendental.

A natural question is whether the above condition on β to be a Pisot or a Salem number can be relaxed to include all algebraic numbers lying outside the closed unit disc.

As a first result, we establish that, for any given algebraic number β with $|\beta| > 1$ and for any bounded sequence of rational integers $\mathbf{a} = (a_k)_{k\geq 1}$, the real number $\sum_{k=1}^{+\infty} a_k/\beta^k$ is either in $\mathbf{Q}(\beta)$, or is transcendental, provided that the Diophantine exponent of $\mathbf{a}$ is sufficiently large in terms of β. Throughout this paper, we denote by $M(\beta)$ the Mahler measure of the algebraic number β. We recall that if b is the leading coefficient of the minimal polynomial of β, and if $\beta_1 := \beta$, and if $\beta_2, \ldots, \beta_d$ denote the algebraic conjugates of β, then

$$M(\beta) = |b| \cdot \prod_{i=1}^{d} \max\{|\beta_i|, 1\}.$$

Theorem 1. *Let β be an algebraic number with $|\beta| > 1$. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be a bounded sequence of rational integers. Assume that*

$$\mathit{Dio}(\mathbf{a}) > \frac{\log M(\beta)}{\log |\beta|} \cdot \tag{2.1}$$

Then, the real number

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$$

either lies in $\mathbf{Q}(\beta)$, or is transcendental.

Note that $M(\beta) = \beta$ when β is a Pisot or Salem number. Consequently, Theorem 1 implies Theorem AB, and thus Theorem ABL.

The main drawback of Theorem 1 is that it gives no information about the representations of the elements lying in the number field $\mathbf{Q}(\beta)$. This is really unfortunate, since, as was already mentionned in Section 1, the β-expansion of 1 plays a crucial role for the study of the β-shift. Of course, if the sequence $\mathbf{a}$ is eventually periodic, then α clearly lies in $\mathbf{Q}(\beta)$, but the converse is in general not true. Indeed, even if we only consider β-expansions (instead of the more general representations considered in Theorem 1), there generally exist some elements in $\mathbf{Q}(\beta)$ with a non-eventually periodic β-expansion. This is actually the case if β is neither a Pisot nor a Salem number [34].

In order to remove this inconvenience, we now restrict our attention to the case of β-expansions. We are then able to modify the proof of Theorem 1 in order to include all algebraic numbers. This yields the following combinatorial dichotomy for the β-expansions of algebraic numbers in an algebraic base β.

Theorem 2. *Let $\beta > 1$ be a real algebraic number. Let α be an algebraic number in $[0,1]$ such that*

$$d_\beta(\alpha) = 0.a_1 a_2 \dots a_k \dots$$

Then, either the sequence $\mathbf{a} = (a_k)_{k\geq 1}$ is eventually periodic, and thus

$$Dio(\mathbf{a}) = +\infty,$$

or

$$Dio(\mathbf{a}) \leq \frac{\log M(\beta)}{\log \beta}.$$

The proofs of Theorems 1 and 2 are postponed to Section 6.

3. Complexity and β-expansions with zero entropy

A classical measure of disorder for an infinite word $\mathbf{a}$ defined over a finite alphabet $\mathcal{A}$ of cardinality $|\mathcal{A}|$ is the complexity function $p_{\mathbf{a}}$ which associates with every positive integer n the number $p_{\mathbf{a}}(n)$ of distinct blocks of n consecutive letters occurring in $\mathbf{a}$. Then, the topological entropy of the sequence $\mathbf{a}$ is defined by

$$H(\mathbf{a}) := \lim_{n\to\infty} \frac{\log_{|\mathcal{A}|}(p_{\mathbf{a}}(n))}{n}\cdot$$

Now, given two real numbers $\beta > 1$ and α in $[0,1]$, we define the complexity function $p_{\alpha,\beta}$ of α with respect to the base β as the complexity function of the β-expansion of α, that is, we set:

$$p_{\alpha,\beta}(n) := p_{d_\beta(\alpha)}(n), \quad \text{for any } n \geq 1.$$

Then, both Hypotheses $\mathcal{H}_1$ and $\mathcal{H}_2$ predict the following dichotomy for algebraic parameters α and β:

- either $(p_{\alpha,\beta}(n))_{n\geq 1}$ is bounded, which corresponds to an eventually periodic orbit $(T_\beta^n(\alpha))_{n\geq 1}$;
- or $p_{\alpha,\beta}(n)$ is maximal for every integer n, which means that every admissible block of digits occurs, and thus corresponds to a dense orbit.

In the latter case, the entropy of $d_\beta(\alpha)$ is equal to $\log_{\lfloor\beta\rfloor+1}\beta$, the entropy of the β-shift [30, 32]; thus, we have $\lim_{n\to\infty}(\log p_{\alpha,\beta}(n))/n = \log_{\lfloor\beta\rfloor+1}\beta$. Our hypotheses would imply that the sequence $(p_{\alpha,\beta}(n))_{n\geq 1}$ grows exponentially with n as soon as it is unbounded.

