On the greatest prime factor of $(ab+1)(bc+1)(ca+1)$

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1. Introduction

For any rational number $r > 0$, we denote by $P(r)$ the greatest prime factor of r, with the convention that $P(1) = 1$. Győry, Sárközy & Stewart [4] conjectured that if $a > b > c$ are positive integers, then

$$P((ab+1)(bc+1)(ca+1)) \longrightarrow \infty \tag{1}$$

as a tends to infinity.

Very recently, Stewart & Tijdeman [7] showed that (1) holds under the additional hypothesis $(\log a / \log(c+1)) \to \infty$. More precisely, using Baker's theory of linear forms in logarithms, they proved the following result.

Theorem A. *There exists an effectively computable positive numerical constant κ_1 such that, if $a \geq b > c$, then*

$$P((ab+1)(bc+1)(ca+1)) > \kappa_1 \log(\log a / \log(c+1)).$$

Further, Győry & Sárközy [3] confirmed the conjecture in the special case when at least one of the rational numbers a, b, c, a/b, b/c, c/a has bounded prime factors. Their result can be reformulated in the following qualitative form (we denote by $|I|$ the cardinality of a finite set I).

Theorem B. *Let $\mathcal{A}$ be a finite set of triples (a, b, c) of integers $a > b > c > 0$. Let $p_1 < \ldots < p_s$ be distinct primes, and let S denote the set of positive rational numbers composed by the p_i's. Assume that for all triples (a, b, c) in $\mathcal{A}$, at least one of the numbers a, b, c, a/b, b/c or c/a is contained in S. Then, there exists a triple (a, b, c) in $\mathcal{A}$ such that*

$$P((ab+1)(bc+1)(ca+1)) \geq \kappa_2 \log|\mathcal{A}| \log\log|\mathcal{A}|, \tag{2}$$

provided that $\log|\mathcal{A}| \geq s/\kappa_2^2$, where κ_2 is an effectively computable positive numerical constant.

Theorem B is an immediate consequence of a lower bound for the number of distinct prime factors of $(ab+1)(bc+1)(ca+1)$, which proof depends on a powerful estimate of

Evertse about the number of solutions of S-unit equations, obtained by ineffective methods. That is the reason why the lower bound in (2) depends on the cardinality of $\mathcal{A}$ and not on the size of its elements.

In the present work, we slightly generalize Theorem B, using a totally different approach, which e.g. leads to an explicit lower bound for $P((ab+1)(bc+1)(ca+1))$ in terms of the maximum of a, b and c, subject that $P(a)$ is bounded from above. As for the proof of Theorem A, we use Baker's theory. However, our main tool is an estimate for simultaneous linear forms in logarithms due to Loxton [5].

2. Statement of the results

Theorem 1. *Let $a \geq b > c$ be positive integers. Let α denote any element of the set $a, b, c, a/b, b/c, c/a$. Then there exists an effectively computable positive numerical constant κ_3 such that*

$$P(\alpha\,(ab+1)(bc+1)(ca+1)) \geq \kappa_3\,\log\log a.$$

Theorem B can be deduced from Theorem 1, but with a weaker lower bound. Indeed, let define the sets $\mathcal{A}$ and S as in Theorem B, and set $M = \max_{(a,b,c)\in\mathcal{A}} a$. Assume that for all triples (a,b,c) in $\mathcal{A}$, we have $a \in S$. Since, obviously, $|\mathcal{A}| \leq M^3$, it follows from Theorem 1 that if $p_s < \kappa_3 \log\log |\mathcal{A}|^{1/3}$, then there exists a triple (a,b,c) in $\mathcal{A}$ (e.g. a triple with $a = M$), such that

$$P((ab+1)(bc+1)(ca+1)) \geq \kappa_3\,\log\log|\mathcal{A}|^{1/3}.$$

As mentioned in the introduction, we prove Theorem 1 by using estimates of Loxton for simultaneous linear forms in logarithms. One of the interests of our work is to present a new application of this nice result, which, to our knowledge, has only been utilized in [5], [1] and [2].

In reply to the conjecture proposed by Győry & Sárközy [3], we formulate the following open problem.

Conjecture. *Let $a \geq b > c$ be positive integers. There exists an effectively computable positive numerical constant κ_4 such that*

$$P((ab+1)(bc+1)(ca+1)) \geq \kappa_4\,\log\log a.$$

We note that, in view of Lemma 2, this conjecture is true when $a = b$. This is also true, by Theorem 1, when one of the numbers a, b, c, a/b, b/c, c/a has all its prime factors less than $\kappa_3 \log\log a$, and, by Theorem A, when $\log c < (\log a)^{1-\delta}$ for some fixed positive constant δ. Further, if true, our conjecture would imply Győry & Sárközy's one.

3. Auxiliary lemmas

Our main tool is the following result of Loxton [5].

