# The Irrationality Exponents of Computable Numbers


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The irrationality exponent $a$ of a real number $x$ is the supremum of the set of real numbers $z$ for which the inequality

$$0 < |x - p/q| < 1/q^z$$

is satisfied by an infinite number of integer pairs $(p, q)$ with $q > 0$. Rational numbers have irrationality exponent equal to 1, irrational numbers have it greater than or equal to 2. The Thue–Siegel–Roth theorem states that the irrationality exponent of every irrational algebraic number is equal to 2. Almost all real numbers (with respect to the Lebesgue measure) have irrationality exponent equal to 2. The Liouville numbers are precisely those numbers having infinite irrationality exponent.

For any real number $a$ greater than or equal to 2, Jarník (1931) used the theory of continued fractions to give an explicit construction, relative to $a$, of a real number $x_a$ such that the irrationality exponent of $x_a$ is equal to $a$. For $a = 2$, we can take $x_2 = \sqrt{2}$. For $a > 2$, we construct inductively the sequence of partial quotients of $x_a = [0; a_1, a_2, \ldots]$.

For $n \geq 1$, set $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$. Take $a_1 = 2$ and $a_{n+1} = \lfloor q_n^{a-2} \rfloor$, for $n \geq 1$, where $\lfloor \cdot \rfloor$ denotes the integer part function. Then, the theory of continued fractions (see Schmidt, 1980) directly gives that the irrationality exponent of $x_a$ is equal to $a$.

The theory of computability defines a computable function from non-negative integers to non-negative integers as one which can be effectively calculated by some algorithm. The definition extends to functions from one countable set to another, by fixing enumerations of those sets. A real number $x$ is computable if there is a base and a computable function that gives the digit at each position of the expansion of $x$ in that base. Equivalently, a real number is computable if there is a computable sequence of rational numbers $(r_j)_{j\geq 0}$ such that $|x - r_j| < 2^{-j}$ for $j \geq 0$.

The construction cited above shows that for any computable real number $a$ there is a computable real number $x_a$ whose irrationality exponent is equal to $a$. What of the inverse question? Are there computable numbers with non-computable irrationality exponents? Theorem 1 gives a characterization of the irrationality exponents of computable real numbers.

**Theorem 1.** *A real number $a$ greater than or equal to 2 is the irrationality exponent of some computable real number if and only if $a$ is the upper limit of a computable sequence of rational numbers.*

A real number $x$ is said to be right-computably enumerable (see Soare, 1969) if and only if the set of rational numbers $r$ such that $r > x$ is computably enumerable, which is to say that $x$ is right-computably enumerable if and only if there is an algorithm to output a listing $(p_n, q_n)_{n\geq 0}$ of all integer pairs whose quotients are greater than $x$. By only enumerating rational numbers smaller than any previously enumerated, one can show that $x$ is right-computably enumerable if and only if there is a computable strictly decreasing sequence of rational numbers with limit $x$. The set of left-computably enumerable real numbers is defined similarly but with non-decreasing sequences.

The computable real numbers are exactly those that are both, right and left, computably enumerable. There are numbers that are just left-computably enumerable or just right-computably enumerable. For example, if $A$ is a computably enumerable but not computable subset of the non-negative integers, such as could be obtained from the Halting Problem (see Soare, 1969), then the real number $x_A = \sum_{n=1}^{\infty} a_n 2^{-n}$, where for each $n \geq 1$, $a_n = 1$ if $n \in A$ and $a_n = 0$ otherwise, is left-computably enumerable but not computable.

The theory of computability extends to algorithms that use external data sets in the course of their computations. For historical reasons, these external data sets are called oracles. Here, an oracle $A$ is simply an infinite set of non-negative integers and algorithms relative to $A$ are allowed to use information about membership in $A$. In practical terms, any method to examine data and extract statistical information, such as mean or median, provides an example of an algorithm relative to an oracle. If a function can be calculated by an algorithm relative to oracle $A$ we say that it is computable relative to $A$. Similarly, if a set can be calculated by an algorithm relative to oracle $A$ we say it is computable relative to $A$. For example, the function mapping $n$ to the density of a given set $A$ below $n$, that is, the number of elements of $A$ less than $n$ divided by $n$, is computable relative to $A$. And the set $\{2n : n \in A\}$ is computable relative to $A$. In contrast, there are properties of a set which are not necessarily computable relative to it. For example, there is a set $A$ of integers such that the set $\big\{n : (\exists k \in \mathbb{Z})\big[kn \in A\big]\big\}$ is computably enumerable relative to $A$ but not computable relative to $A$, because computations are finite and can only make use of finitely many facts about $A$. In general, no finite amount of information about elements in $A$ is sufficient to draw the infinite conclusion that $A$ omits every multiple of $n$.

There is a distinguished oracle in the theory of computability that encodes all the truths of first-order Peano arithmetic that can be expressed with just one block of existential quantifiers. It is called $0'$. There are many possible equivalent presentations of oracle $0'$ (see Soare, 1987, Chapter IV). For example, the assertion that a polynomial $p$ in several variables with integer coefficients has an integer-valued solution is expressible with just one block of existential quantifiers; written formally it would appear as $(\exists z_1 \ldots \exists z_k)[p(z_1, \ldots, z_k) = 0]$, where $(\exists z_1 \ldots \exists z_k)$ is the leading block of existential quantifiers.[1] By a theorem of Matiyasevich, we can fix a computable enumeration of all the polynomials in several variables with integer coefficients and we can take oracle $0'$ to be exactly the set of positive integers $n$ such that the $n$th polynomial in this enumeration has an integer-valued solution.