In this section, though we are very far away from confirming such a strong dichotomy, we show how to derive from Theorem 2 a first result in this direction.

Theorem 3. *Let $\beta > 1$ be a real algebraic number. Assume that there exists a positive constant c such that*

$$\frac{\log M(\beta)}{\log \beta} < 1 + \frac{1}{c}\cdot \tag{3.1}$$

Let α be an algebraic number in $[0,1]$. Then, either $(p_{\alpha,\beta}(n))_{n\geq 1}$ is bounded, or

$$\liminf_{n\to+\infty} \frac{p_{\alpha,\beta}(n)}{n} > c.$$

This result is especially interesting when β is a Pisot or a Salem number, as shown by the following two corollaries.

Corollary 1. *Let β be a Pisot or a Salem number and α be an algebraic number in $[0,1]$. Then, we have the following dichotomy: either the sequence $(p_{\alpha,\beta}(n))_{n\geq 1}$ is bounded, or*

$$\liminf_{n\to+\infty} \frac{p_{\alpha,\beta}(n)}{n} = +\infty.$$

Proof. If β is a Pisot or a Salem number β, we have $M(\beta) = 1$. Thus, condition (3.1) is fulfilled for every positive real number c. The result directly follows from this observation. □

It is also interesting to consider the complexity of the β-expansions from a more algorithmic point of view. We refer the reader to [6] for a definition of a finite automaton.

Corollary 2. *Let β be a Pisot or a Salem number and α be an algebraic number in $[0,1]$. Then, the β-expansion of α can be generated by a finite automaton if and only if it is eventually periodic.*

For instance, Corollary 2 implies that the expansion of $1/\sqrt{2}$ with respect to the base $(1+\sqrt{5})/2$ cannot be generated by a finite automaton.

Corollaries 1 and 2 for Pisot numbers were already obtained in [2], but the proofs given there depend on a result of K. Schmidt [34].

Proof. It is well-known that an eventually periodic sequence can always be generated by a finite automaton (see for instance Theorem 5.4.2 in [6]): this proves the 'if part'. On the other hand, if the β-expansion of α can be generated by a finite automaton, then we have $p_{\alpha,\beta}(n) = O(n)$ (see for instance Theorem 10.3.1 in [6]). We thus infer from Corollary 1 that $(p_{\alpha,\beta}(n))_{n\geq 1}$ is bounded, or equivalently, that $d_\beta(\alpha)$ is eventually periodic. This ends the proof. □

Proof of Theorem 3. Let $\beta > 1$ be a real algebraic number and c be a positive constant such that (3.1) holds. Let α be an algebraic number in $[0,1]$. Let $\mathbf{a} = (a_k)_{k\geq 1}$ be such that

$$d_\beta(\alpha) = 0.a_1a_2\ldots a_k\ldots$$

The sequence $\mathbf{a}$ takes its values in $\{0,1,\ldots,\lfloor\beta\rfloor\}$, and by definition $p_{\mathbf{a}} = p_{\alpha,\beta}$.

We assume that $p_{\mathbf{a}}(n) \leq cn$ for infinitely many integers $n \geq 1$, and we shall prove that the sequence $\mathbf{a}$ is then eventually periodic, or, equivalently, that $(p_{\alpha,\beta}(n))_{n\geq 1}$ is bounded (see for instance Chap. 1 of [31]).

Let $(n_k)_{k\geq 1}$ be an increasing sequence of integers such that $p_{\mathbf{a}}(n_k) \leq cn_k$ for every $k \geq 1$. For $\ell \geq 1$, denote by $A(\ell)$ the prefix of $\mathbf{a} := a_1a_2\ldots$ of length ℓ. Let $k \geq 1$ be an integer. By the *Schubfachprinzip*, there exists (at least) one word W_k of length n_k which has (at least) two occurrences in $A((c+1)n_k)$. Thus, there are possibly empty words B_k, D_k, E_k and a non-empty word C_k such that

$$A((c+1)n_k) = B_k W_k D_k E_k = B_k C_k W_k E_k.$$

We have now to distinguish two cases.

