On the family of Diophantine triples $\{k-1, k+1, 16k^3-4k\}$

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Abstract

It is proven that if $k \geq 2$ is an integer and d is a positive integer such that the product of any two distinct elements of the set

$$\{k-1,\, k+1,\, 16k^3-4k,\, d\}$$

increased by 1 is a perfect square, then $d = 4k$ or $d = 64k^5 - 48k^3 + 8k$. Together with a recent result of Fujita, this shows that all Diophantine quadruples of the form $\{k-1, k+1, c, d\}$ are regular.

1 Introduction

A Diophantine m-tuple is a set of m positive integers such that the product of any two of them increased by 1 gives a perfect square. Diophantus himself studied sets of positive rationals with the same property, while the first Diophantine quadruple, namely the set $\{1, 3, 8, 120\}$, was found by Fermat ([4, 5, 12]). In 1969, Baker and Davenport [1] proved that the Fermat set cannot be extended to a Diophantine quintuple. There are several generalizations of the result of Baker and Davenport. In 1997, Dujella [6] proved that the Diophantine triples of the form $\{k-1, k+1, 4k\}$, for $k \geq 2$, cannot be extended to a Diophantine quintuple ($k = 2$ gives the Baker–Davenport result), while in 1998, Dujella and Pethő [9] proved that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. Recently, Fujita [11] obtained a result which is common generalization of the results from [6] and [9]. Namely, he proved that the Diophantine pairs of the form $\{k-1, k+1\}$, for $k \geq 2$ cannot be extended to a Diophantine quintuple.

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A folklore conjecture is that there does not exist a Diophantine quintuple. An important progress towards its resolution was done in 2004 by Dujella [8], who proved that there are only finitely many Diophantine quintuples. The stronger version of this conjecture states that if $\{a,b,c,d\}$ is a Diophantine quadruple and $d > \max\{a,b,c\}$, then $d = a+b+c+2abc+2\sqrt{(ab+1)(ac+1)(bc+1)}$. Diophantine quadruples of this form are called regular. Equivalently, a Diophantine quadruple $\{a,b,c,d\}$ is regular if and only if $(a+b-c-d)^2 = 4(ab+1)(cd+1)$ (see [10]).

If $\{k-1,k+1,c\}$ is a Diophantine triple, then $c = c_\nu$ for some $\nu \geq 1$, where

$$c_\nu = \frac{1}{2(k^2-1)}\Big\{(k+\sqrt{k^2-1})^{2\nu+1} + (k-\sqrt{k^2-1})^{2\nu+1} - 2k\Big\}$$

(see [6, 11]). As already noticed, in [11], Fujita proved that the Diophantine pair $\{k-1,k+1\}$ cannot be extended to a Diophantine quintuple. In particular, he proved that if $\{k-1,k+1,c_\nu,d\}$ is a Diophantine quadruple and $\nu \neq 2$, then $d = c_{\nu-1}$, or $d = c_{\nu+1}$, or $d = c_2$. In the present paper, we prove that this statement is valid also for $\nu = 2$.

Theorem 1 *If $k \geq 2$ is an integer and d is a positive integer such that the product of any two distinct elements of the set*

$$\{k-1,\ k+1,\ 16k^3-4k,\ d\}$$

increased by 1 is a perfect square, then $d = 4k$ or $d = 64k^5 - 48k^3 + 8k$.

It follows from Theorem 1 that all Diophantine quadruples of the form $\{k-1,k+1,c,d\}$ are regular.

The difficulty with the case $\nu = 2$ is that the gap between $k+1$ and $c_2 = 16k^3 - 4k$ is too small for the applications of results on simultaneous Diophantine approximations (theorem of Bennett [3] and its modification by Fujita [11]). Let us mention that Fujita was able to handle this case for $k > 5 \cdot 10^{20}$ (see [11, Remark 20]). Our improvements come from two directions. We improved lower bounds for the solutions by more delicate application of the congruence method. Roughly, we improved the bound $\log d > 4k \log k$ to $\log d > 4k^{1.5} \log k$ (assuming that $d \neq c_1 = 4k$ and $d \neq c_3 = 64k^5 - 48k^3 + 8k$). Another improvement comes from the application of the best available result on linear forms in three logarithms of algebraic numbers, due to Mignotte [14] (instead of the theorem of Baker and Wüstholz [2]). This leads to the proof of our result for $k > 5.4 \cdot 10^8$, so that the number of remaining cases is not too large, and can be handled by computers.