Lemma 1. *Let $n \geq 1$ be an integer, and let $\alpha_1, \ldots, \alpha_n$ be non-zero multiplicatively independent rational numbers. Consider the linear forms*

$$\Lambda_i = b_{i1} \log \alpha_1 + \ldots + b_{in} \log \alpha_n \quad (i = 1, 2),$$

where the b_{ij}'s denote rational integers. We assume that the matrix (b_{ij}) has rank two and that the height of α_j (resp. b_{ij}) is at most $A_j \, (\geq 4)$ (resp. $B \, (\geq 4)$). Then, we have

$$\max\Big\{|\Lambda_1|, |\Lambda_2|\Big\} > \exp\Big\{-(16n)^{200n} \, (\log A_1 \ldots \log A_n)^{2/3} \, \log B\Big\}.$$

Proof : This is a weaker form of Theorem 4 in [5]. We note that the numerical constant can be subsequently reduced, but this is irrelevant in the present work. □

We also need a lower bound for the prime factor of a quadratic polynomial.

Lemma 2. *Let g denote a quadratic polynomial with integral coefficients and with distinct roots. Then there exists an effectively computable positive constant $\kappa_5 = \kappa_5(g)$, depending only on g, such that, for every integer $x \geq 2$, we have*

$$P(g(x)) \geq \kappa_5 \log\log x.$$

Proof : This is Corollary 7 of Schinzel [6]. □

4. Proofs

We prove the theorem for $\alpha = a$ and we briefly indicate how one should modify our arguments in order to get the five other results. In the sequel $\kappa_6, \ldots, \kappa_{11}$ denote effectively computable positive numerical constants.

In view of Theorem A, we can assume that $b > c \geq 16$. Hence, setting

$$\Lambda_1 := \frac{ac+1}{ac} \quad \text{and} \quad \Lambda_2 := \frac{(ac+1)(bc+1)}{c^2\,(ab+1)},$$

we obtain that $0 < \log \Lambda_1 < a^{-1}$ and $0 < \log \Lambda_2 < b^{-1}$.

Denote by $2 =: p_1 < p_2 < \ldots < p_t < \ldots$ the sequence of primes ranged in increasing order, and assume that $p_r = P(a\,(ab+1)(bc+1)(ca+1))$. Then, there exist rational integers l_i and m_i, $1 \leq i \leq r$, with absolute values bounded by $6 \log a$, such that

$$\log \Lambda_1 = \sum_{i=1}^{r} \, l_i \, \log p_i - \log c,$$

$$\log \Lambda_2 = \sum_{i=1}^{r} \, m_i \, \log p_i - 2 \log c.$$

If c has no prime divisor greater than p_r, then

$$P(a\,(ab+1)(bc+1)(ca+1)) = P(ac\,(ab+1)(bc+1)(ca+1)) \geq P(ac\,(ac+1)),$$

which, by Lemma 2, is greater or equal to $\kappa_6 \log\log a$.

If this is not the case, $\log \Lambda_1$ and $\log \Lambda_2$ are not proportional unless we have $\Lambda_1^2 = \Lambda_2$, i.e. $(ac+1)(ab+1) = a^2\,(bc+1)$. But this is excluded since $a > 1$.

Hence, we may assume that the hypotheses of Lemma 1 are satisfied, and we get

$$\begin{aligned} \log\log\max\{\Lambda_1, \Lambda_2\} &> -(17r)^{200r} \left(\prod_{i=1}^{r} \log p_i \right) (\log c)^{2/3}\, \log\log a \\ &> -e^{\kappa_7\, p_r}\, (\log c)^{2/3}\, \log\log a, \end{aligned} \tag{3}$$

by the prime number theorem. Since $\log\log\max\{\Lambda_1, \Lambda_2\} < -\log b$, we infer from (3) that

$$p_r \geq \kappa_8\, \log\left(\frac{\log b}{\log^3 \log a} \right). \tag{4}$$

Combining (4) with Theorem A, we obtain

$$\begin{aligned} 2\,p_r &\geq \kappa_1\, \log\left(\frac{\log a}{\log(c+1)} \right) + \kappa_8\, \log\left(\frac{\log b}{\log^3 \log a} \right) \\ &\geq \kappa_9\, \log\log a, \end{aligned}$$

which is the desired estimate.

In order to prove Theorem 1 for $\alpha = b$ and $\alpha = c$, we introduce $\Lambda_3 := (bc+1)/bc$, which satisfies $0 < \log \Lambda_3 < b^{-1}$. We work with Λ_2 and Λ_3 and proceed exactly as for the case $\alpha = a$. Hence, we get

$$P(b\,(ab+1)(bc+1)(ca+1)) \geq \kappa_{10}\, \log\bigl(\log b / \log^3 \log a\bigr)$$

and

$$P(c\,(ab+1)(bc+1)(ca+1)) \geq \kappa_{11}\, \log\bigl(\log b / \log^3 \log a\bigr),$$

and we conclude by applying Theorem A.

The proof for the remaining cases is similar. If $\alpha = a/b$ (*resp.* $\alpha = b/c$, $\alpha = c/a$), we use Λ_1 and Λ_3 (*resp.* Λ_2 and Λ_3, Λ_1 and Λ_2). $\square$

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