We first assume that $|C_k| \geq |W_k|$. Then, there exists a possibly empty word F_k such that

$$A((c+1)n_k) = B_k W_k F_k W_k E_k.$$

We set $U_k = B_k$ and $V_k = W_k F_k$ and $w_k = |W_k F_k W_k|/|W_k F_k|$. Then, $U_k V_k^{w_k}$, wich is equal to $B_k W_k F_k W_k$, is a prefix of $\mathbf{a}$. Moreover, $|V_k^{w_k}| > n_k$ and

$$\frac{|U_k V_k^{w_k}|}{|U_k V_k|} = 1 + \frac{|W_k|}{|B_k W_k F_k|} \geq 1 + \frac{n_k}{cn_k} = 1 + \frac{1}{c}\cdot$$

Now, we assume that $|C_k| < |W_k|$. Then, the two occurrences of W_k overlap. In such a case, there exists a real number $w_k > 2$ such that

$$C_k W_k = C_k^{w_k}.$$

We set $U_k = B_k$ and $V_k = C_k$. Then, $U_k V_k^{w_k}$, wich is equal to $B_k C_k W_k$, is a prefix of $\mathbf{a}$. Moreover, $|V_k^{w_k}| = |C_k W_k| > |W_k| = n_k$ and

$$\frac{|U_k V_k^{w_k}|}{|U_k V_k|} = \frac{|U_k V_k W_k|}{|U_k V_k|} = 1 + \frac{|W_k|}{|B_k C_k|} \geq 1 + \frac{n_k}{cn_k} = 1 + \frac{1}{c}\cdot$$

Thus, we have shown that for every positive integer k there exist two finite words U_k and V_k and a positive real number w_k such that:

(*i*) $U_k V_k^{w_k}$ is a prefix of $\mathbf{a}$;

(*ii*) $|U_k V_k^{w_k}|/|U_k V_k| \geq 1 + 1/c$;

(*iii*) $|V_k^{w_k}| > n_k$.

Moreover, thanks to (*iii*), we can always extract a subsequence of positive integers $(k_n)_{n \geq 1}$ such that $|V_{k_n}^{w_{k_n}}|$ increases with n. Consequently, the sequence $\mathbf{a}$ satisfies Condition $(*)_{1+1/c}$, and thus $\mathrm{Dio}(\mathbf{a}) \geq 1 + 1/c$. Then, (3.1) implies that

$$\mathrm{Dio}(\mathbf{a}) > \frac{\log M(\beta)}{\log \beta},$$

and we infer from Theorem 2 that $\mathbf{a}$ is an eventually periodic sequence. This ends the proof. $\square$

4. Blanchard's classification of β-shifts and transcendental numbers

In this section, we give a continuum of explicit transcendental numbers lying in Class C_4 of the classification of β-shifts introduced by Blanchard in [12]. We first briefly recall this classification and some related results.

Following [12], we have:

- β belongs to C_1 if the β-shift is of finite type, or, equivalently, if $d_\beta(1)$ is finite [30];
- β belongs to C_2 if $d_\beta(1)$ is eventually periodic but not finite (this implies that the β-shift is sofic [10]);
- β belongs to C_3 if $d_\beta(1)$ does not contain arbitrarily large strings of 0's and if $d_\beta(1)$ is not eventually periodic (this implies that the β-shift is specified [10]);
- β belongs to C_4 if $d_\beta(1)$ does not contain some admissible words, but contains arbitrarily large strings of 0's (this implies that the β-shift is synchronizing [10]);
- β belongs to C_5 if $d_\beta(1)$ contains all admissible words.

Elements of Class $C_1 \cup C_2$ are called *Parry numbers*. The term *simple Parry numbers* usually denotes elements that belong to Class C_1. Classes C_1 and C_2 are both countable and only countain algebraic integers. Every Pisot number is a Parry number [34, 10]. The fact that all Salem numbers would be Parry numbers is a particular instance of a conjecture due to K. Schmidt [34]. This was proved by Boyd [14] for all Salem numbers of degree 4. However, the same author considered in [15] a heuristic suggesting the existence of Salem numbers of degree 8 that are not Parry numbers. This casts doubt on the Schmidt conjecture.

At the end of [12], Blanchard asked whether there are transcendental numbers in every Class C_3, C_4 and C_5. A partial answer is given by Allouche, Bertrand-Mathis and Mauduit in the Appendix of the same paper (related results are recalled below). Later, Schmeling [33] proved that C_3, C_4 and C_5 all have Hausdorff dimension 1, and, consequently, mostly countain transcendental numbers. However, the method used in [33] does not provide any explicit example of such numbers. Let us also note that Hypothesis $\mathcal{H}_2$ implies that algebraic numbers can only belong to $C_1 \cup C_2$ or to C_5.

Explicit natural examples of transcendental numbers in C_3 were given by Allouche and Cosnard [5], and by Chi and Kwon [16]. Both results are a consequence of the so-called Mahler's method introduced in [27].

We are now interested in Class C_4. We say that a real number $\beta > 1$ is a *self-lacunary number* if there exist a positive real number δ and a sequence $(u_n)_{n\geq 1}$ of positive integers satisfying the following lacunarity condition:

$$u_1 = 1 \text{ and } \frac{u_{n+1}}{u_n} \geq 1 + \delta, \quad \text{for } n \geq 1, \tag{4.1}$$

such that

$$1 = \sum_{n=1}^{+\infty} \frac{1}{\beta^{u_n}}. \tag{4.2}$$

As a consequence of a nice result of Corvaja and Zannier [17], that rests on Schmidt's Subspace Theorem, we obtain the following result. This provides a continuum of explicit and natural transcendental numbers in Class C_4.