Let us mention that the case $k = 2$ was solved by Dujella and Pethő [9]. Thus, we may assume that $k \geq 3$. Moreover, using results of Fujita [11], it is easy to handle the cases with, say, $k \leq 1000$ (we will give the details in the last section). Therefore, to simplify some technical details, we will assume that $k > 1000$.

2 Preliminaries

We will use some general results on extendability of Diophantine triples from [7, 8]. Let $\{a, b, c\}$ be a Diophantine triple with $a < b < c$. Furthermore, let positive integers r, s, t be defined by $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. In order to extend $\{a, b, c\}$ to a Diophantine quadruple $\{a, b, c, d\}$, we have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2, \tag{1}$$

with positive integers x, y, z. Eliminating d from (1) we get the following system of Pellian equations

$$az^2 - cx^2 = a - c\,, \tag{2}$$
$$bz^2 - cy^2 = b - c\,. \tag{3}$$

By [7, Lemma 1], all solutions of (2) are given by $z = v_m^{(i)}$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = sz_0^{(i)} + cx_0^{(i)}, \quad v_{m+2}^{(i)} = 2sv_{m+1}^{(i)} - v_m^{(i)}\,,$$

and $|z_0^{(i)}| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}}$. Similarly, all solutions of (3) are given by $z = w_n^{(j)}$, where

$$w_0^{(j)} = z_1^{(i)}, \quad w_1^{(j)} = tz_1^{(j)} + cy_1^{(j)}, \quad w_{n+2}^{(j)} = 2tw_{n+1}^{(j)} - w_n^{(j)}\,.$$

and $|z_1^{(j)}| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$.

The initial terms $z_0^{(i)}$ and $z_1^{(j)}$ are almost completely determined in the following lemma (see [8, Lemma 8]).

Lemma 1

1) *If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$. *Furthermore,* $|z_0| = 1$ *or* $|z_0| = cr - st$ *or* $|z_0| < \min\{0.869\, a^{-5/14} c^{9/14},\ 0.972\, b^{-0.3} c^{0.7}\}$.

2) *If the equation* $v_{2m+1} = w_{2n}$ *has a solution, then* $|z_0| = t$, $|z_1| = cr - st$ *and* $z_0 z_1 < 0$.

3) *If the equation* $v_{2m} = w_{2n+1}$ *has a solution, then* $|z_0| = cr - st$, $|z_1| = s$ *and* $z_0 z_1 < 0$.

4) *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $|z_0| = t$, $|z_1| = s$ *and* $z_0 z_1 > 0$.

(We have omitted the superscripts (i) and (j), and we will continue to do so.)

In our case, we have $a = k-1$, $b = k+1$, $c = c_2 = 16k^3 - 4k$, $r = k$, $s = 4k^2 - 2k - 1$ and $t = 4k^2 + 2k - 1$. Therefore, $cr - st = 8k^2 - 1$, and we can easily check that $|z_0| \le \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < k^2 \sqrt{\frac{32\sqrt{k}}{\sqrt{k-1}}} < 8k^2 - 1$ and $|z_1| < k^2\sqrt{32} < 8k^2 - 1$. Furthermore, the third possibility in **1)** appears only if there is a positive integer $d_0 < c$ such that $\{a, b, c, d_0\}$ is an irregular Diophantine quadruple. But this is not possible in our case, since $d_0 = c_1 = 4k$, and the quadruple $\{a, b, c_1, c_2\}$ is regular.

Hence, the only possibilities which may occur in our case are **1)** with $|z_0| = 1$ and **4)**.

The exponential equation $v_m = w_n$ can be in standard way transformed into a logarithmic inequality.

