On the Diophantine equation $\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1}$

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Abstract. We study the Diophantine equation $\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1}$ in integers $x>1$, $y>1$, $m>1$, $n>1$ with $x \neq y$. We show that, for given x and y, this equation has at most two solutions. Further, we prove that it has finitely many solutions (x,y,m,n) with $m>2$ and $n>2$ such that $\gcd(m-1,n-1)>1$ and $(m-1)/(n-1)$ is bounded.

1. Introduction

Goormaghtigh [7] observed in 1917 that

$$31=\frac{2^5-1}{2-1}=\frac{5^3-1}{5-1} \quad \text{and} \quad 8191=\frac{2^{13}-1}{2-1}=\frac{90^3-1}{90-1}$$

are two solutions of the Diophantine equation

$$\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1} \qquad \text{in integers } x>1,\ y>1,\ m>2,\ n>2 \text{ with } x\neq y. \tag{1}$$

There is no restriction in assuming that $y>x$ in (1) and thus we have $m>n$. This equation asks for integers having all their digits equal to one with respect to two distinct bases and we still do not know whether or not it has finitely many solutions. Even if we fix one of the four variables, it remains an open question to prove that (1) has finitely many solutions.

However, when either the bases x and y, or the base x and the exponent n, or the exponents m and n are fixed, then it is proved that (1) has finitely many solutions (see [3] for references). In the first two cases, thanks to Baker's theory of linear forms in logarithms, we can compute explicit (huge) upper bounds for the size of the solutions. As for the number of solutions, Shorey [14] proved that for two integers $y>x$, the Diophantine equation

$$\frac{x^m-1}{x-1}=\frac{y^n-1}{y-1} \qquad \text{in integers } m>1,\ n>1. \tag{2}$$

has at most 17 solutions, independently of x and y. One of the purposes of the present work is to considerably improve this estimate by showing that (2) has at most one solution provided that y is large enough and that otherwise (2) has at most two solutions.

2000 *Mathematics Subject Classification :* 11D61.

When the exponents m and n are fixed, Davenport, Lewis and Schinzel [6] proved that (1) has finitely many solutions, but their proof rests on a theorem of Siegel and it is ineffective. However, when $\gcd(m-1, n-1) > 1$, they are able to replace Siegel's result by an effective argument due to Runge.

Theorem DLS. *Equation (1) with* $\gcd(m-1, n-1) > 1$ *implies that* $\max(x, y)$ *is bounded by an effectively computable number depending only on* m *and* n.

Recently, this has been improved by Nesterenko & Shorey [13] as follows.

Theorem NS. *Let* $d \geq 2$, $r \geq 1$ *and* $s \geq 1$ *be integers with* $\gcd(r, s) = 1$. *Assume that* $m-1 = dr$ *and* $n-1 = ds$. *If* (x, y, m, n) *satisfy (1), then* $\max\{x, y, m, n\}$ *is bounded by an effectively computable number depending only on* r *and* s.

This is the first result of the type where there is no restriction on the bases x and y and the exponents m and n extend over an infinite set. In the present work, we show that the assertion of Theorem NS continues to be valid when the ratio $(m-1)/(n-1)$ is bounded.

Acknowledgement. This work was done when the first author was visiting the Tata Institute of Fundamental Research, Mumbai. He would like to thank the mathematical department for its hospitality and the CNRS for its financial support.

2. Statement of the results

Our first result deals with the number of solutions of equation (2) and improves a previous estimate of Shorey [14].

Theorem 1. *Let* $y > x > 1$ *be integers. If* $\gcd(x, y) > 1$ *or if* $y \geq 10^{11}$, *then (2) has at most one solution. Further, if* $y \geq 7$, *then (2) has at most two solutions. Finally, the only solutions of (2) with* $y \leq 6$ *are given by* $(x, y, m, n) = (2, 5, 5, 3)$ *or* $(2, 6, 3, 2)$.

M. Mąkowski and A. Schinzel [12] proved that (2) with $y \leq 10$ and $m > n > 2$ has only the solution $(x, y, m, n) = (2, 5, 5, 3)$, however, for sake of completeness, we give a proof of the last statement of Theorem 1. Our proof is based on an idea of Le [10] and it combines the theory of linear forms in logarithms together with a strong gap principle proved by elementary means. As is apparent from the proof, one can derive several other interesting statements including the following.