Theorem 4. *Every self-lacunary number is transcendental and belongs to class C_4.*

The special case where $u_n = 2^{n-1}$ is suggested by Mauduit in [12] as an example providing a non-specified subshift associated to a transcendental number. The transcendence of the associated β follows from a classical result by Mahler [27].

Proof. Let β be a self-lacunary number and assume that the parameters δ and $\mathbf{u} = (u_n)_{n\geq 1}$ occurring in (4.1) are fixed.

We first prove that β belongs to Class C_4. Let us recall the so-called Parry condition introduced in [30]. We say that a sequence $\mathbf{a}$ satisfies the Parry condition if, for every positive integer n, $\mathbf{a}$ is greater than or equal to $S^n(\mathbf{a})$ with respect to the lexicographic order, where S denotes the shift. It was proved by Parry [30] that a sequence $\mathbf{a}$ with values in a finite subset of the non-negative integers is the β-expansion of 1 for some β if and only if it satisfies the Parry condition.

Let us now consider the sequence $\mathbf{a} = (a_k)_{k\geq 1}$ defined by $a_k = 1$ if k belongs to the sequence $\mathbf{u}$, and $a_k = 0$ otherwise. Since $u_1 = 1$ and $u_{n+1} - u_n$ increases with n, we easily check that $\mathbf{a}$ satisfies the Parry condition. We thus infer from (4.2) that

$$d_\beta(1) = 0.a_1a_2\dots a_k\dots$$

Since the gap $u_{n+1} - u_n$ between two consecutive 1's increases with n, the sequence $\mathbf{a}$ cannot be eventually periodic and thus β cannot belong to C_1 or to C_2. Moreover, the existence of arbitrarily large blocks of consecutive 0's implies that β cannot lie in Class C_3. We thus have to prove that β cannot belong to Class C_5.

Let us assume that β is an element of C_5. By definition, all admissible words occur in $d_\beta(1)$. Since the topological entropy of the β-shift is equal to $\log_{\lfloor\beta\rfloor+1}\beta$ (see [32]), we thus have

$$\lim_{n\to\infty} \frac{\log(p_{1,\beta}(n))}{n} = \log_{\lfloor\beta\rfloor+1}\beta. \tag{4.3}$$

On the other hand, the fact that the gap between two consecutive 1's increases with n easily implies that

$$p_{\mathbf{a}}(n) = p_{1,\beta}(n) = O(n^2).$$

Such a result is for instance a straightforward consequence of Theorem 1.2 in [21]. This contradicts (4.3). Hence, β belongs to C_4.

The transcendence of β follows immediately from Corollary 5 of [17]. This ends the proof. $\square$

We end this Section with an additional remark. To find an explicit example of a transcendental number lying in Class C_5 seems to be a rather difficult task. At this point, no explicit construction of an element lying in C_5 is even known. The latter problem shares some similarity with the one of the construction of an absolutely normal number.

Nevertheless, for any real number $\beta > 1$, it is possible to give explicit examples of numbers with a normal β-expansion thanks to a Champernowne-like construction (see [11, 23]).

5. Lacunarity in $d_\beta(1)$ for algebraic β

In the previous section, we have seen, through Blanchard's classification, that the occurrences of consecutive 0's in $d_\beta(1)$ play a crucial role in the study of the β-shift. This motivates the following problem first investigated in [37].

Let $\beta > 1$ be a real number and let $\mathbf{a} = (a_k)_{k\geq 1}$ be the infinite sequence such that $d_\beta(1) = 0.a_1a_2\ldots$ In particular, we assume that β is not a simple Parry number. Let us assume that there exist a sequence of positive integers $(r_n)_{n\geq 1}$ and an increasing sequence of positive integers $(s_n)_{n\geq 1}$ such that

$$a_{s_n+1} = a_{s_n+2} = \ldots = a_{s_n+r_n} = 0,$$

for every positive integer n. The problem is then to estimate the gaps occurring in $d_\beta(1)$ when β is an algebraic number, that is, to estimate the asymptotic behaviour of the ratio r_n/s_n. Of course, if β is a Parry number, then the sequence $(r_n)_{n\geq 1}$ is bounded, and this ratio tends to 0 when n tends to infinity.

With our notation, the main result of [37] can be reformulated as follows. It mainly shows that $d_\beta(1)$ cannot be 'too lacunary' when β is an algebraic number.

Theorem VG. *Let $\beta > 1$ be a real algebraic number. Then, with the above notation, we have*

$$\limsup_{n\to\infty} \frac{r_n}{s_n} \leq \frac{\log M(\beta)}{\log \beta} - 1.$$

As explained below, Theorem VG is a special case of Theorem 2.

