Dynamics of some contracting linear functions modulo 1

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Abstract. In various problems in signal theory, the following family of functions of $[0,1[$ into itself arises naturally :

$$T_{\gamma,\alpha} : x \mapsto \gamma x + \alpha \text{ modulo } 1, \text{ with } 0 < \gamma < 1, 0 \leq \alpha < 1.$$

The purpose of the present work is to describe very precisely the asymptotic behaviour of the iterates of the functions $T_{\gamma,\alpha}$. Further, we give a continued fraction type algorithm which allows us to obtain, for a given pair (γ,α), the numerical informations on the dynamic of $T_{\gamma,\alpha}$.

1 Introduction

The problem considered here is motivated by questions arising naturally in signal theory (see for example [5]). Indeed, in order to describe certain observations, one is led to define the following sequences (s_n) and (x_n) :

$$s_n = A \sin\big(2\pi(nf + f_0)\big),$$

$$x_{n+1} = \{\gamma\, x_n + s_n\},$$

where A, γ, f_0 and f are real numbers, with f irrational and $0 < \gamma \leq 1$. Here, and in the rest of the paper, $\{x\}$ denotes the fractional part of the real number x. We would like to study the existence of the limit of the mean sequence $m_n = \frac{1}{n}\sum_{k=1}^{n} x_k$ and of the correlation sequence $c_n = \frac{1}{n}\sum_{k=1}^{n} x_k\, x_{k+1} - m_n^2$. Both questions are not yet solved and they look very difficult.

A preliminary step is the study of the dynamic associated to the recurrence $x_{n+1} = \{\gamma\, x_n + s_n\}$ when $f = 0$, *i.e.* when $s_n =: \alpha$ is a constant input. The case $\gamma = 1$ corresponds to the rotations of the circle, and hence is well known. However, for $0 < \gamma < 1$, the associated map, say $T_{\gamma,\alpha} : x \mapsto \{\gamma\, x + \alpha\}$, contracts the torus $[0, 1[$, and it seems that the asymptotic behaviour of the iterates of $T_{\gamma,\alpha}$ has not been studied previously. In a joint work with J.-P. Conze [3], we completely solve this question and, for any fixed $\gamma \in]0, 1[$ and any given rational $p/q \in]0, 1[$, we determine precisely the set of values of the parameter α for which $T_{\gamma,\alpha}$ behaves asymptotically like the rotation $x \mapsto x + p/q$. Some of the results of [3] were announced in [2].

This expository paper is organized as follows. We first recall the basic results about the rotations of the torus, and introduce some (classical) notations used in the statement of the main theorem. Then, we state our main result, as well as a slight generalization, and we briefly discuss two different ways to prove them. Finally, we describe an algorithm which, for any given pair (γ, α) of real numbers with $0 < \gamma, \alpha \leq 1$, computes the asymptotic behaviour of the iterates of $T_{\gamma,\alpha}$, and we conclude by studying an example.

2 Notations

We denote by X the unit interval $[0,1[$ and by $[x]$ the integer part of the real number x. For any $\alpha \in [0,1[$, we denote by $T_\alpha : X \to X$ the rotation of the torus given by $T_\alpha x = \{x+\alpha\}$ for $x \in X$, and by $\mathcal{R}$ the set of all the rotations.

One can code the rotation T_α using the partition of the interval $[0,1[$ in the two sub-intervals $[0, 1-\alpha[$ and $[1-\alpha, 1[$. Namely, let $\varepsilon : X \to X$ be the function defined by

$$\varepsilon(x) = 0, \text{ if } x \in [0, 1-\alpha[, \quad \text{and } \varepsilon(x) = 1, \text{ if } x \in [1-\alpha, 1[.$$

For any $x \in X$, the sequence $(\varepsilon(T_\alpha^{k-1} x))_{k \in \mathbf{Z}}$ is a sequence of “0” and “1” coding the point x. Applying the relation $x + \alpha = T_\alpha x + \varepsilon(x)$ at the point $T_\alpha^{n-1} x$, we get by induction that

$$x + n\alpha = T_\alpha^n x + \varepsilon(T_\alpha^{n-1} x) + \varepsilon(T_\alpha^{n-2} x) + \ldots + \varepsilon(x).$$