Theorem 2. *Let* $y > x > 1$ *be coprime integers and assume that the smallest integer* $s \geq 1$ *such that* $x^s \equiv 1 \pmod{y}$ *satisfies*

$$s > 48^{11} (\log y)^2 (\log\log y). \tag{3}$$

Then equation (2) has no solution.

Remark 1 : We point out that for given y sufficiently large, Theorem 2 solves (2) for a wide set of integers x, indeed for at least $\varphi(y) - \varphi(y)^{2/3}$ integers x, with $1 \leq x \leq y$ and gcd $(x, y) = 1$. Here, φ denote the Euler totient function. A proof of this assertion is given just after the proofs of Theorem 1 & 2.

Remark 2 : Theorem 1 implies that the equations

$$\frac{2^m - 1}{2 - 1} = \frac{5^n - 1}{5 - 1} \quad \text{and} \quad \frac{2^m - 1}{2 - 1} = \frac{90^n - 1}{90 - 1}$$

have exactly one solution in integers $m > 1$ and $n > 1$, namely $(m, n) = (5, 3)$ and $(13, 3)$, respectively.

Remark 3 : M. Mąkowski and A. Schinzel [11] proved that (2) with $y \leq 10$ and $m > n > 2$ has only the solution $(x, y, m, n) = (2, 5, 5, 3)$, however, for sake of completeness, we give a proof of the last statement of Theorem 1.

In Theorem NS quoted in the Introduction, the exponents m and n are allowed to vary such that the ratio $(m-1)/(n-1)$ is constant. This condition also implies that y cannot be too large compared with x. We now present two new results under a similar hypothesis. The first one can be seen as an improvement of Theorem NS, although it does not imply the latter. The second one is of a different nature, namely, there is no restriction on the exponents m and n and the bases x and y extend to an infinite set.

Theorem 3. *Let $\alpha > 1$. Equation (1) with* gcd $(m-1, n-1) \geq 4\alpha + 6 + \frac{1}{\alpha}$ *and* $(m-1)/(n-1) \leq \alpha$ *implies that* max (x, y, m, n) *is bounded by an effectively computable number depending only on α.*

Theorem 3 does not contain Theorem NS because of the condition imposed on $\gcd(m-1, n-1)$. If r and s are fixed, we may remove this condition by Theorem DLS and then Theorem NS follows from Theorem 3. In the course of the proof of Theorem 3, we need an auxiliary result (namely, Lemma 4 below) which enables us to considerably improve Theorem 2 of [13].

Theorem 4. *Let (x, y, m, n) be a solution of (1) with $y > x$. Then we have*

$$\gcd(m-1, n-1) \leq 33.4\, m^{1/2}.$$

Remark 4 : Theorem 2 of [13] only asserts that there exist an effectively computable absolute constant C such that $\gcd(m-1, n-1) \leq C m^{4/5} (\log m)^{3/5}$. Further, its proof combines the theory of linear forms in logarithms together with sharp upper bounds for the size of the solutions of (1) obtained by Runge's method, whereas the proof of Theorem 4 depends only on estimates for linear forms in two logarithms.

Theorem 5. *Let $a > 1$. Let $y > x > 1$ be integers such that x divides $y - 1$ and $y \leq x^a$. If (x, y, m, n) satisfies (1), then*

$$n < m \leq 14000\, a^2 (\log 3a)^2$$

and

$$x < n, \quad y < n^a.$$

Remark 5 : It follows from Theorem 5 that for a given $a > 1$, equation (1) has only finitely many solutions (x, y, m, n) with $y \leq x^a$ and $x \mid (y-1)$.

3. Auxiliary lemmas

We begin with the following theorem of Baker and Wüstholz [2] on linear forms in logarithms.

Lemma 1. *Let $\alpha_1, \ldots, \alpha_d$ be positive rational numbers of heights not exceeding $A_1, \ldots, A_d$, respectively, where $A_j \geq e$ for $1 \leq j \leq d$. Put*

$$\Omega = \prod_{j=1}^{d} \log A_j.$$

Then the inequalities

$$0 < |b_1 \log \alpha_1 + \cdots + b_d \log \alpha_d| < \exp\bigl(-(16d)^{2(d+2)} \Omega \log B\bigr)$$

have no solution in integers $b_1, \cdots, b_d$ of absolute values not exceeding B, where $B \geq e$.

In order to prove Theorems 1 & 2, we need an explicit estimate for the size of the solutions of (2).