Proof of Theorem VG. Let β be a real algebraic number and let $\mathbf{a} = (a_k)_{k\geq 1}$ be an infinite sequence such that $d_\beta(1) = 0.a_1a_2\ldots$ Let the sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ be as above. If β is a Parry number, then the result immediatly follows since the sequence $(r_n)_{n\geq 1}$ is bounded. We now assume that β is not a Parry number. Thus $\mathbf{a}$ is not eventually periodic. Let U_n be the prefix of $\mathbf{a}$ of length s_n, and let $V_n = 0$. By assumption, $U_nV_n^{r_n}$ is a prefix of $\mathbf{a}$, and $|U_nV_n^{r_n}|/|U_nV_n| = 1 + r_n/(s_n+1)$. It follows that

$$\mathrm{Dio}(\mathbf{a}) \geq 1 + \limsup_{n\to\infty} \frac{r_n}{s_n}\cdot$$

On the other hand, since 1 is an algebraic number and $\mathbf{a}$ is not eventually periodic, we infer from Theorem 2 that

$$\mathrm{Dio}(\mathbf{a}) \leq \frac{\log M(\beta)}{\log \beta}\cdot$$

This ends the proof. □

6. Proofs of Theorems 1 and 2

Before beginning the proof of Theorem 1, we quote a version of Schmidt's Subspace Theorem, as formulated by Evertse [19].

We normalize absolute values and heights as follows. Let $\mathbf{K}$ be an algebraic number field of degree d. Let $M(\mathbf{K})$ denote the set of places on $\mathbf{K}$. For x in $\mathbf{K}$ and a place v in $M(\mathbf{K})$, define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/d}$ if v corresponds to the embedding $\sigma : \mathbf{K} \hookrightarrow \mathbf{R}$;

(ii) $|x|_v = |\sigma(x)|^{2/d} = |\overline{\sigma}(x)|^{2/d}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \overline{\sigma} : \mathbf{K} \hookrightarrow \mathbf{C}$;

(iii) $|x|_v = (N\mathfrak{p})^{-ord_{\mathfrak{p}}(x)/d}$ if v corresponds to the prime ideal $\mathfrak{p}$ of $O_{\mathbf{K}}$.

These absolute values satisfy the product formula

$$\prod_{v \in M(\mathbf{K})} |x|_v = 1 \qquad \text{for } x \text{ in } \mathbf{K}^*.$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be in $\mathbf{K}^n$ with $\mathbf{x} \neq \mathbf{0}$. For a place v in $M(\mathbf{K})$, put

$$|\mathbf{x}|_v = \left(\sum_{i=1}^{n} |x_i|_v^{2d} \right)^{1/(2d)} \qquad \text{if } v \text{ is real infinite;}$$

$$|\mathbf{x}|_v = \left(\sum_{i=1}^{n} |x_i|_v^{d} \right)^{1/d} \qquad \text{if } v \text{ is complex infinite;}$$

$$|\mathbf{x}|_v = \max\{|x_1|_v, \ldots, |x_n|_v\} \quad \text{if } v \text{ is finite.}$$

Now define the *height* of $\mathbf{x}$ by

$$H(\mathbf{x}) = H(x_1, \ldots, x_n) = \prod_{v \in M(\mathbf{K})} |\mathbf{x}|_v.$$

We stress that $H(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the number field $\mathbf{K}$ containing the coordinates of $\mathbf{x}$, see e.g. [19].

We use the following formulation of the Subspace Theorem over number fields. In the sequel, we assume that the algebraic closure of $\mathbf{K}$ is $\overline{\mathbf{Q}}$. We choose for every place v in $M(\mathbf{K})$ a continuation of $|\cdot|_v$ to $\overline{\mathbf{Q}}$, that we denote also by $|\cdot|_v$.

Theorem E. *Let $\mathbf{K}$ be an algebraic number field. Let $m \geq 2$ be an integer. Let S be a finite set of places on $\mathbf{K}$ containing all infinite places. For each v in S, let $L_{1,v}, \ldots, L_{m,v}$ be linear forms with algebraic coefficients and with*

$$\textit{rank}\, \{L_{1,v}, \ldots, L_{m,v}\} = m.$$

Let ε be real with $0 < \varepsilon < 1$. Then, the set of solutions $\mathbf{x}$ in $\mathbf{K}^m$ to the inequality

$$\prod_{v \in S} \prod_{i=1}^{m} \frac{|L_{i,v}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq H(\mathbf{x})^{-m-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbf{K}^m$.

For a proof of Theorem E, the reader is referred to [19], where a quantitative version is established (in the sense that an explicit bound for the number of exceptional subspaces is given).

Proof of Theorem 1. Let β be an algebraic number with $|\beta| > 1$. Consider a bounded sequence $\mathbf{a} = (a_k)_{k\geq 1}$ of rational integers satisfying the condition (2.1). There exists a real number $\rho > \log M(\beta)/\log|\beta|$ such that $\mathbf{a}$ satisfies Condition $(*)_\rho$. We assume that the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$, and $(w_n)_{n\geq 1}$ occurring in the definition of Condition $(*)_\rho$ are fixed. Set also $r_n = |U_n|$ and $s_n = |V_n|$, for any $n \geq 1$.