We will simply denote by $(\varepsilon_k)_{k \in \mathbf{Z}}$ the sequence corresponding to $x = 0$. Then, by definition, we have $\varepsilon_k = \varepsilon(T_\alpha^{k-1} 0)$, for all $k \in \mathbf{Z}$. In order to point out that this sequence depends on α, we also denote it by (ε_k^α). We point out that for any α, we have $\varepsilon_1^\alpha = 0$ and, for $k \geq 1$, we get

$$\varepsilon_k^\alpha = [k\alpha] - [(k-1)\alpha].$$

The sequence (ε_k^α) gives the dynamic of the map T_α on the orbit of the point 0. If we denote by A_0 (*resp.* A_1) the rotation $x \mapsto x + \alpha$ (*resp.* $x \mapsto x + \alpha - 1$), we have

$$T_\alpha^n(0) = A_{\varepsilon_n^\alpha} \ldots A_{\varepsilon_1^\alpha}(0). \tag{1}$$

In the sequel, we call the sequence (ε_k^α) the coding sequence of the rotation T_α. This is also called the Sturmian sequence of α. For a rational $\alpha = p/q$, with p/q irreducible, the sequence $(\varepsilon_k^{p/q})$ is q-periodic. Moreover, the set $\{T_{p/q}^n 0\}_{n \in \mathbf{Z}}$ is finite and has exactly q distinct elements. If α is irrational, then the set $\{T_\alpha^n 0\}_{n \in \mathbf{Z}}$ is dense in $[0,1[$, and even equidistributed. The following observation appears to be the key of the proofs of our main results.

Lemma 1. *Let p/q and p'/q' be rational numbers with $p'q - pq' = 1$ and set $r = (p+p')/(q+q')$. Let $(\varepsilon_k^{p/q}$ (resp. $(\varepsilon_k^{p'/q'})$) be the coding sequence of the* *rotation $T_{p/q}$ (resp. $T_{p'/q'}$). Then the coding sequence (ε_k^r) of the rotation T_r is given by*

$$\varepsilon_\ell^r = \varepsilon_\ell^{p/q} \text{ for all } 1 \le \ell \le q,$$

and

$$\varepsilon_{q+\ell}^r = \varepsilon_\ell^{p'/q'} \text{ for all } 1 \le \ell \le q'.$$

We observe that, for $n \geq 1$, we have $[n\alpha] = \sum_1^n \varepsilon_k$ and $n\alpha = \{n\alpha\} + \sum_1^n \varepsilon_k$. This furnishes a method of binary quantification for α, since α can be written, if $\alpha = p/q \in [0,1[$ is rational :

$$\alpha = \frac{1}{q}[\varepsilon(T_\alpha^{q-1}0)+\varepsilon(T_\alpha^{q-2}0)+...+\varepsilon(0)] = \lim_n \frac{1}{n}[\varepsilon(T_\alpha^{n-1}0)+\varepsilon(T_\alpha^{n-2}0)+...+\varepsilon(0)],$$

and if α is irrational :

$$\alpha = \lim_n \frac{1}{n}[\varepsilon(T_\alpha^{n-1}0) + \varepsilon(T_\alpha^{n-2}0) + ... + \varepsilon(0)].$$

3 Results

The problem of the asymptotic behaviour of iterates of smooth perturbations of rotation maps has been extensively studied. The more famous example is given by the family of applications over X

$$A_{\alpha,\varepsilon} : \quad x \mapsto \{x + \alpha + \varepsilon \sin 2\pi x\}, \tag{2}$$

for $0 \leq \alpha < 1$ and $0 \leq \varepsilon < 1/(2\pi)$, which leads to the so-called "Arnold's tongues" (see [1], [4], [7]).

The question considered here is on an essentially different nature, since the perturbated maps are no more onto. Let $0 < \gamma, \alpha \leq 1$ be two real numbers and set

$$T_{\gamma,\alpha} : \ x \longmapsto \{\gamma x + \alpha\},$$

where γ and α are two real parameters with $\gamma \in]0,1]$ and $\alpha \in [0,1[$. We observe that the maps $T_{\gamma,\alpha}$ are injective and that, for $\gamma = 1$, we obtain the rotation T_α. We denote by $\mathcal{C}$ the set of these maps $T_{\gamma,\alpha}$. As we are now interested in the contracting case, we assume in the sequel that $0 < \gamma < 1$.