Lemma 2. *Let $y > x > 1$ be integers and let (m, n) be a solution of (2). Then we have*

$$m \leq 2 \times 48^{10} (\log y)^2 (\log m) + 1. \tag{4}$$

Proof : We re-write (2) as

$$\frac{x^m}{x-1} - \frac{y^n}{y-1} = \frac{1}{x-1} - \frac{1}{y-1},$$

thus we get

$$0 < 1 - y^n x^{-m} \left(\frac{x-1}{y-1}\right) < x^{-m},$$

which implies that

$$0 < \left| n \log y - m \log x + \log\left(\frac{x-1}{y-1}\right) \right| < 2x^{-m}. \tag{5}$$

Now we apply Lemma 1 with $d = 3$, $A_1 = y$, $A_2 = x + 1$, $A_3 = y$ and $B = m$ to derive

$$\left| n \log y - m \log x + \log\left(\frac{x-1}{y-1}\right) \right| > \exp\bigl(-(48)^{10} (\log y)^2 \log(x+1) \log m\bigr)$$

which, combined with (5), yields the estimate stated in the lemma. □

Apart from Lemma 1, we shall also need the following refinement, due to Mignotte[12], of a theorem of Laurent, Mignotte & Nesterenko [9] on linear forms in two logarithms.

Lemma 3. *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put

$$D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}] \,/\, [\mathbf{R}(\alpha_1, \alpha_2) : \mathbf{R}].$$

Let a_1, a_2, h, k be real positive numbers, and ρ a real number > 1. Put $\lambda = \log \rho$, $\chi = h/\lambda$ and suppose that $\chi \geq \chi_0$ for some number $\chi_0 \geq 0$ and that

$$\begin{aligned} h &\geq D\Big(\log\Big(\frac{b_1}{a_2} + \frac{b_2}{a_1}\Big) + \log \lambda + f(\lceil K_0 \rceil)\Big) + 0.023, \\ a_i &\geq \max\{1, \rho\,|\log \alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\}, \quad (i = 1, 2), \\ a_1 a_2 &\geq \lambda^2 \end{aligned}$$

where

$$f(x) = \log \frac{(1 + \sqrt{x-1})\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log \frac{3}{4} + \frac{\log \frac{x}{x-1}}{x-1},$$

and

$$K_0 = \frac{1}{\lambda}\left(\frac{\sqrt{2 + 2\chi_0}}{3} + \sqrt{\frac{2(1+\chi_0)}{9} + \frac{2\lambda}{3}\Big(\frac{1}{a_1} + \frac{1}{a_2}\Big) + \frac{4\lambda\sqrt{2+\chi_0}}{3\sqrt{a_1 a_2}}}\right)^2 a_1 a_2.$$

Put

$$v = 4\chi + 4 + 1/\chi \quad \text{and} \quad m = \max\{2^{5/2}(1+\chi)^{3/2}, (1+2\chi)^{5/2}/\chi\}.$$

Then we have the lower bound

$$\begin{aligned} \log|\Lambda| \geq &-\frac{1}{\lambda}\left(\frac{v}{6} + \frac{1}{2}\sqrt{\frac{v^2}{9} + \frac{4\lambda v}{3}\Big(\frac{1}{a_1} + \frac{1}{a_2}\Big) + \frac{8\lambda m}{3\sqrt{a_1 a_2}}}\right)^2 a_1 a_2 \\ &- \max\Big\{\lambda(1.5 + 2\chi) + \log\Big(\big((2+2\chi)^{3/2} + (2+2\chi)^2\sqrt{k^*}\big)A + (2+2\chi)\Big), D\log 2\Big\} \end{aligned}$$

where

$$A = \max\{a_1, a_2\} \quad \text{and} \quad k^* = \frac{1}{\lambda^2}\left(\frac{1+2\chi}{3\chi}\right)^2 + \frac{1}{\lambda}\left(\frac{2}{3\chi} + \frac{2}{3}\frac{(1+2\chi)^{1/2}}{\chi}\right).$$

Proof : This is Theorem 2 of [12]. □

We apply Lemma 3 for deriving the following result.

