On the approximation by algebraic numbers with bounded degree

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Abstract. In this expository paper, we give an overview of the different results related to Wirsing's problem on the approximation by algebraic numbers with bounded degree. We present a detailed proof of the best known estimate, due to Bernik & Tishchenko, and show how their idea can be used to give a new proof of a result of Wirsing.

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1. Introduction

Very recently, several authors, including Diaz, Laurent, Roy and Waldschmidt, have obtained new, important results on the approximation of a complex number by algebraic numbers (for references, see the paper of Waldschmidt in this volume), which complement pioneering work of Wirsing [7]. They, however, do not settle the conjecture of Schmidt, which claims that, for every integer $n \geq 1$ and for any real number θ not algebraic of degree less or equal to n, there exist a numerical constant $c = c(n, \theta)$ and infinitely many real algebraic numbers α with

$$0 < |\theta - \alpha| < c\,\mathrm{H}(\alpha)^{-n-1}.$$

Here, $\mathrm{H}(\alpha)$ denotes the naïve height of α (see definition below). For $n = 1$, the conjecture reduces to Dirichlet's theorem, and, for $n = 2$, it was proved by Davenport & Schmidt [3], with any constant c strictly larger than $160/9 \max\{1, |\theta|^2\}$.

Since this problem becomes of growing interest, it seems to be worth making a brief survey of the different attempts toward its resolution. In this note, we recall the two important results of Wirsing [7] and we show how Bernik & Tishchenko [1] succeeded in slightly improving one of them. Further, using their idea, we give a short proof of the other statement of Wirsing, which seems to be new.

2. Notation and results

In all the present work, we use the following notation. Let $P(X)$ be a non-zero polynomial with integral coefficient and write

$$P(X) = a_d\, X^d + \ldots + a_1\, X + a_0 = a_d\, \prod_{i=1}^{d} (X - \alpha_i),$$

where $a_d > 0$. We denote by $\mathrm{H}(P)$ its naïve height, defined by

$$\mathrm{H}(P) = \max_{0 \le j \le d} |a_j|.$$

If α is an algebraic number, its height, denoted by $\mathrm{H}(\alpha)$, is, by definition, the height of its minimal defining polynomial. Further, if α is a root of the polynomial $P(X)$, a result of Gelfond (see *e.g.* [5]) asserts that

$$2^{-d}\, \mathrm{H}(\alpha) \le \mathrm{H}(P). \tag{1}$$

Definition. *Let $n \ge 1$ be an integer and θ be a complex number. We denote by $w_n(\theta)$ the supremum of the real numbers w for which there exist a numerical constant $c = c(n, \theta)$ and infinitely many polynomial $P(X) \in \mathbb{Z}[X]$, with degree less or equal to n and satisfying*

$$0 < |P(\theta)| \le c\, \mathrm{H}(P)^{-w}.$$

Likewise, we denote by $w_n^*(\theta)$ the supremum of the real numbers w^* for which there exist a numerical constant $c^* = c^*(n, \theta)$ and infinitely many algebraic numbers α with degree less or equal to n and satisfying

$$0 < |\theta - \alpha| \le c^*\, \mathrm{H}(\alpha)^{-w^* - 1}.$$

Further, we denote by w_n (resp. w_n^*) the infimum of the set composed by the real numbers $w_n(\theta)$ (resp. by the real numbers $w_n^*(\theta)$), where θ runs through the transcendental real numbers.

With the above notation, Schmidt's conjecture becomes

Conjecture S. *For all integer $n \ge 1$ and for all real θ not algebraic of degree $\le n$, we have $w_n^*(\theta) \ge n$.*

As already noticed, Conjecture S remains unproved for $n \ge 3$, and even the following weaker statement, conjectured by Wirsing [7], is still open.

Conjecture W. *Let $\varepsilon > 0$ be arbitrarily small. For all integer $n \ge 1$ and for all real θ not algebraic of degree $\le n$, we have $w_n^*(\theta) \ge n - \varepsilon$.*

Observe that Conjecture W and Conjecture S are not equivalent in view of the constant c^* occurring in the definition of $w_n^*(\theta)$.

Let $n \ge 1$ be an integer and θ be a real number. It follows from Minkowski's theorem that $w_n(\theta) \ge n$, provided that θ is not algebraic of degree $\le n$. Further, it is easily seen that $w_n(\theta) \ge w_n^*(\theta)$ (*cf.* [4], page 74). However, it is a rather hard problem to give a non trivial lower bound for $w_n^*(\theta)$ in terms of $w_n(\theta)$. The first result in this setting was obtained by Wirsing in 1961 [7].