We have to prove that the real number

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k} \tag{6.1}$$

either lies in $\mathbf{Q}(\beta)$, or is transcendental. In order to achieve this, we assume that α is algebraic and we aim at proving that α is in $\mathbf{Q}(\beta)$. The key fact is the observation that α admits infinitely many good approximants in the number field $\mathbf{Q}(\beta)$ obtained by truncating the series and completing it by periodicity. Precisely, for every positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$\begin{aligned} b_h^{(n)} &= a_h \quad \text{for } 1 \leq h \leq r_n + s_n, \\ b_{r_n+h+js_n}^{(n)} &= a_{r_n+h} \quad \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0. \end{aligned} \tag{6.2}$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is eventually periodic, with preperiod U_n and with period V_n. Set

$$\alpha_n = \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k}, \tag{6.3}$$

and observe that

$$\alpha - \alpha_n = \sum_{k=r_n+\lceil w_n s_n\rceil+1}^{+\infty} \frac{a_k - b_k^{(n)}}{\beta^k}. \tag{6.4}$$

We recall now Lemma 1 from [2].

Lemma 1. *For any integer n, there exists an integer polynomial $P_n(X)$ of degree at most $r_n + s_n - 1$ such that*

$$\alpha_n = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}.$$

Further, the coefficients of $P_n(X)$ are bounded in absolute value by $2\max_{k\geq 1}|a_k|$.

Set $\mathbf{K} = \mathbf{Q}(\beta)$ and denote by d the degree of $\mathbf{K}$. We assume that α is algebraic, and we consider the following linear forms, in three variables and with algebraic coefficients. For the place v corresponding to the embedding of $\mathbf{K}$ defined by $\beta \hookrightarrow \beta$, set $L_{1,v}(x,y,z) = x$, $L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = \alpha x + \alpha y + z$. It follows from (6.4) and Lemma 1 that

$$|L_{3,v}(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))|_v = |\alpha(\beta^{r_n}(\beta^{s_n}-1)) - P_n(\beta)|^{1/d} \ll \frac{1}{\beta^{(w_n-1)s_n/d}}, \tag{6.5}$$

where we have chosen the continuation of $|\cdot|_v$ to $\overline{\mathbf{Q}}$ defined by $|x|_v = |x|^{1/d}$. Here and throughout this section, the constants implied by the Vinogradov symbol $\ll$ depend (at most) on α, β, and $\max_{k\geq 1}|a_k|$, but are independent of n.

Denote by S'_∞ the set of all other infinite places on $\mathbf{K}$ and by S_0 the set of all finite places v on $\mathbf{K}$ for which $|\beta|_v \neq 1$. Observe that S_0 is empty if β is an algebraic unit. For any v in $S_0 \cup S'_\infty$, set $L_{1,v}(x,y,z) = x$, $L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = z$. Denote by S the union of S_0 and the infinite places on $\mathbf{K}$. Clearly, for any v in S, the linear forms $L_{1,v}$, $L_{2,v}$ and $L_{3,v}$ are linearly independent.

To simplify the exposition, set

$$\mathbf{x}_n = (\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta)).$$

We wish to estimate the product

$$\Pi_n := \prod_{v\in S}\prod_{i=1}^{3} \frac{|L_{i,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v} = \prod_{v\in S} |\beta^{r_n+s_n}|_v\, |\beta^{r_n}|_v\, \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3}$$

from above. By the product formula and the definition of S, we immediately get that

$$\Pi_n = \prod_{v\in S} \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3}. \tag{6.6}$$

Since the polynomial $P_n(X)$ has integer coefficients, for any infinite place v in S'_∞, we have

$$|L_{3,v}(\mathbf{x}_n)|_v \ll (r_n+s_n)\cdot(\max\{1,|\beta|_v\})^{r_n+s_n}. \tag{6.7}$$

Furthermore, we have

$$|L_{3,v}(\mathbf{x}_n)|_v = |P_n(\beta)|_v \ll (\max\{1,|\beta|_v\})^{r_n+s_n} \tag{6.8}$$

for any place v in S_0. Let b denote the leading coefficient of the minimal defining polynomial of β. By using the auxiliary result from [38], page 74, we infer from (6.8) that

$$\prod_{v\in S_0} |L_{3,v}(\mathbf{x}_n)|_v \ll b^{-(r_n+s_n)/d}. \tag{6.9}$$

Combining (6.5), (6.6), (6.7), (6.8) and (6.9), we get that

$$\begin{aligned}\Pi_n &\ll (r_n+s_n)^d\, |\beta|^{-(r_n+w_ns_n)/d}\, M(\beta)^{(r_n+s_n)/d} \prod_{v\in S} |\mathbf{x}_n|_v^{-3} \\ &\ll (r_n+s_n)^d\, |\beta|^{-(r_n+w_ns_n)/d}\, M(\beta)^{(r_n+s_n)/d}\, H(\mathbf{x}_n)^{-3},\end{aligned} \tag{6.10}$$

since $|\mathbf{x}_n|_v = 1$ if v does not belong to S.