For a real number $x$, if the set of rational numbers $r$ such that $r > x$ is computably enumerable relative to $0'$ then we say that $x$ is right-computably enumerable relative to $0'$. The real numbers computable relative to $0'$ are exactly those that are both, right and left,

[1] There are assertions which are not expressible with just one block of existential quantifiers. For example, the assertion that there are arbitrarily large integers $m$ such that $m$ and $m + 2$ are prime, requires a universal quantifier: $(\forall n)(\exists m > n)[\ldots]$).

computably enumerable relative to $0'$. In Lemma 4, we give other equivalences. In particular, $x$ is right-computably enumerable relative to $0'$ if and only if it is the upper limit of a computable sequence of rational numbers, which is the condition cited in Theorem 1.

Now, consider the case of the irrationality exponent of a computable real number $x$. If $x$ is rational, its irrationality exponent is equal to 1. If $x$ is irrational algebraic, its irrationality exponent is equal to 2. In these cases, the irrationality exponents are clearly right-computably enumerable relative to $0'$. Now, suppose that $x$ is not algebraic. Then, for every pair of rational numbers $p/q$ and $b$, the quantity $|x-p/q|$ is not equal to $1/q^b$. Consequently, it is computable to determine whether $|x-p/q|$ is less than $1/q^b$ by computing both quantities to sufficient precision to determine which is larger. This implies that the set of rational numbers $b$ for which there are only finitely many rational numbers $p/q$ such that $|x-p/q| < 1/q^b$ is computably enumerable relative $0'$: Since $x$ is computable, given a rational number $b$ and an integer $k$, the existential statement "there are integers $p$ and $q$ such that $q$ is greater than $k$ and $|x-p/q| < 1/q^b$" constitutes a single query to $0'$. Then, we can examine all pairs $b$ and $k$ (fix one enumeration) and list $b$ upon discovery of some $k$ for which this query to $0'$ is answered negatively; that is, there are no $p$ and $q$ such that $q$ is greater than $k$ and $|x-p/q| < 1/q^b$. It follows that, if the irrationality exponent of $x$ is finite, then it is right-computably enumerable in $0'$.

Thus, to complete the proof of Theorem 1, we only need to show that for every real number $a$ greater than 2, if $a$ is right-computably enumerable in $0'$, then there is a computable real number $x$ such that $x$ has irrationality exponent equal to $a$. Since there are numbers that are right-computably enumerable in $0'$ that are not computable, the proof of this direction of the theorem, the existence direction, necessarily involves approximations of sets and real numbers which cannot be directly computed. It also immediately implies the following corollary.

**Corollary 2.** *There are computable real numbers whose irrationality exponent is not computable.*

Similarly and more generally, there are computable real numbers whose irrationality exponents have no monotonous approximations by a computable sequence of rational numbers.

We give two proofs of Theorem 1. The first one is more combinatorial and is based on a construction given by Bugeaud (2008). The second one, more geometric and based on a construction given by Jarník (1929), also yields the following corollary.

**Corollary 3.** *For each real number $a$ greater than or equal to 2 and right-computably enumerable relative to $0'$, there is a computable Cantor-like construction whose limit set contains uncountably many real numbers with irrationality exponent equal to $a$, and countably many of them are computable.*

In fact, the natural measure on this Cantor set concentrates on the set of numbers with the prescribed irrationality exponent.

The next lemma states three equivalent, and useful, formulations of the property of right-computable enumerability relative to $0'$.

**Lemma 4.** *For any real number $a$, the following properties are equivalent.*

1. *There is a computable sequence $(a_j)_{j\geq 0}$ of rational numbers such that* $\limsup_{j\geq 0} a_j = a$.

2. *There is strictly decreasing sequence $(b_j)_{j\geq 0}$ of rational numbers, that is computable relative to $0'$ and has limit equal to $a$.*

3. *There is a computable doubly-indexed sequence $(a(j,s))_{j,s\geq 0}$ of rational numbers satisfying that, for each $j \geq 0$, the sequence $(a(j,s))_{s\geq 0}$ is eventually constant and the sequence $\left(\lim_{s\to\infty} a(j,s)\right)_{j\geq 0}$ is strictly decreasing with limit $a$. Without loss of generality, the following can be assumed:*

   *(a) The number $a(0,0)$ is an integer greater than or equal to $a(j,s)$, for $j \geq 0$ and $s \geq 0$.*

   *(b) For each $j \geq 0$, $a(j,0) = a(0,0)$.*

   *(c) For each $s \geq 0$, the sequence $(a(j,s))_{j\geq 0}$ is strictly decreasing.*

*Proof.* $(1 \Rightarrow 2)$ If $a$ is rational, then the sequence $(b_j)_{j\geq 0} = (a+1/2^j)_{j\geq 0}$ verifies Condition 2. Assume that $a$ is not rational and that $(a_j)_{j\geq 0}$ is a computable sequence of rational numbers with limit supremum equal to $a$. Let $M$ be an integer strictly greater than each of the values $a_j$, for $j \geq 0$ (this value $M$ may not be found computably in $0'$, but it does exist). Define $b_0 = M$ and $j_0 = 0$.