Similarly as for diffeomorphisms of the torus (see [4], Theorem 14.6), one can define a *rotation number* for the maps $T_{\gamma,\alpha}$.

Proposition 1. *Let $f : X \to X$ be an injective map and assume that there exists a continuous, increasing map $F : [0,1[\to \mathbf{R}$ such that for all $x \in [0,1[$ we have $f(x) = \{F(x)\}$. We still denote by F the extension of F to $\mathbf{R}$ given by $F(x) = [x] + F(\{x\})$, for all $x \in \mathbf{R}$. Then, F is called a lift of f and the limit*

$$\lim_{n\to\infty} \frac{F^n(x)}{n} =: \rho$$

exists and does not depend on $x \in \mathbf{R}$: the real $\rho \in [0,1]$ is called the rotation number of f.

The dynamic of the iterates of the map $T_{\gamma,\alpha}$ is given in terms of the parameters γ and α by the following result of [2].

Theorem 1. *Let q and p be two relatively prime integers with $1 \le p \le q$, and let define the interval $I_q^p(\gamma)$ by :*

$$I_q^p(\gamma) = \left[\frac{P_q^p(\gamma)}{1+\gamma+\ldots+\gamma^{q-1}}, \frac{P_q^p(\gamma)+\gamma^{q-1}-\gamma^q}{1+\gamma+\ldots+\gamma^{q-1}} \right],$$

where P_q^p is the polynomial

$$P_q^p(X) = \sum_{k=0}^{q-1} \varepsilon_{-k}^{p/q} X^k. \tag{3}$$

Then, the application $T_{\gamma,\alpha}$ has a periodic attracting orbit with the same dynamic as the rotation $T_{1,p/q}$ if, and only if, $\alpha \in I_q^p(\gamma)$. Say differently, the rotation number of $T_{\gamma,\alpha}$ equals p/q if, and only if, $\alpha \in I_q^p(\gamma)$.

We point out that unlike the case of the rotations, the applications $T_{\gamma,\alpha}$ have almost surely a rational rotation number.

Corollary 1. *Let $\gamma \in]0,1[$. The Lebesgue measure of the set of parameters $\alpha \in]0,1[$ for which the rotation number of $T_{\gamma,\alpha}$ is rational is equal to one.*

Since the proof of Corollary 1 is short and easy, we put it in here. The sum of the lengths of the intervals $I_q^p(\gamma)$ defined in Theorem 1 is given by

$$\mu(\gamma) = \sum_{q=1}^{\infty} \frac{\Phi(q)(\gamma^{q-1}-\gamma^q)}{1+\gamma+\ldots+\gamma^{q-1}},$$

where μ is the Lebesgue measure on $[0,1]$ and $\Phi(q)$, the Euler totient function, counts the number of positive integers p less than q and coprime with q. Then, Theorem 308 of [6] yields $\mu(\gamma) = 1$.

A comparison with the asymptotic behaviour of the applications $A_{\alpha,\varepsilon}$ (see (2)) may be in order here. Indeed, let $0 < \varepsilon < 1/(2\pi)$ and set

$$\mu_\varepsilon = \sum_{1\le p\le q \text{ and } (p,q)=1} \mu\{\alpha \in [0,1[\mid \rho(A_{\alpha,\varepsilon}) = p/q\}.$$

Then we have (see [1]) $0 < \mu_\varepsilon < 1$ and $\mu_\varepsilon \to 0$ when $\epsilon \to 0$.

One can also describe very precisely the set of values of α such that the rotation number of $T_{\gamma,\alpha}$ is irrational.

Theorem 2. *Let $\gamma \in]0,1[$ and let $\alpha \in]0,1[$ be an irrational number. Then there exists a unique real number $\tau := \tau_\gamma(\alpha)$ such that the rotation number of the function $T_{\gamma,\tau}$ is equal to α. The map $\gamma \mapsto \tau_\gamma(\alpha)$ is analytic and is given by*

$$\tau_\gamma(\alpha) = (1-\gamma)\sum_{k=0}^{\infty} \varepsilon_{-k}^{\alpha}\gamma^k.$$

Further, we have $\lim_{\gamma\to 1} \tau_\gamma(\alpha) = \alpha$.