Since $\rho > \log M(\beta)/\log|\beta|$, there exists $\delta > 0$ such that $\rho > (1+\delta)\log M(\beta)/\log|\beta|$. Now, since $\mathbf{a}$ satisfies the condition $(*)_\rho$, we get that

$$|\beta|^{-(r_n+w_n s_n)} M(\beta)^{r_n+s_n} \ll M(\beta)^{-\delta(r_n+s_n)}. \tag{6.11}$$

Furthermore, it follows from Lemma 1 that

$$H(\mathbf{x}_n) \ll (2M(\beta))^{r_n+s_n}. \tag{6.12}$$

By (6.10), (6.11) and (6.12), there exists a positive real number ε for which

$$\Pi_n \ll H(\mathbf{x}_n)^{-3-\varepsilon}.$$

It then follows from Theorem E that the points $(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))$ lie in a finite number of proper subspaces of $\mathbf{K}^3$. Thus, there exist a non-zero triple (x_0, y_0, z_0) in $\mathbf{K}^3$ and an infinite set of distinct positive integers integers $\mathcal{N}$ such that

$$x_0\beta^{r_n+s_n} - y_0\beta^{r_n} - z_0P_n(\beta) = 0, \tag{6.13}$$

for every n lying in $\mathcal{N}$. Dividing (6.13) by $\beta^{r_n+s_n}$ and letting n tend to infinity along $\mathcal{N}$ we obtain

$$x_0 - z_0\alpha = 0. \tag{6.14}$$

Since (x_0, y_0, z_0) is a non-zero triple, we easily derive from (6.13) and (6.14) that $z_0 \neq 0$. It follows that α lies in the field $\mathbf{Q}(\beta)$, which ends the proof of Theorem 1. $\square$

We now prove Theorem 2. In order to do this, we need the following auxiliary result. We keep the notation of the proof of Theorem 1. In particular, α and α_n are respectively defined by (6.1) and (6.3). The only change with respect to Theorem 1 is that the sequence $\mathbf{a} = (a_k)_{k\geq 1}$ corresponds to the β-expansion of α and thus the a_k are non-negative integers at most equal to $\lfloor\beta\rfloor$. Furthermore, we assume that $\mathbf{a}$ is not eventually periodic and that it satisfies (2.1). We assume that α is an algebraic number, and we aim at deriving a contradiction.

Lemma 2. *Under the previous assumption, we have*

$$\alpha \neq \alpha_n,$$

for every positive integer n.

Proof. The key observation is that for every non-negative integer r we have

$$\sum_{k\geq r+1} \frac{a_k}{\beta^k} \leq \frac{1}{\beta^r}\cdot \tag{6.15}$$

Note that such an inequality is in general not satisfied by an arbitrary expansion in base β. However, it holds when one considers the β-expansion, as a by-product of the fact that the β-expansion arises from the greedy algorithm.

Let n be a positive integer. We first infer from (6.2) and the fact that the sequence $(a_k)_{k\geq 1}$ is not eventually periodic that there exists a positive integer $j_n > r_n + s_n$ satisfying:

(i) $a_k = b_k^{(n)}$ for $1 \leq k < j_n$;

(ii) $a_{j_n} \neq b_{j_n}^{(n)}$.

Since the coefficients a_k are non-negative and infinitely many of them are positive, we infer from (6.15) that

$$\sum_{k=j_n-s_n+1}^{j_n} \frac{a_k}{\beta^k} < \sum_{k\geq j_n-s_n+1} \frac{a_k}{\beta^k} \leq \frac{1}{\beta^{j_n-s_n}},$$

and thus we get

$$a_{j_n-s_n+1}\beta^{s_n-1} + a_{j_n-s_n+2}\beta^{s_n-2} + \ldots + a_{j_n} < \beta^{s_n}. \tag{6.16}$$

We also recall that (6.2) easily implies that

$$b_k^{(n)} = b_{k+js_n}^{(n)}, \quad \text{if } k > r_n \text{ and } j \geq 0. \tag{6.17}$$