Lemma 4. *Let $\alpha > 1$ and $d > 1$ be an integer. Suppose that (x, y, m, n) with $y > x$ is a solution of (1). Assume that*

$$\gcd(m-1, n-1) = d, \quad \frac{m-1}{n-1} \leq \alpha.$$

Then we have

$$d \leq 743\left(\alpha + \frac{1}{2}\right).$$

Proof : Let $\alpha > 1$ and $d > 1$ be an integer. We suppose that (1) with $\gcd(m-1, n-1) = d$ and $(m-1)/(n-1) \leq \alpha$ is satisfied and we put

$$m - 1 = dr, \quad n - 1 = ds$$

where r and s are positive integers. In view of the last assertion of Theorem 1, whose proof is independent of Lemma 3, we may assume that $y \geq 7$. We write (1) as

$$\frac{x}{x-1}\, x^{rd} - \frac{y}{y-1}\, y^{sd} = \frac{1}{x-1} - \frac{1}{y-1}$$

which implies that

$$0 < \log \frac{x(y-1)}{y(x-1)} - d \log \frac{y^s}{x^r} < y^{-sd}. \tag{6}$$

Now we apply Lemma 3 with $b_1 = d$, $b_2 = 1$, $\alpha_1 = y^s/x^r$ and $\alpha_2 = (x(y-1))/(y(x-1))$ in order to get a lower bound for $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$. We observe that $\mathrm{h}(\alpha_1) \leq s \log y$, $\mathrm{h}(\alpha_2) \leq 2 \log y$ and that α_1 and α_2 are multiplicatively independent. Further, we put

$$\rho = 1 + \frac{3 \log y}{4 \log(1 + \frac{1}{x-1})}.$$

Then we may take

$$a_1 = (2s + 3/4) \log y \quad \text{and} \quad a_2 = (19 \log y)/4.$$

Thus, we check that

$$\log x \leq \log\left(1 + \frac{3}{4}(x-1)\log y\right) \leq \lambda := \log \rho \leq \frac{4}{3} \log y \tag{7}$$

and

$$a_1 a_2 \geq 7\lambda^2, \quad a_1 a_2/\lambda \geq 19. \tag{8}$$

By (2), we observe that $y^{n-1} \leq 2x^{m-1}$ which implies that

$$\frac{\log y}{\log x} \leq \alpha + \frac{1}{2}. \tag{9}$$

We may assume that $d \geq 30$. By (7) and since $\lceil K_0 \rceil \geq 16$, in order to apply Lemma 3, we have to choose the parameter h such that

$$h \geq \log d + 0.3.$$

Assume first that $\lambda \geq \log d + 0.3$ and apply Lemma 3 with $h = \lambda$ and $\chi = 1$. Thus $v = 9$, $m = 16$ and, using that $y \geq 7$ in the definition of ρ, we infer that $k^* \leq 2.4$. Now, we notice that $\lambda/a_1 + \lambda/a_2 \leq 0.77$ and use (8), to obtain that

$$\log \Lambda > -\frac{19.65}{\lambda} a_1 a_2 - 3.5\lambda - \log(56.7 a_1 + 4). \tag{10}$$

Finally, using $y \geq 7$, we conclude from (7), (9), (10) and (6) that

$$d \leq 96\left(2 + \frac{3}{4s}\right)\left(\alpha + \frac{1}{2}\right).$$

We observe that the right hand side of the preceding inequality does not exceed $264(\alpha + 1/2)$. Therefore, we may suppose that $\lambda < \log d + 0.3$, that is, in view of (7)

$$\log\left(1 + \frac{3 \log y}{4 \log(1 + \frac{1}{x-1})}\right) < \log d + 0.3. \tag{11}$$

Next, we show that

$$d \leq 135 \log y. \tag{12}$$

For the proof of (12), we assume that $d > 135 \log y$ and we shall arrive at a contradiction. We apply Lemma 3 once again, but with another choice of the radius ρ. Namely, we take $\rho = e^4$. Then $\lambda = 4$. We set

$$h = \log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \frac{57}{20}$$

and we see that $h \geq 6$, thus we may choose $\chi_0 = \frac{3}{2}$. We observe that $a_1 a_2 \geq 49$ and $\lceil K_0 \rceil \geq 27$. Further, we have $v \leq h + \frac{14}{3}$, $m \leq 2^{5/2}(1+\chi)^{3/2}$ and we put $H = h + \frac{14}{3}$. Now, Lemma 3 yields

$$\begin{aligned} \log \Lambda &\geq -\frac{3}{40} H^2 a_1 a_2 - 6 - 2h - \log(2h^2 a_1 a_2) \\ &\geq -\frac{17}{200} H^2 a_1 a_2. \end{aligned}$$

since $H \geq \frac{32}{3}$ and $a_1 a_2 \geq 49$. Combining the preceding estimate with (6), we get

$$\frac{d}{\log y} \leq \frac{57}{50}\left(\log\left(\frac{4d}{19 \log y} + \frac{1}{5}\right) + \frac{451}{60}\right)^2.$$

This is not possible and the proof of (12) is complete.