It is easy to check that for $0 < \gamma < 1$, the rotation number of $T_{\gamma,\alpha}$ is strictly less than this of the rotation T_α.

It turns out that Theorem 1 can in a natural way be generalized to a wider class of contracting maps on the torus, namely to maps with two distinct slopes.

Let $\theta \in [0,1]$, u, v be three parameters with $0 < u < 1-\theta < v < 1$ and denote by $\gamma_0, \alpha_0, \gamma_1, \alpha_1$ the four reals defined by

$$\gamma_0 = (1-v)/\theta, \alpha_0 = v, \gamma_1 = u/(1-\theta), \alpha_1 = -u\theta/(1-\theta).$$

They satisfy

$$-1 < \alpha_1 < 0 < \alpha_0, \gamma_0, \gamma_1 < 1, \quad \alpha_0 > \alpha_1 + \gamma_1$$

and also the relation

$$\alpha_1\gamma_0 = (\alpha_0 - 1)\gamma_1.$$

We consider the applications

$$[0,\theta] \overset{A_{\gamma_0,\alpha_0}}{\longrightarrow} [v,1], \text{ where } A_{\gamma_0,\alpha_0}(x) = \gamma_0 x + \alpha_0,$$

$$[\theta,1] \overset{A_{\gamma_1,\alpha_1}}{\longrightarrow} [0,u], \text{ where } A_{\gamma_1,\alpha_1}(x) = \gamma_1 x + \alpha_1.$$

In the sequel, we will write A_0 and A_1 these affine maps (compare with (1)).

The map $T_{\gamma_0,\gamma_1,\alpha_0} : X \longrightarrow X$, which we denote by T when there is no source of confusion, is defined as the unique application which coincides with A_0 on $[0,\theta[$ and with A_1 on $[\theta,1[$.

We denote by $\mathcal{S}$ the set of these transformations $T_{\gamma_0,\gamma_1,\alpha_0}$ contracting, piecewise linear over $[0,1[$, and we observe that $\mathcal{R} \subset \mathcal{C} \subset \mathcal{S}$. The following theorem provides a generalization of Theorem 1 to the set $\mathcal{S}$.

Theorem 3. *Let γ_0 and γ_1 be such that $0 < \gamma_0, \gamma_1 \leq 1$. To any rational p/q irreducible, $1 \leq p \leq q$, there is an associated compact interval $I_q^p(\gamma_0,\gamma_1)$ such that, if $\alpha_0 \in I_q^p(\gamma_0,\gamma_1)$, then the rotation number of the map $T_{\gamma_0,\gamma_1,\alpha_0}$ is equal to p/q. Further, setting $\varepsilon_\ell = \varepsilon_\ell^{p/q}$, we have*

$$I_q^p(\gamma_0,\gamma_1) = \left[\frac{A_q^p(\gamma_0,\gamma_1)}{B_q^p(\gamma_0,\gamma_1)}, \frac{C_q^p(\gamma_0,\gamma_1)}{D_q^p(\gamma_0,\gamma_1)}\right],$$

where

$$\begin{aligned} A_q^p(\gamma_0,\gamma_1) &= 1 + \sum_{k=1}^{q-1} \varepsilon_k \, \gamma_{\varepsilon_{q-1}} \cdots \gamma_{\varepsilon_k} \, , \\ B_q^p(\gamma_0,\gamma_1) &= 1 + \sum_{k=1}^{q-1} \gamma_{\varepsilon_{q-1}} \cdots \gamma_{\varepsilon_k} \, , \\ C_q^p(\gamma_0,\gamma_1) &= 1 + \sum_{k=2}^{q} \varepsilon_k \gamma_{\varepsilon_2} \cdots \gamma_{\varepsilon_k} - \gamma_{\varepsilon_1} \cdots \gamma_{\varepsilon_q} \, , \\ D_q^p(\gamma_0,\gamma_1) &= 1 + \sum_{k=2}^{q} \gamma_{\varepsilon_2} \cdots \gamma_{\varepsilon_k} \, . \end{aligned} \tag{1}$$

It is now time to give some details of the proofs. For the sake of simplicity, we will focus on Theorem 1, since the proofs of Theorem 3 rest on the same ideas, but are a little more technical. The key ingredient for both methods is Lemma 1.