Let $(c_k)_{k\geq 0}$ be a computable enumeration of the rational numbers. For $n > 0$, let $j_n$ be the least $j > j_{n-1}$ for which there is a $k < j$ such that $b_{n-1}$ is greater than $c_k$ and $c_k$ is greater than the supremum of $(a_j)_{j\geq j_n}$. Let $b_n = c_k$ for the least such $k$.

Since $a$ is irrational, $j_n$ is well-defined and, since $j_n$ and $b_n$ are the least integers satisfying "for-all" properties, they can be computed uniformly in $0'$. Thus, $(b_n)_{n\geq 0}$ is computable relative to $0'$.

By construction, $(b_n)_{n\geq 0}$ is strictly decreasing and all of its elements are greater than $a$. Let $b$ be the limit of $(b_n)_{n\geq 0}$. For a contradiction, suppose that $b$ is greater than $a$ and consider $c_{k^*}$ for $k^*$ the least index of a rational number strictly between $b$ and $a$. Let $n^*$ be greater than $k^*$ and also so large that $c_{k^*}$ is greater than the supremum of $(a_j)_{j\geq j_{n^*}}$. For every $n$ greater than or equal to $n^*$, $c_{k^*}$ satisfies the for-all property used to define $b_n$. But then $(b_n)_{n\geq n^*}$ must be contained in $\{c_k : k < k^*\}$, a contradiction.

$(2 \Rightarrow 3)$ Assume $(b_j)_{j\geq 0}$ is computable relative to $0'$ with limit $a$. Let $b_j[s]$ the computable approximation of the value $b_j$ such that the questions to the oracle $0'$ are answered using the set of numbers less than $s$ that are enumerated into $0'$ by computations of length less than $s$, if that computation produces a value, and let $b_j[s]$ be 2, otherwise. It follows that, for each $j \geq 0$, there is an integer $s_j$ such that for every $s \geq s_j$, $b_j[s] = b_j$.

Let $M$ be an integer greater than $b_0$. For each $s \geq 0$, we define the sequence $(a(j,s))_{j\geq 0}$. We let $a(0,s) = M$. For $j > 0$, we let

$$a(j,s) = b_j[s]$$

provided that for all $k < j$ it holds $b_k[s] > b_{k+1}[s] > 2$. If this condition fails, then we let

$$a(j,s) = (a(j-1,s)+2)/2,$$

the midpoint between $a(j-1,s)$ and 2. By construction, $a(j,s)$ satisfies conditions $(a)$, $(b)$ and $(c)$. Set $\tilde{s}_j = \max\{s_k : k \leq j\}$. Then, for each $k \leq j$, we have $b_k[s] = b_k$ for every $s \geq \tilde{s}_j$. By hypothesis, $(b_j)_{j\geq 0}$ is strictly decreasing. Then, for each $s \geq \tilde{s}_j$, we deduce that $a(j,s) = b_j$. This ensures that the sequence $(a(j,s))_{s\geq 0}$ is eventually constant and that $(\lim_{s\to\infty} a(j,s))_{j\geq 0}$ is strictly decreasing with limit $a$.

($3 \Rightarrow 1$) Assume $(a(j,s))_{j,s\geq 0}$ is a sequence of rational numbers such that for each $j \geq 0$ the sequence $(a(j,s))_{s\geq 0}$ is eventually constant and the sequence $\left(\lim_{s\to\infty} a(j,s)\right)_{j\geq 0}$ is strictly decreasing with limit $a$. Let $\ell(s)$ be the computable function defined by $\ell(0) = 0$ and, for $s \geq 1$, let $\ell(s)$ be the least $j$ less than or equal to $s-1$ such that $a(j,s-1) \neq a(j,s)$, if there is such, and let $\ell(s)$ be $s-1$ otherwise. We define the computable sequence $(a_s)_{s\geq 0}$ by

$$a_s = a(\ell(s), s).$$

By assumption on $(a(j,s))_{j,s\geq 0}$, we deduce that $\lim_{s\to\infty} \ell(s) = \infty$. Further, there is an arbitrarily large $t$ with $a_t = a(\ell(t),t) = \lim_{s\to\infty} a(\ell(t),s)$. Thus, $(a_s)_{s\geq 0}$ and $\left(\lim_{s\to\infty} a(j,s)\right)_{j\geq 0}$ have a common subsequence. Since $\left(\lim_{s\to\infty} a(j,s)\right)_{j\geq 0}$ is strictly decreasing with limit $a$, we get that $\limsup_{s\geq 0} a_s$ is greater than equal to $a$. Dually, given any number $b$ greater than $a$, we can fix $j$ so that $\lim_{s\to\infty} a(j,s) < b$ and fix $t$ so that for all $s > t$, $\ell(s) > j$. Then, for all $s > t$,