We combine (7), (11) and (12) to conclude that

$$\frac{3}{4}(x-1)\log y < 183\log y.$$

Thus $x \leq 244$. Finally, we apply (9) and (12) to conclude that $d \leq 743(\alpha+1/2)$. $\square$

In addition to Lemma 4, the proof of Theorem 3 uses an irrationality measure [15] of certain algebraic numbers derived from a Theorem of Baker [1].

Lemma 5. *Let A, B, K and n be positive integers such that $A > B, K < n, n \geq 3$ and $\omega = (B/A)^{1/n}$ is not a rational number. For $0 < \phi < 1$, put*

$$\delta = 1 + \frac{2-\phi}{K}, \quad \sigma = \frac{\delta}{1-\phi}$$

and

$$u_1 = 40^{n(K+1)(\sigma+1)/(K\sigma-1)}, \quad u_2^{-1} = K2^{K+\sigma+1}40^{n(K+1)}.$$

Assume that

$$A(A-B)^{-\delta}u_1^{-1} > 1.$$

Then

$$\left|\omega - \frac{p}{q}\right| > \frac{u_2}{Aq^{K(\sigma+1)}}$$

for all integers p and q with $q > 0$.

Proof : This is Lemma 1 of Shorey & Nesterenko [15]. We notice that this has been refined by Hirata-Kohno in [8] but the statement of [15] is sufficient for our purpose. $\square$

Our last auxiliary result is an estimate from the theory of p-adic linear forms in (two) logarithms. For this, we need some notation.

Let $m > 1$ be an integer and write $m = p_1^{u_1} \ldots p_w^{u_w}$, where $p_1 < \ldots < p_w$ are distinct prime numbers and the u_i's are positive integers. Let x be a non-zero integer and let p be a prime. We recall that the p-adic valuation of x, denoted by $v_p(x)$, is the greatest non-negative integer v such that p^v divides x. Analogously, we define the m-adic valuation of x, which we denote by $v_m(x)$, to be the greatest non-negative integer v such that m^v divides x. We observe that

$$v_m(x) = \min_{1\leq i\leq w}\left[\frac{v_{p_i}(x)}{u_i}\right],$$

where $[\cdot]$ denotes the integer part. Further, if a/b is a non-zero rational number with a and b coprime, we set $v_m(a/b) = v_m(a) - v_m(b)$.

We let x_1/y_1 and x_2/y_2 be two non-zero rational numbers with $x_1/y_1 \neq \pm 1$. Lemma 6 provides an upper bound for the m-adic valuation of

$$\Lambda = \left(\frac{x_1}{y_1}\right)^{b_1} - \left(\frac{x_2}{y_2}\right)^{b_2},$$

where b_1 and b_2 are positive integers, assuming that

$$v_{p_i}\left(\frac{x_1}{y_1}-1\right)\geq u_i,\ \ v_{p_i}\left(\frac{x_2}{y_2}-1\right)\geq 1 \quad \text{for all prime } p_i,\ 1\leq i\leq w$$

and that either m is odd or 4 divides m. Further, let $A_1>1, A_2>1$ be real numbers such that

$$\log A_i \geq \max\{\log|x_i|, \log|y_i|, \log m\},\ \ (i=1,2),$$

and put

$$b' = \frac{b_1}{\log A_2}+\frac{b_2}{\log A_1}.$$

Lemma 6. *With the previous notation and under the above hypothesis, if moreover m, b_1 and b_2 are relatively prime, then we have the upper estimate*

$$v_m(\Lambda)\leq \frac{66.8}{(\log m)^4}\big(\max\{\log b'+\log\log m+0.64, 4\log m\}\big)^2\log A_1\log A_2.$$

Proof : This is Theorem 2 of [4]. □

4. Proofs

Proofs of Theorems 1 and 2 :