Theorem W. *Let $n \geq 1$ be an integer and θ be a real number, which is not algebraic of degree $\leq n$. Then we have*

$$w_n^*(\theta) \geq \frac{w_n(\theta)}{w_n(\theta) - n + 1} \tag{2}$$

and

$$w_n^* \geq \frac{n+2}{4} + \frac{\sqrt{n^2+4n-4}}{4}. \tag{3}$$

Remark : The main interest of estimate (2) is to establish that $w_n^*(\theta) = n$ whenever we know that $w_n(\theta)$ is exactly equal to n. Hence, it yields, together with Sprindžuk's proof of Mahler's conjecture (see *e.g.* [4] page 280), $w_n^*(\theta) = n$ for almost all (in the sense of Lebesgue measure) real numbers θ. Further, we have $w_n^*(\theta) = n$ for all real algebraic numbers θ of degree $\geq n+1$, in view of a result of Schmidt, deduced from his powerful Subspace Theorem (see *e.g.* [4] page 279).

Bombieri & Mueller [2] have developed a new approach for estimating $w_n^*(\theta)$ in terms of $w_n(\theta)$, but their result - which they do not state explicitely, since they are interested only in a particular aspect of this problem, namely the approximation of real algebraic numbers of degree $n+1$ by algebraic numbers of degree $\leq n$ - is, however, slightly inferior to (2).

Theorem BM. *Under the hypothesis of Theorem W, we have*

$$w_n^*(\theta) \geq \frac{n}{w_n(\theta) - n + 1}.$$

The only known improvement of Theorem W is due to Bernik & Tishchenko [1], who, thanks to a clever observation, obtained the following result.

Theorem BT. *Under the hypothesis of Theorem W, we have*

$$w_n^* \geq \frac{n}{4} + \frac{\sqrt{n^2+16n-8}}{4}.$$

The aim of the present work is to present the idea of Bernik & Tishchenko, and to show how it leads to a new proof of estimate (2) of Theorem W.

We should also mention that Tishchenko [6] has sharpened Theorem BT in the range $3 \leq n \leq 10$. However, his proofs seem to contain some unclear arguments.

3. Proofs

Let n be an integer greater or equal to 2. Let θ be a real number, which is not algebraic of degree less or equal to n. Assume, without loss of generality, that θ satisfy

$$0 < \theta < \frac{1}{n!}. \tag{4}$$

In all what follows, the notation $a \gg b$ means that there exists a real λ, depending only on θ and n, such that $a \geq \lambda\, b$.

Let A and H_A be real numbers such that, for all algebraic number α of degree $\leq n$ and with height $\geq H_A$, we have

$$|\theta - \alpha| \geq \mathrm{H}(\alpha)^{-A}. \tag{5}$$

It follows from Dirichlet's Theorem that $A \geq 2$, independently of n. Further, we point out that $w_n^*(\theta) \geq A - 1$.

The next lemma is classical and can be found *e.g.* in [2] (Lemma 1).

Lemma. *Let $P(X)$ be a non-zero polynomial of degree $k \geq 1$ with complex coefficients. Let $\xi \in \mathbf{C}$ such that $P'(\xi) \neq 0$. Then the closest root β of P to ξ satisfies*

$$|\xi - \beta| \leq k\, \frac{|P(\xi)|}{|P'(\xi)|}. \tag{6}$$

Let $\varepsilon > 0$ and $H > 0$ be real numbers. By Minkowski's theorem, there exists a non-zero polynomial $P(X) := a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$, with integer coefficients, such that

$$|P(\theta)| \leq H^{-n-\varepsilon}, \quad |a_1| \leq H^{1+\varepsilon},$$

$$|a_2|, \ldots, |a_n| \leq H.$$

Decreasing H if necessary, we observe that one of the inequalities concerning the $|a_i|$ $(i \geq 1)$ is in fact an equality. Further, we have $\mathrm{H}(P) > |a_0|$ by (4).

The idea of Bernik & Tishchenko is to consider the quantity $|P'(\theta)|$: when $|P'(\theta)|$ is not too small, the Lemma asserts that P has a root very close to θ.

If $\mathrm{H}(P) = |a_1|$, we deduce from the hypothesis (4) that $|P'(\theta)| \geq \mathrm{H}(P)/2$ and, by (5) and (6), we get

$$|P(\theta)| \gg |\theta - \alpha|\, |P'(\theta)| \gg \mathrm{H}(P)^{1-A}, \tag{7}$$

where α denotes the root of P closest to θ. Further, we have

$$|P(\theta)| \leq \mathrm{H}(P)^{-(n+\varepsilon)/(1+\varepsilon)},$$

which, combined with (7), yields

$$\varepsilon \geq \frac{n+1-A}{A-2}.$$

Thus, if ε is sufficiently small, then the height of the polynomial P obtained by applying Minkowski's theorem is strictly less than $|a_1|$ and satisfies $\mathrm{H}(P) = \max\{|a_2|, \ldots, |a_n|\}$. In other words, under the hypothesis

$$\varepsilon < \frac{n+1-A}{A-2}, \tag{8}$$

we have constructed a non-zero polynomial $P(X) \in \mathbb{Z}[X]$, satisfying $\mathrm{H}(P) \leq H$ and $|P(\theta)| \leq H^{-n-\varepsilon}$, hence such that

$$|P(\theta)| \leq \mathrm{H}(P)^{-n-\varepsilon}. \tag{9}$$

We will exploit this remark in two different ways.