$$a_s = a(\ell(s), s) < a(j,s) = \lim_{s\to\infty} a(j,s) < b$$

and so

$$\limsup_{s\geq 0} a_s < b,$$

as required. □

# 1 First proof of Theorem 1

*First proof of Theorem 1.* Let $b \geq 2$ be an integer. Recently, Bugeaud (2008) constructed a class $\mathcal{C}$ of real numbers whose irrationality exponent can be read off from their base-$b$ expansion. The class $\mathcal{C}$ includes the real numbers of the form

$$\xi_{\mathbf{n}} = \sum_{j\geq 1} b^{-n_j},$$

for a sequence $\mathbf{n} = (n_j)_{j\geq 1}$ of positive integers satisfying $n_{j+1}/n_j \geq 2$ for every large integer $j$. To obtain good rational approximations to $\xi_{\mathbf{n}}$, we simply truncate the above sum. Thus, we set

$$\xi_{\mathbf{n},J} = \sum_{j=1}^{J} b^{-n_j} = \frac{p_J}{b^{n_J}}, \quad J \geq 1.$$

It then follows from

$$\left|\xi_{\mathbf{n}} - \frac{p_J}{b^{n_J}}\right| < \frac{2}{(b^{n_J})^{n_{J+1}/n_J}}, \quad J \geq 1,$$

that the irrationality exponent $\mu(\xi_{\mathbf{n}})$ of $\xi_{\mathbf{n}}$ satisfies

$$\mu(\xi_{\mathbf{n}}) \geq \limsup_{j\to\infty} \frac{n_{j+1}}{n_j}\ .$$

Shallit (1982) proved that the continued fraction expansion of some rational translate of any such $\xi_{\mathbf{n}}$ can be given explicitly, and Bugeaud (2008) proved that its irrationality exponent is given precisely by

$$\mu(\xi_{\mathbf{n}}) = \limsup_{j\to\infty} \frac{n_{j+1}}{n_j} ,$$

and hence can be read off from its expansion in base $b$. This means that the denominators of the best rational approximations to $\xi_{\mathbf{n}}$ are (except finitely many of them) powers of $b$.

Consequently, given a real number $a \geq 2$ for which there is a computable sequence $(a_j)_{j\geq 0}$ of rational numbers such that $\limsup_{j\to\infty} a_j = a$, it is sufficient to construct a computable strictly increasing sequence $\mathbf{n} = (n_j)_{j\geq 1}$ of positive integers satisfying $n_{j+1}/n_j \geq 2$ and

$$\limsup_{j\to\infty} \frac{n_{j+1}}{n_j} = a,$$

which we do as follows. By substituting 2 for any smaller values, we may assume that each $a_j$ is greater than or equal to 2. We construct the desired sequence $\mathbf{n}$ by induction as follows. Let $n_1 = 2$. Given $n_1, \ldots, n_j$, let $n_{j+1}$ be the least $n$ such that $n/n_j \geq a_{j+1}$. By construction, for all $j$, $n_{j+1}/n_j \geq 2$. Consequently, $n_j \geq 2^j$. Since $(n_{j+1} - 1)/n_j < a_{j+1}$, $n_{j+1}/n_j - a_{j+1}$ is less than or equal to $1/2^j$. It follows directly that $\limsup_{j\to\infty} n_{j+1}/n_j$ is equal to $\limsup_{j\to\infty} a_j = a$. $\square$

## 2 Second proof of Theorem 1

For each real number $a$ greater than 2, Jarník (1931) gave a Cantor-like construction of a fractal subset $K$ of $[0, 1]$ such that the uniform measure $\nu$ on $K$ has the property that the set of real numbers with irrationality exponent equal to $a$ has $\nu$-measure equal to 1. Thus, for all real numbers $b$ greater than $a$, the set of real numbers in $K$ with irrationality exponent equal to $b$ has $\nu$-measure equal to 0.

**Lemma 5** (Jarník (1931))**.** *For every real number $a$ greater than or equal to 2, the set of numbers with irrationality exponent equal to $a$ has Hausdorff dimension $2/a$.*

We note that Jarník (1929) and Besicovitch (1934) independently established that the set of real numbers with irrationality exponent greater than or equal to $a$ has Hausdorff dimension $2/a$. Actually, Lemma 5 is not explicitly stated in Jarník (1931); however, it is an immediate consequence of the results of that paper.

In the following and throughout this text, we denote by $|I|$ the length of the interval $I$.

**Lemma 6** (Mass Distribution Principle)**.** *Let $\nu$ be a finite measure, $d$ a positive real number and $X$ a set with Hausdorff dimension less than $d$. Suppose that there is a positive real number $C$ such that for every interval $I$, $\nu(I) < C\ |I|^d$. Then we have $\nu(X) = 0$.*

**Lemma 7.** *Let $\mathbf{a}$ be a strictly decreasing sequence of rational numbers greater than 2 which is computable relative to $0'$ and has limit equal to $a$, greater than 2. There is a Cantor-like construction of a fractal $K$, with uniform measure $\nu$, and a function $C$, computable relative to $0'$, from $\mathbb{Q} \cap (0, 2/a)$ to $\mathbb{Q}$ such that for each rational number $d < 2/a$, for every interval $I$, $\nu(I) \leq C(d)|I|^d$.*

*Proof.* We follow the proof of Jarník's Theorem as presented in Falconer (2003). Let $\mathbf{a}$ be $(a_j)_{j\geq 0}$. Fix a computable doubly-indexed sequence $(a(j,s))_{j,s\geq 0}$ of rational numbers such that for all $j$, $\lim_{s\to\infty} a(j,s) = a_j$. Without loss of generality, we assume that for every $s$, the sequence $a(j,s)_{j\geq 0}$ is strictly decreasing, $a(0,0)$ is an integer and for all $s$, $a(0,s) = a(0,0)$. Further, we fix a rational number $\beta$ greater than 2 and assume that $\beta$ is a lower bound for the numbers $a(j,s)$.