Let us denote $\alpha_1 = s + \sqrt{ac}$, $\alpha_2 = t + \sqrt{ac}$, $\alpha_3 = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}$, $\alpha_4 = \frac{\sqrt{b}(k\sqrt{c} \pm t\sqrt{a})}{\sqrt{a}(k\sqrt{c} \pm s\sqrt{b})}$. We recall the following result from [11] (see also [7, Lemma 5]):

Lemma 2 ([11], Lemma 8)

(i) *If* $v_{2m} = w_{2n}$ *has a solution with* $m \ge 1$ *and* $z_0 = z_1 = \pm 1$, *then we have*

$$0 < 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3 < 1.2{\alpha_1}^{-4m}.$$

(ii) *If* $v_{2m+1} = w_{2n+1}$ *has a solution with* $m \ge 0$ *and* $z_0 = \pm t$, $z_1 = \pm s$ $(z_0 z_1 > 0)$, *then we have*

$$0 < (2m+1) \log \alpha_1 - (2n+1) \log \alpha_2 + \log \alpha_4 < 4.1k^2{\alpha_1}^{-4m-2}.$$

Lemma 3 *Assume that* $v_m = w_n$ *with* $n \ne 0$. *Then*

$$0 \le m - n < \frac{0.51n}{k \log k}.$$

Proof. Since m and n have the same parity, [8, Lemma 3] implies that $m \geq n$. On the other hand, since $\alpha_3, \alpha_4 > 1$, from Lemma 2, we find that

$$m \log \alpha_1 - n \log \alpha_2 < 0.065 \cdot k^{-2}.$$

Hence,

$$\begin{aligned}
\frac{m-n}{n} &< \frac{\log(\alpha_2/\alpha_1)}{\log \alpha_1} + \frac{0.065}{nk^2 \log \alpha_1} \\
&< \frac{\log(\frac{8k^2+4k-1}{8k^2-4k-2})}{\log(8k^2-4k-2)} + \frac{0.065}{nk^2 \log(8k^2-4k-2)} \\
&< \frac{8k+1}{(8k^2-4k-2)\log(8k^2-4k-2)} + \frac{0.033}{nk^2 \log k} < \frac{0.51}{k \log k},
\end{aligned}$$

since $k > 1000$. ■

3 Even indices

In this section, we are studying the equation

$$v_{2m} = w_{2n},$$

under the assumption that $|z_0| = 1$. We intend to show that if $n \neq 0$, then m and n are large compared with k. By [11, Lemma 10], we have $m \equiv 0$ or $\pm 1 \pmod{2k}$. This gives the lower bound $m \geq 2k-1$ for m. In what follows, we will improve this bound.

Our starting point is the following congruence relation which follows from [7, Lemma 4]:

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{4c}. \tag{4}$$

In our case, the congruence (4) becomes

$$\pm(k-1)m^2+(4k^2-2k-1)m \equiv \pm(k+1)n^2+(4k^2+2k-1)n \pmod{64k^3-16k}$$

or

$$\begin{aligned}
&\pm k(m^2-n^2) - (4k^2-1)(m-n) + 2k(m+n) \\
&\qquad \equiv \pm(m^2+n^2) \pmod{64k^3-16k}.
\end{aligned} \tag{5}$$

Assume that $n < 5.6k^{1.5}$. Then the both sides in congruence (5) are (in absolute value) less than $63k^3$. Indeed, by Lemma 3, we have $(m^2+n^2) \leq$

$2.001n^2 < 63k^3$, $k(m^2-n^2) = k(m-n)(m+n) < k \cdot \frac{0.51n}{k\log k} \cdot 2.001n < 0.148n^2 < 5k^3$, $(4k^2-1)(m-n) < 4k^2 \cdot \frac{0.51n}{k\log k} < 0.3kn < 0.06k^3$, $2k(m+n) \leq 2.001kn < 0.36k^3$. Hence, we have an equality in (5).

Let $m = n + \alpha$. Then

$$0 \leq \alpha < \frac{2.856\sqrt{k}}{\log k}, \tag{6}$$

by Lemma 3. Furthermore, the congruence (5) implies

$$\alpha^2 - 2m\alpha + 2m^2 \equiv \pm\alpha \pmod{k}. \tag{7}$$

If we now insert $m \equiv 0, \pm 1$ in (7), then by (6) we obtain equalities instead of congruences in (7) in all three cases. This gives $\alpha^2 = \pm\alpha$ and $\alpha^2 \mp \alpha + 2 = \pm\alpha$, respectively. And this implies $\alpha \in \{0, 1, 2\}$.