Let $y>x\geq 2$ be integers and denote by d their greatest common divisor. Set $x_0=x/d$ and $y_0=y/d$. Assume that (2) has two distinct solutions (m_2,n_2) and (m_1,n_1) with $m_2>m_1>1$. Then we get

$$(dy_0-1)(dx_0)^{m_2}-(dx_0-1)(dy_0)^{n_2}=dy_0-dx_0 \tag{13}$$

and

$$(dy_0-1)(dx_0)^{m_1}-(dx_0-1)(dy_0)^{n_1}=dy_0-dx_0. \tag{14}$$

We first observe that d and x_0 are coprime. Indeed, let p be a prime such that $p^a\|d$ and $p|x_0$. Thus p^{2a} must divide the left-hand side of (13) since $m_2\geq 2$ and $n_2\geq 2$. But p^{a+1} divides dx_0, hence p must divide y_0 contradicting $\gcd(x_0,y_0)=1$. Similarly, we prove that d and y_0 are coprime. Using (14) to replace dy_0-dx_0 in (13) and dividing (13) by $(dx_0-1)(dy_0)^{n_1}$, we get

$$1=\frac{(dx_0)^{m_1}(dy_0-1)(1-(dx_0)^{m_2-m_1})}{(dy_0)^{n_1}(dx_0-1)}+(dy_0)^{n_2-n_1}. \tag{15}$$

Since d and x_0y_0 are coprime, this implies that d divides the right-hand side of (15), hence $d=1$. Consequently, if x and y have a common factor, then (2) has at most one solution.

In the sequel of the proof of Theorem 1, we always assume that x and y are coprime. Further, we assume that there exist pairs of positive integers (m_3, n_3), (m_2, n_2) and (m_1, n_1), with $m_3 > m_2 > m_1 \geq 1$ and

$$\frac{x^{m_j} - 1}{x - 1} = \frac{y^{n_j} - 1}{y - 1}, \qquad j = 1, 2, 3.$$

Since $y > x$, we clearly have $m_{j+1} - m_j \geq 2$ for $j = 1, 2$. Further

$$n_{j+1} - n_j \geq 2, \qquad j = 1, 2. \tag{16}$$

Indeed, if for example $n_2 = n_1 + 1$, then we get

$$\frac{x^{m_2} - 1}{x - 1} = \frac{x^{m_1} - 1}{x - 1} + y^{n_1},$$

and x divides y, which is a contradiction. This proves (16).

Let $\delta = \gcd(x-1, y-1)$ and put $a = (y-1)/\delta$, $b = (x-1)/\delta, c = (y-x)/\delta$. Let s denote the smallest integer ≥ 1 such that

$$x^s \equiv 1 (\bmod\ by^{n_1}).$$

Then, for $j = 1$, 2, we have

$$ax^{m_{j+1}} - by^{n_{j+1}} = c \tag{17}$$

and

$$ax^{m_j} - by^{n_j} = c. \tag{18}$$

Therefore

$$x^{m_{j+1} - m_j} \equiv 1 (\bmod\ by^{n_j}), \tag{19}$$

thus $m_3 - m_2$ and $m_2 - m_1$ are multiples of s.

Further, (17) and (18) with $j = 1$ yield

$$1 = \frac{ax^{m_1}(1 - x^{m_2 - m_1})}{by^{n_1}} + y^{n_2 - n_1},$$

from which we deduce that y and $(1 - x^{m_2 - m_1})/(by^{n_1})$ are coprime. By the definition of s, we deduce that

$$\gcd\left(y, \frac{x^s - 1}{by^{n_1}}\right) = 1. \tag{20}$$

Write $m_3 - m_2 = su$, with $u \geq 1$ integer. By (19), we see that by^{n_2} divides $x^{su} - 1$, thus $y^{n_2 - n_1}$ divides

$$(1 + x^s + \ldots + x^{s(u-1)}) \frac{x^s - 1}{by^{n_1}}. \tag{21}$$

¿From (20) and (21), we then obtain that

$$1 + x^s + \ldots + x^{s(u-1)} \equiv 0 \ (\bmod\ y^{n_2 - n_1}). \tag{22}$$

Let p be a prime and assume that $p^\alpha \| y$. By the definition of s, there exists an integer $\lambda \geq 1$ such that $x^s = 1 + \lambda p^{\alpha n_1}$. We infer from (22) that $p^{\alpha(n_2-n_1)}$ divides $\big((1+\lambda p^{\alpha n_1})^u - 1\big)/(\lambda p^{\alpha n_1}) = u + u(u-1)\lambda p^{\alpha n_1}/2 + \ldots$ If p is odd or if $\alpha n_1 \geq 2$, we get that $p^{\alpha(n_2-n_1)}$ divides u. If $p = 2$, then x is odd and b must be even, thus λ is also even and, arguing as above, we see that $2^{\alpha(n_2-n_1)}$ divides u. Consequently, $y^{n_2-n_1}$ divides u and we get