We fix some notation to be applied in the course of our eventual construction. For a positive integer $q$ and a real number $b$ greater than $\beta$, let

$$G_q(b) = \left\{ x \in \left(\frac{1}{q^b}, 1 - \frac{1}{q^b}\right) : \exists p \in \mathbb{Z}, \left|\frac{p}{q} - x\right| < \frac{1}{q^b} \right\}.$$

For $M$ a sufficiently large positive integer according to $\beta$, and $p_1$ and $p_2$ primes such that $M < p_1 < p_2 < 2M$, the sets $G_{p_1}(b)$ and $G_{p_2}(b)$ are disjoint and the distance between any point in $G_{p_1}(b)$ and any point in $G_{p_2}(b)$ is greater than or equal to

$$\frac{1}{4M^2} - \frac{2}{M^b} \geq \frac{1}{8M^2}.$$

For such $M$ the set

$$H_M(b) = \bigcup_{\substack{p \text{ prime} \\ M<p<2M}} G_p(b)$$

is the disjoint union of the intervals composing the sets $G_p(b)$, so $H_M(b)$ is made up of intervals of length less than or equal to $2/M^b$ which are separated by gaps of length at least $1/(8M^2)$. If $I \subseteq [0,1]$ is any interval with $3/|I| < M < p < 2M$ then at least $p|I|/3 > M|I|/3$ of the intervals in $G_p(b)$ are completely contained in $I$. By the prime number theorem, for sufficiently large $M$ the number of primes between $M$ and $2M$ is at least $M/(2\log M)$. Thus, for such $M$ and $I$, at least $M^2|I|/(6\log M)$ intervals of $H_M(b)$ are contained in $I$. With $M_1$ sufficiently large as above and larger than $3 \times 2^{a(0,0)}$, let

$$M_k = M_{k-1}^k = M_1^{k!}, \quad (k \geq 1).$$

For a positive integer $k$, let $j$ be the least integer less than $k$ such that $a(j+1,k) \neq a(j+1,k-1)$, if such exists, and let $j$ be $k-1$, otherwise. That is, $j$ is the greatest index less than $k$ such that the approximation to $\mathbf{a}$ remains unchanged at positions less than or equal to $j$ from step $k-1$ to step $k$. Let

$$b_k = a(j,k).$$

Let $E_0 = [0,1]$ and for $k = 1, 2, \ldots$ let $E_k$ consist of those intervals of $H_{M_k}(b_k)$ that are completely contained in $E_{k-1}$. By discarding intervals if necessary, we arrange that all intervals in $E_{k-1}$ are split into the same number of intervals in $E_k$. The intervals of $E_k$ are of length at least $1/(2M_k)^{b_k}$ and are separated by gaps of length at least

$$g_k = \frac{1}{8M_k^2}.$$

Thus, each interval of $E_{k-1}$ contains at least $m_k$ intervals of $E_k$ where $m_1 = 1$ and

$$m_k = \frac{M_k^2}{(2M_{k-1})^{b_k} 6\log M_k} \geq \frac{cM_k^2}{(M_{k-1})^{b_k}\log M_k},$$

if $k \geq 2$ and $c = 1/(2^{a(0,0)}6)$. Let

$$K = \bigcap_{k \geq 1} E_k.$$

Define a mass distribution $\nu$ on $K$ by assigning a mass of $1/(m_1 \times \ldots \times m_k)$ to each of the $m_1 \times \ldots \times m_k$ many $k$-level intervals. Let $S$ be a subinterval of $[0,1]$. For a lighter notation we write $2\epsilon$ to denote the length $|S|$ of $S$. We estimate $\nu(S)$. Let $k$ be the integer such that $g_k \leq 2\epsilon < g_{k-1}$. The number of $k$-level intervals that intersect $S$ is

- at most $m_k$, since $S$ intersects at most one $(k-1)$-level interval.
- at most $1 + 2\epsilon/g_k \leq 4\epsilon/g_k$ since the $k$-level intervals have gaps of at least $g_k$ between them.