A first way for proving Theorem 1 is given in details in [3]. Roughly speaking, for a fixed $\gamma \in]0,1[$ and for a polynomial P_q^p as given in (3), we compute the orbits of the point 0, hence

$$T^n_{\gamma, P_q^p(\gamma)/(1+\ldots+\gamma^{q-1})}(0) \text{ and } T^n_{\gamma,(P_q^p(\gamma)+\gamma^{q-1}-\gamma^q)/(1+\ldots+\gamma^{q-1})}(0),$$

for all integers $n \geq 0$. This method allows us to determine explicitely the points of the limit cycle of $T_{\gamma,\alpha}$, when α belongs to the interval $I_q^p(\gamma)$. Indeed, for $0 \leq u \leq \gamma^{q-1} - \gamma^q$, the asymptotic orbit of $T_{\gamma,(P_q^p(\gamma)+u)/(1+\ldots+\gamma^{q-1})}$ is composed by the points

$$\frac{R_\ell(\gamma) + u/(1-\gamma)}{1 + \ldots + \gamma^{q-1}},$$

where, for $1 \leq \ell \leq q$, the polynomial R_ℓ is the rest in the Euclidean division of $(1 + X + \ldots + X^{\ell-1}) \times P_q^p(X)$ by the polynomial $1 + X + \ldots + X^{q-1}$.

The second approach for proving Theorem 1 does not yield such an information. This is briefly described in [2], and more details are available from the author. It depends on the observation that the asymptotic behaviour of the iterates of $T = T_{\gamma,\alpha}$ is deeply related to the position of the critical point $\theta = (1-\alpha)/\gamma$, which is the preimage of 0 by T. More precisely, one has the following result.

Lemma 2. *Let $n_0 = \inf\{n \geq 0 \mid \theta \notin T^n(X)\} + 1$. For all integers $n < n_0$, $T^n(X)$ is a disjoined union of $n+1$ intervals. If n_0 is finite, then, for all $n \geq n_0$, $T^n(X)$ is the union of exactly n_0 disjoined intervals.*

Proof : Let $n \geq 0$ be given and assume that $T^n(X)$ is the disjoined union of ℓ intervals $I_1, ..., I_\ell$. Then, if $\theta \in I_\ell$, we have $T^{n+1}(X) = \bigcup_{i=1}^{\ell-1} T(I_i) \cup J_1 \cup J_2$, where the $T(I_i)$ are intervals and where J_1 and J_2 are respectively of the shape $[0, .[$ and $[., 1[$. These $\ell + 1$ intervals are disjoined, since the map T is injective; in this case, an additional iteration yielded an additional interval. If $\theta \notin T^n(X)$, then $T^{n+1}(X)$ is the distinct union of the ℓ intervals $T(I_i)$. We conclude by noticing that $T^0(X) = X$ is composed by a unique interval. $\square$

The idea is then to write the condition $\theta \notin T^n(X)$ in terms of inequalities involving γ and α and to conclude by an induction based on Lemma 1.

4 An algorithm

Finally, we describe a continued fraction algorithm which computes the rotation number of any given function $T_{\gamma,\alpha}$. The existence of such an algorithm is not surprising at all, since it is well known that, for $0 < \alpha < 1$, induction of the rotation $x \mapsto x + \alpha$ on the interval $[1 - \alpha, 1]$ allows us to recover the continued fraction expansion of α. Here, and in the sequel, we mean by continued fraction expansion a (signed) continued fraction expansion, *i.e.* an expansion of the shape

$$a = a_0 \pm \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{a_3 \pm \cfrac{1}{\cdots}}}}.$$

The sign essentially depends on the choice of the interval on which we induce and we merely describe here an algorithm leading to an expansion with the minus sign.