This can be now inserted in (the equality) (5). We easily obtain contradictions in all cases except for $\alpha = 0$ and $z_0 = z_1 = -1$, when the only nontrivial case is $m = n = 2k$, and for $\alpha = 2$ and $z_0 = z_1 = 1$ when the case $m = 2k+1$, $n = 2k-1$ has to be considered. Let us look closer at these remaining case.

By Lemma 2, we have

$$\log \alpha_3 > 2n\Big(\log \frac{t + \sqrt{bc}}{s + \sqrt{ac}}\Big) \geq \frac{1.8n}{k} = 3.6,$$

and $\alpha_3 > 36$. But, on the other hand,

$$\alpha_3 = \frac{\sqrt{b}}{\sqrt{a}} \cdot \frac{\sqrt{c} - \sqrt{a}}{\sqrt{c} - \sqrt{b}} \leq \Big(1 + \frac{2}{k}\Big)\Big(1 + \frac{1}{3k^2}\Big) < 1 + \frac{3}{k} < 2.$$

Assume now that $m = 2k+1$ and $n = 2k-1$. Then Lemma 2 implies

$$(2n+4)\log(s + \sqrt{ac}) - 2n\log(t + \sqrt{bc}) < 0.001.$$

We obtain $4\log(7k^2) < \frac{1.8n}{k} + 1 < 4$, a contradiction.

Hence, we proved

Proposition 1 *If $v_{2m} = w_{2n}$ with $n \neq 0$, then $m \geq n \geq 5.6k^{1.5}$.*

4 Odd indices

In this section, we consider the equation

$$v_{2m+1} = w_{2n+1},$$

with $z_0 = \pm t$, $z_1 = \pm s$, under the assumption that $n \neq 0$. By [11, Lemma 10], we have $m \equiv 0, -1 \pmod{k}$.

¿From [7, Lemma 4], we have the following congruence

$$\pm(am(m+1) - bn(n+1)) \equiv rst(n-m) \pmod{8k^3 - 2k}.$$

Since $rst \bmod (8k^3 - 2k) = k$, we obtain

$$\begin{aligned} k(m-n)(m+n+1) &- m(m+1) \\ &\equiv n(n+1) \pm k(n-m) \pmod{8k^3 - 2k}. \end{aligned} \tag{8}$$

Assume that $n < 2.45k^{1.5}$. Note that by Lemma 3 it holds $m \leq 1.0001n$. We have the following estimates for the terms in (8): $k(m-n)(m+n+1) \leq k \cdot \left(\frac{0.51(n+\frac{1}{2})}{k \log k}\right) \cdot 2.002n < 0.15n^2 < k^3$, $m(m+1) < 1.002n^2 < 6.1k^3$, $n(n+1) < 1.001n^2 < 6.1k^3$, $k(m-n) < kn < k^3$. Hence, we have an equality in (8).

Let us proceed as in the case of even indices by denoting $m = n + \alpha$, and inserting it in (8). We obtain

$$\alpha^2 - (2m+1)\alpha + 2m(m+1) \equiv 0 \pmod{k},$$

which, by $m \equiv 0, -1 \pmod{k}$, implies $\alpha \in \{0, 1, 3\}$. ¿From $m = n$ we obtain $2n(n+1) = 0$ and $n = 0$. For $m = n+1$, we have that $k(2n+2 \pm 1) = (n+1)(2n+2)$ which implies $n = 0$. Similarly, for $m = n+3$, we find that $3k(2n+4 \pm 1) = 2n^2 + 8n + 12 = (2n+3)(2n+5) + 9$. Hence, $(2n+3)|9$ or $(2n+5)|9$. The first possibility gives $n = 3$ and $k = 2$, which is the case already solved in [9]. The second possibility leads to $n = 2$ and $3k = 4$, a contradiction.

Hence, we proved

Proposition 2 *If $v_{2m+1} = w_{2n+1}$ with $n \neq 0$, then $m \geq n \geq 2.45k^{1.5}$.*

5 Linear forms in logarithms

We intend to apply a recent result due to Mignotte [14] to the linear forms from Lemma 2. However, in order to check the conditions of Mignotte's theorem, we will use information obtained by the application of the following result of Matveev.