Each $k$-level interval has measure $1/(m_1 \times \ldots \times m_k)$ so that

$$\nu(S) \ \leq \ \frac{\min(4\epsilon/g_k, m_k)}{m_1 \times \ldots \times m_k} \ \leq \ \frac{(4\epsilon/g_k)^s \ m_k^{1-s}}{m_1 \times \ldots \times m_k},$$

for every $s$ between 0 and 1. Hence,

$$\nu(S) \ \leq \ \frac{2^s}{(m_1 \times \ldots \times m_{k-1}) \ m_k^s g_k^s}(2\epsilon)^s.$$

Thus, $\nu(S)$ is at most

$$1\frac{M_1^{b_2} \log M_2}{cM_2^2} \ \frac{M_2^{b_3} \log M_3}{cM_3^2} \cdots \frac{M_{k-2}^{b_{k-1}} \log M_{k-1}}{cM_{k-1}^2} \ \frac{2^s}{m_k^s g_k^s} \ (2\epsilon)^s =$$
$$\frac{M_1^{b_2} \log M_2}{cM_2^2} \ \frac{M_2^{b_3} \log M_3}{cM_3^2} \cdots \frac{M_{k-2}^{b_{k-1}} \log M_{k-1}}{cM_{k-1}^2} \left( \frac{M_{k-1}^{b_k} \log M_k}{cM_k^2} \right)^s (8M_k^2)^s 2^s \ (2\epsilon)^s =$$
$$\big(\log M_2 \ldots \log M_{k-1}\big)\big(M_1^{b_2} M_2^{b_3-2} \ldots M_{k-2}^{b_{k-1}-2}\big)(\log M_k)^s (16)^s c^{-k+2-s} M_{k-1}^{b_k s-2} (2\epsilon)^s.$$

We want to verify that for every $j$ and for every $s < 2/a_j$ there is a $C$ such that $\nu(S) < C(2\epsilon)^s$. It suffices to show that there is a $C$ such that for every $k$,

$$\big(\log M_2 \ldots \log M_{k-1}\big)\big(M_1^{b_2} M_2^{b_3-2} \ldots M_{k-2}^{b_{k-1}-2}\big)(\log M_k)^s (16)^s \ c^{-k+2-s} < \ C \ M_{k-1}^{2-b_k s}. \quad (*)$$

Fix $k_0$ such that for every $k \geq k_0$, $a(j+1,k) = a(j+1,k_0)$. Thus, for every $k \geq k_0$, $a(j+1,k) = a_{j+1}$. Then, define $\delta > 0$ as follows so that for every $k \geq k_0$,

$$2 - \left(b_k \frac{2}{a_j}\right) \geq 2 - 2\left(a_{j+1} \frac{2}{a_j}\right) \geq 2 - 2\frac{a_{j+1}}{a_j} = \delta.$$

By the choice of $k_0$ and the definition of $b_k$, for all $k > k_0$, it holds that $b_k < a_{j+1}$. Hence the left hand side of the inequality (*) is at most a constant multiple of

$$\big(\log M_2 \ldots \log M_{k-1}\big)\big(M_1^{a_{j+1}} M_2^{a_{j+1}-2} \ldots M_{k-2}^{a_{j+1}-2}\big)(\log M_k)^s (16)^s \ c^{-k+2-s}.$$

Furthermore, there is a constant $C$ such that

$$\big(\log M_2 \ldots \log M_{k-1}\big)\big(M_1^{a_{j+1}} M_2^{a_{j+1}-2} \ldots M_{k-2}^{a_{j+1}-2}\big)(\log M_k)^s (16)^s \ c^{-k+2-s} < \ C \ M_{k-1}^{\delta}.$$

The above inequality follows by noticing that $M_\ell = M_1^{\ell!}$ for $\ell \geq 1$, taking logarithms on each side and recognizing that the contribution of $M_{k-1}$ is the dominating term for sufficiently large $k$. The value of $C$ is determined by the value $k_0$, which is computable relative to $0'$ as a function of $j$. $\square$

*Second proof of Theorem 1.* Let $a$ be a real number right-computably enumerable in $0'$ and greater than 2 (for $a$ equal 2, taking $x$ equals $\sqrt{2}$ suffices). Fix a computable doubly-indexed sequence $(a(j,s))_{j,s\geq 0}$ of rational numbers satisfying property (3) of Lemma 4. That is, we assume that $\lim_{j\to\infty}\lim_{s\to\infty} a(j,s) = a$, for all $s$ the sequence $(a(j,s))_{j\geq 0}$ is strictly decreasing, for all $j \geq 0$ the sequence $(a(j,s))_{s\geq 0}$ is eventually constant, for all $s$, we have $a(0,s) = a(0,0)$ and $a(1,s) = a(1,0)$. The last condition gives an appropriate initialization of the construction. Let $K$ be the fractal with measure $\nu$ and $C$ be the function associated with this approximation of $a$ in Lemma 7. Fix a computable function $C(r,s) : \mathbb{Q} \times \mathbb{Z}_{>0} \to \mathbb{Q}$ such that for every $r$ in $\mathbb{Q}$, $(C(r,s))_{s\geq 0}$ is eventually equal to $C(r)$. We may also assume that for all $s$, $C(a(1,s),s) = C(a(1,0),0)$.

We compute a real number $x$ in $K$. By recursion on $s$ we construct a sequence of nested intervals $(I(s))_{s\geq 0}$ such that if $I(s)$ is different from $I(s-1)$ then $I(s)$ is an element of the $s$-level of $K$. We define an auxiliary function $\ell(s)$, with infinite limit, to approximate the convergence of the sequence $a(j,s)$. We also define an auxiliary integer-valued function $q(j,s)$ where $j$ is an integer in $[0,\ell(s))$, with the intention that $x$ avoids approximation by rational numbers with denominator $q$ greater than or equal to $q(j,s)$ within $1/q^{a(j,s)}$. This intention will be realized in the construction at step $s$ onwards provided that at every step $t \geq s$, $\ell(t)$ is greater than $j$; in particular, provided that $a(j,s)$ and $C(a(j,s),s)$ have reached their limit values relative to $s$.