3.1. Proof of Theorem BT

We follow the same lines as Wirsing [7], see also the Lecture Notes of Schmidt [4], pages 268–272. Applying Minkowski's theorem, we build sequences $(P_k)_{k\geq 1}$ and $(Q_k)_{k\geq 1}$ of non-zero polynomials with integer coefficients and degree less or equal to n, such that the height of P_k tends to infinity with k,

$$|P_k(\theta)| \leq \mathrm{H}(P_k)^{-n}, \quad \mathrm{H}(Q_k) \ll \mathrm{H}(P_k), \quad |Q_k(\theta)| \ll \mathrm{H}(P_k)^{-n} \quad (k \geq 1) \tag{10}$$

and

$$P_k \text{ and } Q_k \text{ are relatively prime } (k \geq 1).$$

Then, we estimate the resultant of P_k and Q_k, which is a non-zero integer, in terms of the $|\theta - \alpha_k^{(j)}|$, where the $\alpha_k^{(j)}$'s denote the roots of the polynomial $P_k\, Q_k$. An argument due to Wirsing allows us to assert that five cases may occur (see [4], page 269, they are labelled from (i) to (v)) and that, in the worst cases (*i.e.* cases (iv) and (v)), there exists an index j such that

$$|\theta - \alpha_k^{(j)}|^2 \ll |P_k(\theta)|^2 \cdot |Q_k(\theta)| \cdot \mathrm{H}(P_k)^{n-2} \cdot \mathrm{H}(Q_k)^{n-1} \tag{11}$$

or

$$|\theta - \alpha_k^{(j)}|^2 \ll |P_k(\theta)| \cdot |Q_k(\theta)|^2 \cdot \mathrm{H}(P_k)^{n-1} \cdot \mathrm{H}(Q_k)^{n-2}. \tag{12}$$

The estimate $w_n^*(\theta) \geq (n+1)/2$ then follows from (10). More precisely, this approach allows Wirsing to prove that $w_n^*(\theta) \geq (w_n(\theta)+1)/2$, which, combined with (2), yields (3).

Let $\varepsilon > 0$ satisfying (8). Using the above reasoning, we can do again the construction of Wirsing, but, by (9), the polynomials P_k and Q_k moreover satisfy

$$|P_k(\theta)| \leq \mathrm{H}(P_k)^{-n-\varepsilon}, \quad |Q_k(\theta)| \ll \mathrm{H}(P_k)^{-n-\varepsilon} \ (k \geq 1), \tag{13}$$

and (11) and (12) are, in view of (1), (5) and (13), replaced by

$$\mathrm{H}(\alpha_k^{(j)})^{-2A} \ll |\theta - \alpha_k^{(j)}|^2 \ll \mathrm{H}(\alpha_k^{(j)})^{-3-n-3\varepsilon}. \tag{14}$$

For completeness, we mention that, as in the preceding setup, cases (i) and (ii) lead to better estimates and case (iii) cannot happen if H is sufficiently large.

Letting ε tend to its limit value $(n+1-A)/(A-2)$, it follows from (14) that A satisfies the inequation

$$2A^2 - (n+4)A + (3-n) \geq 0,$$

hence

$$A \geq \frac{n+4}{4} + \frac{\sqrt{n^2+16n-8}}{4},$$

as claimed. □

3.2. New proof of the lower bound (2)

Let $\delta > 0$ be a real number. Since our method allows us to construct infinitely many polynomials satisfying (9), we have, by definition of $w_n(\theta)$,

$$n + \varepsilon \leq w_n(\theta) + \delta.$$

Letting δ tend to 0 and ε to its limit value $(n+1-A)/(A-2)$, we get

$$w_n(\theta) \geq n + \frac{n+1-A}{A-2},$$

hence

$$A \geq \frac{2\, w_n(\theta) + 1 - n}{w_n(\theta) + 1 - n},$$

as asserted. □

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References

[1] Bernik, V. and Tishchenko, K. I., Integral polynomials with an overfall of the coefficient values and Wirsing's problem. Dokl. Akad. Nauk. Belarusi 37 (1993), 9–11.

[2] Bombieri, E. and Mueller J., Remarks on the approximation to an algebraic number by algebraic numbers. Mich. Math. J. **33** (1986), 83–93.

[3] Davenport, H. and Schmidt, W. M., Approximation to real numbers by quadratic irrationals. Acta Arith. 13 (1967), 169–176.

[4] Schmidt, W. M., Diophantine Approximation. Lecture Notes in Math. 785, Springer-Verlag, Berlin, 1980.

[5] Schneider T., Introduction aux nombres transcendants. Gauthier-Villars, Paris, 1959.

[6] Tishchenko, K. I., Systems of linearly independent polynomials with height gaps and Wirsing problem. Vestsi Akad. Nauk. Belarusi. Ser. Fis.–Mat. Nauk. 4 (1996), 16–22.

[7] Wirsing E., Approximation mit algebraischen Zahlen beschränkten Grades. J. reine angew. Math. **206** (1961), 67–77.