Lemma 4 ([13], Theorem 2.1) *Let Λ be a linear form in logarithms of l multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational integer coefficients $b_1, \dots, b_l$ ($b_l \neq 0$). Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j, $1 \leq j \leq l$. Define the numbers D, A_j, $1 \leq j \leq l$, and B by $D = [\mathbb{Q}(\alpha_1, \dots, \alpha_l) : \mathbb{Q}]$, $A_j = \max\{D\, h(\alpha_j), |\log \alpha_j|\}$, $B = \max\{1, \max\{\frac{|b_j| A_j}{A_l} : 1 \leq j \leq l\}\}$. Then*

$$\log \Lambda > -C(l) C_0 W_0 D^2 \Omega\,, \tag{9}$$

where $C(l) = \frac{8}{(l-1)!}(l+2)(2l+3)(4e(l+1))^{l+1}$, $C_0 = \log(e^{4.4l+7} l^{5.5} D^2 \log(eD))$, $W_0 = \log(1.5eBD \log(eD))$, $\Omega = A_1 \cdots A_l$.

We apply Lemma 4 to the forms from Lemma 2. We have $l = 3$ and $D = 4$. Furthermore, $A_1 = 2 \log \alpha_1 < 2 \log 2s < 4.1 \log k$, $A_2 = 2 \log \alpha_2 < 4.1 \log k$. Also, $A_1 > 4 \log k$ and $A_2 > 4 \log k$. Estimating the conjugates of α_3, resp. α_4, we find that

$$A_3 \leq \log \left(a^2 (c-b)^2 \frac{b}{a} \cdot \frac{c-a}{c-b} \cdot \frac{\sqrt{b}(k\sqrt{c} + t\sqrt{a})}{\sqrt{a}(k\sqrt{c} - s\sqrt{b})} \right) < 2 \log(2bck) < 11 \log k$$

and $A_3 \geq 2 \log(b(c-a)) > 8 \log k$. Hence, we conclude that $B < 0.52m$. Finally, $\log \Lambda < \log 4.2k^2 \alpha_1{}^{-2m} < -0.9m \log c < -2.7m \log k$. Putting all these estimates in (9), we obtain

$$2.7m \log k < 6.45 \cdot 10^8 \cdot 29.89 \cdot \log(21m) \cdot 184.91 \cdot \log^3 k.$$

Taking into account the inequality $m > 4.9k^{1.5}$, we obtain

$$\frac{m}{\log(21m) \log^2 m} < 5.92 \cdot 10^{11},$$

which implies $m < 3.6 \cdot 10^{16}$ and, finally,

$$k < 3.8 \cdot 10^{10}.$$

We will further reduce this upper bound for k, using following result due to Mignotte on linear forms in three logarithms:

Lemma 5 ([14]) *We consider three non-zero algebraic numbers α_1, α_2 and α_3, which are either all real and > 1 or all complex of modulus one and all $\neq 1$. Moreover, we assume that either the three numbers α_1, α_2 and α_3 are* *multiplicatively independent, or two of these numbers are multiplicatively independent and the third one is a root of unity. Put*

$$\mathcal{D} = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

We also consider three coprime positive rational integers b_1, b_2, b_3, *and the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i *are arbitrary determinations of the logarithm, but which are all real or all purely imaginary. And we assume also that*

$$b_2 |\log \alpha_2| = b_1 |\log \alpha_1| + b_3 |\log \alpha_3| \pm |\Lambda|.$$

We put

$$d_1 = \gcd(b_1, b_2), \quad d_2 = \gcd(b_3, b_2), \quad b_2 = d_1 b_2'' = d_3 b_2'', \quad b_1 = d_1 b_1', \quad b_3 = d_2 b_3''.$$

Let a_1, a_2 *and* a_3 *be real numbers such that*

$$a_i \geq \max\{4, 5.296\ell_i - \log |\alpha_i| + 2\mathcal{D} h(\alpha_i)\}, \quad \textit{where } \ell_i = |\log \alpha_i|,\ i = 1, 2, 3$$

and

$$\Omega = a_1 a_2 a_3 \geq 100.$$

Put

$$b' = \Big(\frac{b_1'}{a_2} + \frac{b_2'}{a_1}\Big)\Big(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\Big)$$

and

$$\log \mathcal{B} = \max\{0.882 + \log b', 10/\mathcal{D}\}.$$

Then either

$$\log \Lambda > -790.95 \cdot \Omega \cdot \mathcal{D}^2 \log^2 \mathcal{B} > -307187 \cdot \mathcal{D}^5 \log^2 \mathcal{B} \cdot \prod_{i=1}^{3} \max\{0.55, h_i, \ell_i/\mathcal{D}\},$$

or the following condition holds:

– *either there exist two non-zero rational numbers* r_0 *and* s_0 *such that*

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \leq 5.61(\mathcal{D} \log \mathcal{B})^{1/3} a_2 \quad \textit{and} \quad |s_0| \leq 5.61(\mathcal{D} \log \mathcal{B})^{1/3} a_1,$$

– *or there exist rational integers* r_1, s_1, t_1 *and* t_2, *with* $r_1 s_1 \neq 0$, *such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisfy

$$|r_1 s_1| \leq \delta \cdot 5.61 (\mathcal{D} \log \mathcal{B})^{1/3} a_3, \quad |s_1 t_1| \leq \delta \cdot 5.61 (\mathcal{D} \log \mathcal{B})^{1/3} a_1,$$

$$|r_1 s_2| \leq \delta \cdot 5.61 (\mathcal{D} \log \mathcal{B})^{1/3} a_2,$$

where

$$\delta = \gcd(r_1, s_1).$$

Moreover, when $t_1 = 0$ *we can take* $r_1 = 1$, *and when* $t_2 = 0$ *we can take* $s_1 = 1$.

Let $\alpha_1 = s + \sqrt{ac}$, $\alpha_2 = t + \sqrt{ac}$, $\alpha_3 = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}$, $\alpha_4 = \frac{\sqrt{b}(k\sqrt{c} \pm t\sqrt{a})}{\sqrt{a}(k\sqrt{c} \pm s\sqrt{b})}$. We may apply Lemma 5 to the numbers $\alpha_1, \alpha_2, \alpha_3$, resp. $\alpha_1, \alpha_2, \alpha_4$. Indeed, these numbers are real and > 1. Moreover, they are multiplicatively independent, since the relation

$$(s + \sqrt{ac})^{i_1} (t + \sqrt{ac})^{i_2} = \Big(\frac{\sqrt{b}(x_0 \sqrt{c} \pm z_0 \sqrt{a})}{\sqrt{a}(y_1 \sqrt{c} \pm z_1 \sqrt{b})} \Big)^{i_3}$$

implies

$$(s - \sqrt{ac})^{i_1} (t - \sqrt{ac})^{i_2} = \Big(\frac{\sqrt{b}(x_0 \sqrt{c} \mp z_0 \sqrt{a})}{\sqrt{a}(y_1 \sqrt{c} \mp z_1 \sqrt{b})} \Big)^{i_3},$$

and by multiplying these two relations we obtain $\left(\frac{b(c-a)}{a(c-b)} \right)^{i_3} = 1$ and $a = b$, a contradiction.

We will assume that $k \geq 2 \cdot 10^7$. Note that this assumption implies that $m \leq k^3$, since otherwise we have $k < (3.6 \cdot 10^{16})^{1/3} < 331000$.

For the quantities appearing in Lemma 5, we may take $a_1 = 8.296 \log \alpha_1$, $a_2 = 8.296 \log \alpha_2$, $a_3 = 4.296 + 21 \log k < 21.3 \log k$, $b' \leq \frac{2m}{a_1} \cdot \frac{1.001n}{a_3}$, $\log \mathcal{B} = 0.882 + \log b' \leq \log(0.018 \cdot \frac{m^2}{\log^2 k})$.

Now, we are checking that two exceptional cases from Lemma 5 cannot appear in our situation (under our assumptions).

1) $r_0 n = s_0 m$

We have

$$
\begin{aligned}
|r_0| \leq 5.61(\mathcal{D}\log\mathcal{B})^{1/3}a_2 &\leq 5.61\Big(4\log\Big(0.018\frac{m^2}{\log^2 k}\Big)\Big)^{1/3}\cdot 8.296\log(2t)\\
&\leq 531\log^{1/3} m\log k,
\end{aligned}
$$

$$
|s_0| \leq 5.61(\mathcal{D}\log\mathcal{B})^{1/3}a_1 \leq 531\log^{1/3} m\log k.
$$