We will employ the following estimate. For a positive integer number $q_0$ and a real number $b$ greater than or equal to 2, let

$$V(q_0,b) = \bigcup_{q\geq q_0} \left\{ x \in \left(\frac{1}{q^b}, 1-\frac{1}{q^b}\right) : \exists p \in \mathbb{Z}, \left|\frac{p}{q} - x\right| < \frac{1}{q^b} \right\}.$$

Suppose that $b_1 > b_2 > a$. By Lemma 7, we can estimate $\nu(V(q_0,b_1))$ by

$$\begin{aligned}
\nu\big(V(q_0,b_1)\big) &\leq \sum_{q\geq q_0} \sum_{0<p<q} C\big(2/b_2\big)\Big(\frac{2}{q^{b_1}}\Big)^{2/b_2} \\
&\leq 2C\big(2/b_2\big) \sum_{q\geq q_0} q\Big(\frac{1}{q^{b_1}}\Big)^{2/b_2} \\
&\leq 2C\big(2/b_2\big) \sum_{q\geq q_0} \frac{1}{q^{2b_1/b_2-1}}.
\end{aligned}$$

Thus, for any $\epsilon > 0$ there is a $q_0$, uniformly computable from $\epsilon$, $b_1$, $b_2$ and $C(2/b_2)$, such that $\nu(V(q_0,b_1))$ is less than $\epsilon$.

*Initial step* 0. Start with $I(0)$ equal to the unit interval and $\ell(0) = 0$.

*Step s, greater than* 0. Let $\ell(s)$ be the least $j$ less than or equal to $s$ such that

$$a(j+1,s-1) \neq a(j+1,s) \text{ or } C\big(2/a(j+1,s),s-1\big) \neq C\big(2/a(j+1,s),s\big)$$

if such exists; otherwise, let $\ell(s)$ be $s$. By our assumptions on $a(j,s)$ and $C(a(1,0),s)$, for every $s>0$, we have that $\ell(s)\geq 1$.

Let $m(s)$ be the $\nu$-measure given to a level-$s$ interval in $K$. We find $h(s)$ so that the following inequality holds for each $j$ such that $0\leq j<\ell(s)$,

$$2C\big(2/a(j,s),s\big)\sum_{q\geq h(s)} 1/q^{\frac{2a(j,s)}{a(j+1,s)}-1} < \frac{1}{s}\,\frac{m(s)}{2^s}.$$

We define $q(j,s)$ for each $j\in[0,\ell(s))$ as follows: if $q(j,s-1)$ is defined then let $q(j,s)=q(j,s-1)$; otherwise, let $q(j,s)=h(s)$.

Let $I(s)$ be the leftmost level-$s$ interval in $K$ that is included in $I(s-1)$ and satisfies

$$\nu\Big(I(s)\cap\bigcup_{0\leq j<\ell(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\Big) < m(s)-2\frac{m(s)}{2^s}$$

if such exists; otherwise, let $I(s)$ be $I(s-1)$. Note that $m(s)\leq\nu(I(s))$.

We now verify that the construction works. Define $\ell_{min}(s)=\min_{t\geq s}\ell(t)$. We show by induction on $s$ that

$$\nu\Big(I(s)\cap\bigcup_{0\leq j<\ell_{min}(s)} V\big(q(j,s),a(j,s)\big)\Big)\leq\nu(I(s))\Big(1-\frac{1}{2^s}\Big).$$

Since $\ell_{min}(0)=0$, the inductive claim holds for $s=0$. Assume the inductive claim for $s-1$:

$$\nu\Big(I(s-1)\cap\bigcup_{0\leq j<\ell_{min}(s-1)} V\big(q(j,s-1),a(j,s-1)\big)\Big)\leq\nu\big(I(s-1)\big)\Big(1-\frac{1}{2^{s-1}}\Big).$$

Consider those integers $j$ such that $j<\ell_{min}(s)$. By the definition of $\ell_{min}$, we have $a(j,s)=\lim_{t\to\infty}a(j,t)$ and $C(2/a(j,s),s)=\lim_{t\to\infty}C(2/a(j,s),t)=C(2/a(j,s))$. Further, by the discussion above,

$$\nu\Big(V\big(h(s),a(j,s)\big)\Big)\leq 2C\big(2/a(j,s),s\big)\sum_{q\geq h(s)}1/q^{\frac{2a(j,s)}{a(j+1,s)}-1}.$$

In the construction we choose $h(s)$ so that for each $j$ less than $\ell(s)$, the term on the right side of this inequality is less than $m(s)/(s2^s)$. This ensures that for each $j$ less than $\ell_{min}(s)$, the same upper bound holds for $\nu(V(h(s),a(j,s)))$.