Unlike the class $\mathcal{C}$, the class $\mathcal{S}$ is stable by induction, and this is the main reason why we have introduced it. Let γ_0, γ_1, α_0 and α_1 be parameters satisfying (4) and (5). As above, we denote by θ the critical point of the transformation $T = T_{\gamma_0,\gamma_1,\alpha_0}$, and we will induce on the interval $[\theta, 1]$. As in the case of the rotations, one can define an algorithm which allows use to know precisely the asymptotic behaviour of the iterates of T. The critical point θ is in $[0, 1]$ if and only if $\alpha_0 + \gamma_0 > 1$, and if θ does not belong to the interval $[0, 1]$, then T contracts the torus X, and the iterates of T converge to its unique fixed point. We then repeat the induction process while the induced transformation has a critical point. The parameters of the induced map of T on $[\theta, 1]$ are computed in [3]; they yield the following algorithm.

`Input` : Two real numbers $0 < \alpha, \gamma \leq 1$.
`Output` : The continued fraction expansion with minus sign of the rotation number of $T_{\gamma,\alpha}$.

```
γ1 := γ ;
while α + γ > 1 do
begin
    u := 1 ;
    z := γ ;
    b := 1 ;
    while u ≤ 1/α do
    begin
        b := b + 1 ;
        u := 1 + γ * u ;
    end ;
    α := (α * u − 1)/(γ + α − 1) ;
    γ := γ^b * γ1 ;
    γ1 := γ/z ;
    print (b) ;
end.
```

By Corollary 1, this algorithm stops for almost all choice of the parameters. If $b_1, b_2, \dots, b_n$ denote the integers successively obtained through the procedure, then the rotation number of T is the rational ρ given by

$$\rho = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_n}}}}.$$

5 An example

We illustrate the preceding sections by the study of a numerical example. We consider the map $T_{\gamma,\alpha}$ given by $\gamma = 0.8$ and $\alpha = 0.5$, hence the map

$$T : x \mapsto \{0.8x + 0.5\}.$$

By an easy calculation, we get

$$T\big([0,1[\big) = [0; 0.3[\cup [0.5; 1[,$$

$$T^2\big([0,1[\big) = [0; 0.3[\cup [0.5; 0.74[\cup [0.9; 1[,$$

$$T^3\big([0,1[\big) = [0; 0.092[\cup [0.22; 0.3[\cup [0.5; 0.74[\cup [0.9; 1[,$$

and

$$T^4\big([0,1[\big) = [0; 0.092[\cup [0.22; 0.3[\cup [0.5; 0.5736[\cup [0.676; 0.74[\cup [0.9; 1[.$$

We observe that the critical point $\theta = (1-\alpha)/\gamma = 0.625$ does not belong to $T^4\big([0,1[\big)$, hence the limit cycle has 5 elements. Further, by Theorem 1, $P_5^2(\gamma) = 1+\gamma^2$, and

$$0.5 \in I_5^2(\gamma) = \left[\frac{1.64}{3.3616}, \frac{1.72192}{3.3616}\right],$$

hence $\rho(T) = 2/5$. As for the algorithm of Section 4, we check that it gives

$$\rho(T) = \frac{1}{3-\dfrac{1}{2}} = \frac{2}{5}.$$

References

1. Arnold, V. I. (1970) Chapitres supplémentaires de la théorie des équations différentielles ordinaires, Editions de Moscou.
2. Bugeaud, Y. (1993) Dynamique de certaines applications contractantes linéaires par morceaux sur $[0,1[$. C. R. Acad. Sci. Paris Sér. I **317**, 575–578.
3. Bugeaud, Y., Conze J.-P. (1999) Calcul de la dynamique de transformations linéaires contractantes mod 1 et arbre de Farey. Acta Arithmetica. To appear.
4. Devaney, R. L. (1989) An Introduction to Chaotic Dynamical Systems. Addison-Wesley : Menlo Park, second edition.
5. Feely, O., Chua L. O. (1991) The effect of integrator leak in $\Sigma - \Delta$ modulation. IEEE Transactions on Circuits and Systems **38**, 1293-1305.
6. Hardy, G. H., Wright, E. M. (1979) An introduction to the theory of numbers. Oxford : Clarendon Press, fifth edition.
7. de Melo, W., van Strien, S. (1993) One-dimensional dynamics. Springer-Verlag, 3. Folge, Band 25.