¿From Lemma 3, we obtain

$$
0 \leq r_0 - s_0 \leq \frac{0.51 s_0}{k\log k}. \tag{10}
$$

a) Assume $r_0 = s_0$. This implies $m = n$.
We have

$$
m\log\frac{t+\sqrt{bc}}{s+\sqrt{ac}} < \log\frac{\sqrt{b}(x_0\sqrt{c}\pm z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}\pm z_1\sqrt{b})} < 0.92.
$$

On the other hand,

$$
m\log\frac{t+\sqrt{bc}}{s+\sqrt{ac}} > m\log\frac{8k^2+4k-2}{8k^2-4k-1} > \frac{m}{2}\cdot\frac{8k-1}{8k^2-4k-1} > 0.4\cdot\frac{m}{k}.
$$

Hence, we obtained that $\frac{m}{k} < 2.3$, a contradiction.
b) If $r_0 \neq s_0$, then (10) implies

$$
1 \leq \frac{0.51 s_0}{k\log k} < \frac{0.51\cdot 531\log^{1/3} m}{k},
$$

and therefore $k < 271\log^{1/3} m < 391\log^{1/3} k$, contradicting our assumption that $k \geq 2\cdot 10^7$.

2) $r_1, s_1, t_1, t_2 \in \mathbb{Z}$, $r_1 s_1 \neq 0$ such that

$$
(t_1 m + r_1)s_1 = r_1 t_2 n. \tag{11}
$$

Furthermore,

$$
|r_1 s_1| \leq \gcd(r_1, s_1)\cdot 5.61(\mathcal{D}\log\mathcal{B})^{1/3}a_3,
$$

which implies $r_1, s_1 \leq 239\log^{1/3} m\log k$. Similarly, we find that $t_1, t_2 \leq 531\log^{1/3} m\log k$.

Assume that $t_1 = 0$. Then $r_1 = 1$ and $s_1 = t_2 n$. We obtain $n \leq s_1 \leq 239 \log^{1/3} m \log k \leq 240 \log^{1/3} n \log k$. Hence,

$$k < \frac{k^{1.5}}{\log k^{1.5}} \leq \frac{n}{\log n} \leq 240 \log k,$$

a contradiction.

From

$$0 \leq m - n \leq \frac{0.51 n}{k \log k}$$

and (11), for $t_1 \neq 0$, it follows that

$$\frac{r_1 s_1}{n} \leq r_1 t_2 - s_1 t_1 \leq \frac{0.51 s_1 t_1}{k \log k} \tag{12}$$

or

$$\frac{r_1 s_1}{n} \geq r_1 t_2 - s_1 t_1 \geq \frac{0.51 s_1 t_1}{k \log k}. \tag{13}$$

Let us consider the case $r_1 t_2 = s_1 t_1$. Then $r_1 = s_1$, $t_1 = t_2$, $t_1(m-n) = -r_1$, and $m \leq |r_1| \leq 239 \log^{1/3} m \log k$. Hence, $k^{3/2} < 345 \log^{4/3} k$, which implies $k < 300$.

Assuming that $r_1 t_2 \neq s_1 t_1$, from (12) and (13), we obtain that $\frac{|r_1 s_1|}{n} \geq 1$ or $\frac{0.51 |s_1 t_1|}{k \log k} \geq 1$.

For $1 \leq \frac{|r_1 s_1|}{n} \leq \frac{239^2 \cdot \log^{2/3} m \log^2 k}{n}$, we obtain $n \leq 60000 \log^{2/3} n \log^2 k$ and $k \leq 60000 \log^2 k$, which implies $k < 2 \cdot 10^7$.

If $\frac{0.51 |s_1 t_1|}{k \log k} \geq 1$, then $0.51 \cdot 239 \cdot 531 \cdot \log^{2/3} m \log^2 k \geq k \log k$. We obtain $k < 134631 \log^{5/3} k$ and $k < 2 \cdot 10^7$.

We have shown that if $k \geq 2 \cdot 10^7$, then the two exceptional cases from Lemma 5 cannot appear. Therefore, Lemma 5 will give us an inequality for $\log \Lambda$.