Now, consider the action of the construction during step $s$. If $I(s)$ is equal to $I(s-1)$, then

$$\begin{aligned}
&I(s)\cap\bigcup_{0\leq j<\ell_{min}(s)} V\big(q(j,s),a(j,s)\big)=\\
&\quad\Big(I(s)\cap\bigcup_{0\leq j<\ell_{min}(s-1)} V\big(q(j,s-1),a(j,s-1)\big)\Big)\cup\\
&\quad\Big(I(s)\cap\bigcup_{\ell_{min}(s-1)\leq j<\ell_{min}(s)} V\big(q(j,s),a(j,s)\big)\Big).
\end{aligned}$$

The first component of the union has $\nu$-measure at most $\nu(I(s))(1-1/2^{s-1})$ and the second component has $\nu$-measure at most $m(s)/2^s$. The union has measure at most $\nu(I(s))(1-1/2^s)$, as required.

Otherwise, $I(s)$ is a proper subinterval of $I(s-1)$ and satisfies

$$\nu\Big(I(s)\cap\bigcup_{0\leq j<\ell(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\Big) < m(s)-2\frac{m(s)}{2^s}.$$

Then,

$$\begin{aligned}
&I(s)\cap\bigcup_{0\leq j<\ell_{min}(s)} V\big(q(j,s),a(j,s)\big) = \\
&\quad\Big(I(s)\cap\bigcup_{0\leq j<\ell_{min}(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\Big)\ \cup \\
&\quad\Big(I(s)\cap\bigcup_{0\leq j<\ell_{min}(s)} V\big(h(s),a(j,s)\big)\Big).
\end{aligned}$$

The $\nu$-measure of the first component of the union is less than

$$m(s)-2\frac{m(s)}{2^s}=\nu\big(I(s)\big)\Big(1-\frac{1}{2^{s-1}}\Big).$$

As in the previous case, the $\nu$-measure of the second component is less than $\nu(I(s))/2^s$. Again, the union has measure at most $\nu(I(s))(1-1/2^s)$, as required.

It remains to show that there are infinitely many $s$ such that $I(s)$ is a proper subinterval of $I(s-1)$. Consider an $s$ such that $\ell(s)$ is equal to $\ell_{min}(s)$. Since

$$\nu\Big(I(s-1)\cap\bigcup_{0\leq j<\ell_{min}(s-1)} V\big(q(j,s-1),a(j,s-1)\big)\Big) < \nu\big(I(s-1)\big)\Big(1-\frac{1}{2^{s-1}}\Big),$$

we may fix an $s$-level subinterval $I$ of $I(s-1)$ such that

$$\nu\Big(I\cap\bigcup_{0\leq j<\ell_{min}(s-1)} V\big(q(j,s-1),a(j,s-1)\big)\Big) < \nu(I)\Big(1-\frac{1}{2^{s-1}}\Big).$$

For this $I$,

$$\begin{aligned}
&I\cap\bigcup_{0\leq j<\ell(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\subseteq \\
&\quad\Big(I\cap\bigcup_{0\leq j<\ell_{min}(s-1)} V\big(q(j,s),a(j,s)\big)\Big)\cup \\
&\quad\Big(I\cap\bigcup_{\ell_{min}(s-1)\leq j<\ell(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\Big).
\end{aligned}$$

For each $j$ such that $\ell(s-1)\leq j<\ell(s)$, $q(j,s)$ is equal to $h(s)$, so the second component of the union is empty. Thus,

$$\begin{aligned}
\nu\Big(I\cap\bigcup_{0\leq j<\ell(s)} V\big(q(j,s),a(j,s)\big)\setminus V\big(h(s),a(j,s)\big)\Big) &< \nu(I)\Big(1-\frac{1}{2^{s-1}}\Big) \\
&= m(s)-2\frac{m(s)}{2^s}.
\end{aligned}$$

Hence, the conditions for the construction to define $I(s)$ to be a proper subinterval of $I(s-1)$ are satisfied, as required.

Consider the sequence given by the closures of the intervals $I(s), s \geq 0$. This is a computable nested sequence of intervals whose lengths approach zero in the limit. Let $x$ be the unique real number in their intersection. By construction, $x$ is computable (as is its base-$b$ expansion, for every integer $b$ greater than or equal to 2.)

We now prove that the irrationality exponent of $x$ is equal to $a$. For each $j \geq 0$, let $b_j = \lim_{s\to\infty} a(j, s)$. The sequence $(b_j)_{j\geq 0}$ is strictly decreasing with limit $a$. The construction ensures that for every $j$, there is a step $s$ such that $I(s)$ is a level-$s$ interval of $K$ containing real numbers that have at least one rational approximation $p/q$ within $1/q^{b_j}$. Thus, the real number $x$ has irrationality exponent greater than or equal to $a$. We now show it can not be greater than $a$. Suppose that $b$ is greater than $a$. Let $j$ be such that $b$ is greater than $b_j$ and let $s$ be such that $\ell_{min}(s)$ is greater than $j$. Then, for all $t > s$, $a(j, t) = a(j, s) = b_j$ and $q(j, t) = q(j, s)$. Further, for any $t > s$, $\nu(I(t) \setminus V(q(j, t), b_j))$ is positive. If there were an integer $q > q(j, s)$ and an integer $p$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^b},$$

then there would be a $t$ greater than $s$ such that

$$I(t) \subset \Big(\frac{p}{q} - \frac{1}{q^{b_j}}, \frac{p}{q} + \frac{1}{q^{b_j}}\Big).$$

But then $I(t) \setminus V(q(j, t), b_j)$ would be empty, a contradiction with the fact that it has positive measure. $\square$

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