$$m_3 - m_2 \geq y^{n_2-n_1}. \tag{23}$$

By (17) and (18), we have also

$$y^{n_{j+1}} - y^{n_j} \equiv 1 \,(\mathrm{mod}\ ax^{m_j})$$

for $j = 1, 2$ and we argue as above to conclude that

$$n_3 - n_2 \geq x^{m_2-m_1}. \tag{24}$$

Assume that (2) has two solutions (m_3, n_3) and (m_2, n_2) with $m_3 > m_2 > 1$. We apply our preceding results with $m_1 = n_1 = 1$ and (23) yields $m_3 > y^{n_2-1}$. By (16), we have $n_2 \geq 3$, thus

$$m_3 > y^2. \tag{25}$$

Combined with Lemma 2, (25) yields $y < 10^{11}$, as claimed.

Assume now that (2) has three solutions (m_4, n_4), (m_3, n_3) and (m_2, n_2) with $m_4 > m_3 > m_2 > 1$. By (23) we get

$$m_4 > y^{n_3-n_2},$$

and by (24), (2) and (16) we see that

$$n_3 - n_2 > x^{m_2-1} \geq y^{n_2-1}/2 \geq y^2/2.$$

Finally, we have

$$m_4 > y^{y^2/2},$$

which, combined with Lemma 2, yields $y \leq 6$.

Now, we solve (2) for any pair (x, y) with $2 \leq x < y \leq 6$. For $(2,3)$ and $(3,4)$, we observe that the equation $|3^u - 2^v| = 1$ has the only solution $(2,3)$ in integers $u > 1$ and $v > 1$. For $(x,y) = (2,4)$, $(2,6)$, $(3,6)$ and $(4,6)$ we conclude by arguing, respectively, modulo 4, 8, 9 and 4 that (2) has no solution other than the one given by $x = 2, y = 6, m = 3, n = 2$. For $(x,y) = (4,5)$, we have to solve the equation $4^u - 3 \cdot 5^v = 1$. Arguing modulo 5, we see that u is even, hence $(4^{u/2} - 1)(4^{u/2} + 1) = 3 \cdot 5^v$ and $u = 2$, $v = 1$. We deal with $(x,y) = (5,6)$ in a similar manner.

Now, for $(x,y) = (2,5)$, we are left with the equation $5^u + 3 = 2^v$. Modulo 8, we see that u is odd, hence $(5^{(u-1)/2}, v)$ is a solution of $5X^2 + 3 = 2^k$. By Theorem 1 of [5], this equation has only two solutions, namely $(X,k) = (1,3)$ and $(5,7)$.

Finally, it remains us to treat the pair $(3,5)$, hence the equation $2 \cdot 3^u - 5^v = 1$. Modulo 3, we see that v is odd, thus $(5^{(v-1)/2}, u)$ is a solution of $1 + 5X^2 = 2 \cdot 3^k$. By

Theorem 2 of [5], this equation has only the solution $(X, k) = (1, 1)$. This completes the proof of Theorem 1.

The proof of Theorem 2 is now easy. Indeed, set $m_1 = n_1 = 1$ and let (m_2, n_2) be a solution of (2). As noticed just below (19), s divides $m_2 - 1$. Thus $s < m_2$ which, together with (4), contradicts (3). Theorem 2 is then a straightforward consequence of Lemma 1. $\square$

Now, we give a proof of Remark 1. Let y be a given positive integer. For an integer $1 \leq x \leq y$ coprime with y, we denote by $\mathrm{ord}_y(x)$ the smallest integer $s \geq 1$ such that $x^s \equiv 1 \pmod{y}$. Further, we write

$$\sum_{d \mid \varphi(y), d < \sqrt{\varphi(y)}} \varphi(d) + \sum_{d \mid \varphi(y), d \geq \sqrt{\varphi(y)}} \varphi(d) = \varphi(y),$$

and we denote by A_1 (*resp.* A_2) the first (*resp.* the second) summation in the above formula. We observe that

$$A_1 \leq D(\varphi(y)) \sqrt{\varphi(y)},$$

where $D(n)$ is the number of divisors of the integer n. Thus, for y large enough, we have $A_1 \leq \varphi(y)^{2/3}$.