Set $a_{j_n} = l$ and $b_{j_n}^{(n)} = m$. We have to distinguish two cases. Let us first assume that $m > l$. Then, m is a positive integer and, since the coefficients $b_k^{(n)}$ are non-negative, we infer from (6.17) and from equalities (i) above that

$$\begin{aligned}\alpha_n = \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} &\geq \sum_{k=1}^{j_n-1} \frac{b_k^{(n)}}{\beta^k} + \frac{b_{j_n}^{(n)}}{\beta^{j_n}} + \frac{b_{j_n+s_n}^{(n)}}{\beta^{j_n+s_n}} + \frac{b_{j_n+2s_n}^{(n)}}{\beta^{j_n+2s_n}} \\ &= \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{m}{\beta^{j_n}} + \frac{m}{\beta^{j_n+s_n}} + \frac{m}{\beta^{j_n+2s_n}} > \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{m}{\beta^{j_n}} + \frac{m}{\beta^{j_n+s_n}},\end{aligned}$$

whereas by Inequality (6.15) we have

$$\alpha = \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k} = \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^n} + \frac{l}{\beta^{j_n}} + \sum_{k\geq j_n+1} \frac{a_k}{\beta^k} \leq \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{l+1}{\beta^{j_n}} \leq \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{m}{\beta^{j_n}}.$$

This yields

$$\alpha_n - \alpha \geq \frac{m}{\beta^{j_n+s_n}} \geq \frac{1}{\beta^{j_n+s_n}}.$$

Now, let us assume that $l > m$. Since $b_{j_n}^{(n)} = m \leq l - 1 = a_{j_n} - 1$, we deduce from (6.16) and equalities (i) above that

$$Q_n(\beta) := b_{j_n-s_n+1}^{(n)}\beta^{s_n-1} + b_{j_n-s_n+2}^{(n)}\beta^{s_n-2} + \ldots + b_{j_n}^{(n)} < \beta^{s_n} - 1. \tag{6.18}$$

On the one hand, we infer from (6.17) that

$$\begin{aligned}\alpha_n &= \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} = \sum_{k=1}^{j_n} \frac{b_k^{(n)}}{\beta^k} + \sum_{k=1}^{+\infty} \frac{Q_n(\beta)}{\beta^{j_n+ks_n}} \\ &= \sum_{k=1}^{j_n} \frac{b_k^{(n)}}{\beta^k} + \frac{Q_n(\beta)}{\beta^{j_n+s_n}} + \sum_{k=2}^{+\infty} \frac{Q_n(\beta)}{\beta^{j_n+ks_n}} \\ &= \sum_{k=1}^{j_n} \frac{b_k^{(n)}}{\beta^k} + \frac{Q_n(\beta)}{\beta^{j_n+s_n}} + \frac{Q_n(\beta)}{\beta^{j_n+s_n}(\beta^{s_n}-1)},\end{aligned}$$

and we then derive from (6.18) that

$$\alpha_n < \sum_{k=1}^{j_n} \frac{b_k^{(n)}}{\beta^k} + \frac{Q_n(\beta)+1}{\beta^{j_n+s_n}}.$$

On the other hand, we have

$$\alpha = \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k} \geq \sum_{k=1}^{j_n} \frac{a_k}{\beta^k} = \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{l}{\beta^{j_n}} \geq \sum_{k=1}^{j_n-1} \frac{a_k}{\beta^k} + \frac{b_{j_n}^{(n)}+1}{\beta^{j_n}} = \sum_{k=1}^{j_n} \frac{b_k^{(n)}}{\beta^k} + \frac{1}{\beta^{j_n}}.$$

This gives

$$\alpha - \alpha_n \geq \frac{1}{\beta^{j_n}} - \frac{Q_n(\beta)+1}{\beta^{j_n+s_n}} = \frac{\beta^{s_n} - Q_n(\beta) - 1}{\beta^{j_n+s_n}}.$$

Then, (6.18) implies that

$$\alpha - \alpha_n > 0,$$

which ends the proof of the lemma. □

We are now ready to complete the proof of Theorem 2.

Proof of Theorem 2. We first follow the proof of Theorem 1 until we get Equalities (6.13) and (6.14). Besides, (6.5) reads

$$|\beta^{r_n}(\beta^{s_n}-1)\alpha - P_n(\beta)| \ll \frac{1}{\beta^{(w_n-1)s_n}}$$

and, combined with (6.14), it gives

$$\left|\beta^{r_n}(\beta^{s_n}-1)\frac{x_0}{z_0} - P_n(\beta)\right| \ll \frac{1}{\beta^{(w_n-1)s_n}},$$

hence,

$$\left|x_0\beta^{r_n+s_n} - x_0\beta^{r_n} - z_0 P_n(\beta))\right| \ll \frac{1}{\beta^{(w_n-1)s_n}}. \tag{6.19}$$

Subtracting (6.13) and (6.19), we obtain

$$|(x_0 - y_0)\beta^{r_n}| \ll \frac{1}{\beta^{(w_n-1)s_n}},$$

for n lying in $\mathcal{N}$. This implies that $y_0 = x_0$. By (6.13) and (6.14), this yelds

$$\alpha = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)} = \alpha_n,$$

for every positive integer n lying in $\mathcal{N}$. Thus we find a contradiction with Lemma 2. This ends the proof. □

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