For any positive integer $x \leq y$ coprime to y, we have that $\mathrm{ord}_y(x)$ divides $\varphi(y)$, thus A_2 is exactly the number of integers x, $1 \leq x \leq y$, with $\gcd(x, y) = 1$, such that there exists an integer $d \geq \sqrt{\varphi(y)}$ with $d \mid \varphi(y)$ and $\mathrm{ord}_y(x) = d$. Such integers satisfy $\mathrm{ord}_y(x) \geq \min_{\ell \geq \sqrt{\varphi(y)}} \varphi(\ell)$. But the latter function is at least $\varphi(y)^{1/3}$ when y is large enough. Thus, the hypothesis of Theorem 2 is satisfied for large y by at least A_2 integers. Since $A_2 \geq \varphi(y) - \varphi(y)^{2/3}$, the remark following Theorem 2 is proved. $\square$

Proof of Theorem 3 :

Let $0 < \phi < 1$ and $\alpha > 1$. We denote by C_1, C_2 and C_3 effectively computable positive numbers depending only on α. Let (x, y, m, n) be a solution of (1) such that gcd $(m-1, n-1) = d \geq 3$ and $(m-1)/(n-1) \leq \alpha$. We write

$$m - 1 = dr, \quad n - 1 = ds$$

where r and s are positive integers. Now we infer from (1) that

$$x \leq 2y^{s/r}, \quad y \leq 2x^{r/s}.$$

By Lemma 4, we know that $d \leq C_1$. By Theorem 1, we may also suppose that $y \geq C_2$ with C_2 sufficiently large. Further, we re-write (1) as

$$x \, \frac{x^{dr}}{x-1} - y \, \frac{y^{ds}}{y-1} = \frac{1}{x-1} - \frac{1}{y-1},$$

which implies that

$$\left|\left(\frac{y(x-1)}{x(y-1)}\right)^{1/d}-\frac{x^r}{y^s}\right|<\frac{1}{y^{ds}}. \tag{26}$$

Now we apply Lemma 5 with $A=x(y-1), B=y(x-1), d=n, \sigma=s$ and $K=[2\alpha]+1$. We may suppose that $K<n$. We choose ϕ, depending only on α, suitably such that

$$1+\frac{1}{\alpha}+sK\left(1+\frac{1}{1-\phi}+\frac{2-\phi}{K(1-\phi)}\right)<\left(4\alpha+6+\frac{1}{\alpha}\right)s.$$

Finally, setting $\delta=1+(2-\phi)/K$ and $u_1=40^{d(K+1)(\delta+1-\phi)/(K\delta+1-\phi)}$, we observe that

$$A(A-B)^{-\delta}u_1^{-1}>C_3y^{1+\frac{1}{\alpha}}(y-x)^{-\delta}>C_3y^{1+\frac{1}{\alpha}-\delta}>1$$

if C_2 is sufficiently large. Hence, we conclude from Lemma 5 that the left hand side of (26) exceeds

$$y^{-s(4\alpha+6+1/\alpha)},$$

hence

$$d<4\alpha+6+\frac{1}{\alpha},$$

as claimed. □

Proof of Theorem 4 :

Set $d=\gcd(m-1,n-1)$. We apply Lemma 4 with $\alpha=(m-1)/(n-1)$. We conclude that $d^2\leq d(n-1)\leq 1114.5\,m$, whence $d\leq 33.4\,m^{1/2}$. □

Proof of Theorem 5 :

Let (x,y,m,n) be a solution of (1) with $x<y\leq x^a$ and $y\equiv 1 \pmod x$. Regarding (1) modulo x, we deduce that $n\equiv 1 \pmod x$, thus $n>x$. We set

$$\Lambda:=(y-1)x^m=(x-1)y^n-(x-y)$$

and we observe that $v_x(\Lambda)\geq m+1$. Further, if $x\neq 2$, then according as $2\|x$ or not we have

$$v_x(\Lambda)\leq v_{x/2}\left(y^n-\left(1-\frac{y-1}{x-1}\right)\right)$$

or

$$v_x(\Lambda)=v_x\left(y^n-\left(1-\frac{y-1}{x-1}\right)\right).$$

We check that the hypotheses of Lemma 6 are satisfied, and, thanks to that Lemma, we obtain for $x>2$ that

$$m+1\leq\frac{66.8}{(\log\frac{x}{2})^4}\left(\max\left\{\log\frac{n+1}{\log y}+\log\log\frac{x}{2}+0.64, 4\log\frac{x}{2}\right\}\right)^2(\log y)^2.$$

Further, $\log y\leq a\log x$ and $\log x/\log\frac{x}{2}\leq\log 6/\log 3\leq 1.631$, hence we get

$$m\leq\max\{2843a^2, 148(\log^2 m+0.64)a^2\}.$$

It follows that $m\leq 14000a^2(\log 3a)^2$, as claimed. □

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