We obtain

$$\begin{aligned} 2.7 m \log k &< 790.95 \cdot 7095.17 \cdot \log^3 k \cdot 16 \cdot \log^2 \left(0.018 \cdot \frac{m^2}{\log^2 k}\right) \\ &< 3.492 \cdot 10^8 \log^3 k \log^2 m, \end{aligned}$$

and

$$\frac{m}{\log^4 m} < 5.92 \cdot 10^7.$$

This implies

$$m < 6 \cdot 10^{13}.$$

Applying Propositions 1 and 2, we find that

- if $v_{2m} = w_{2n}$ with $n \neq 0$, then $k < 3.1 \cdot 10^8$,
- if $v_{2m+1} = w_{2n+1}$ with $n \neq 0$, then $k < 5.4 \cdot 10^8$.

6 Baker-Davenport reduction

It remains to consider the following cases:

(i) $z_0 = z_1 = 1$, $3 \leq k \leq 310000000$;

(ii) $z_0 = z_1 = -1$, $3 \leq k \leq 310000000$;

(iii) $z_0 = t$, $z_1 = s$, $3 \leq k \leq 540000000$;

(iv) $z_0 = -t$, $z_1 = -s$, $3 \leq k \leq 540000000$.

We will apply a version of the reduction procedure due to Baker and Davenport [1], given in the following lemma.

Lemma 6 ([9], Lemma 5) *Suppose that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 10M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

a) *If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{14}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq m \leq M\,.$$

b) *Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$, then there is no solution of inequality* (14) *in integers m and n with*

$$\max\left(\frac{\log(3Aq)}{\log B}, 1\right) < m \leq M\,.$$

We apply Lemma 6 to inequalities from Lemma 2, with $M = 3.6 \cdot 10^{16}$, obtained by the applications of Lemma 4. Note that Lemma 2 is valid also for $k \leq 1000$, while in that case we use the value $M = 10^{21}$ (see [11, Remark 20]). If the first convergent such that $q > 10M$ does not satisfy the conditions **a)** or **b)** of Lemma 6, then we use the next convergent.

We performed all needed computations on the computers of Laboratory for Advanced Computations, Department of Mathematics, University of Zagreb. The computations were carried on a Dual AMD Athlons MP 1800+ with 1Gb RAM memory, under Debian 3.1 (Sarge) operating system with 2.4.19 SMP kernel and running PARI/GP programs written by the authors. Here we summarize the results obtained by the computations.

Case (i): The use of the second convergent was necessary in 3283278 cases (1.06 %), the third convergent was used in 79279 cases (0.03 %), etc., the tenth convergent was needed only in one case (for $k = 145384228$). In all cases we obtained $m \leq 6$. Moreover, $m \leq 2$ for $k \geq 25$; $m \leq 1$ for $k \geq 344$. Running time was 78 hours.

Case (ii): The use of the second convergent was necessary in 32831823 cases (1.06 %), the third convergent was used in 79249 cases (0.03 %), etc., the tenth convergent was needed only in one case (for $k = 145384228$). In all cases we obtained $m \leq 6$. Moreover, $m \leq 2$ for $k \geq 25$; $m \leq 1$ for $k \geq 328$. Running time was 78 hours.

Case (iii): The use of the second convergent was necessary only in 21 cases. Actually, for all $k > 1000$, the condition **b)** of Lemma 6 was satisfied. This is not surprising. Namely, in this case we have $\log\alpha_1 - \log\alpha_2 + \log\alpha_4 = \frac{1}{128}k^{-7} + O(k^{-9})$, and hence $|p - q + r| \approx |\frac{q}{\log\alpha_2}(\log\alpha_1 - \log\alpha_2 + \log\alpha_4)| \approx \frac{q}{256k^7 \log k} < 1$ for $k \gtrsim \sqrt[7]{\frac{q}{256}}$. In all cases we obtained $m \leq 6$. Moreover, $m \leq 2$ for $k \geq 64$; $m \leq 1$ for $k > 1000$. Running time was 134 hours.

Case (iv): The use of the second convergent was necessary in 6819 cases, the third convergent was used in 182 cases, etc., the ninth convergent was needed only in one case (for $k = 154441$). For all $k > 68546778$, the condition **b)** of Lemma 6 was satisfied. In all cases we obtained $m \leq 6$. Moreover, $m \leq 2$ for $k \geq 82$; $m \leq 1$ for $k > 1000$. Running time was 135 hours.

Combining these experimental results, with the lower bounds from Propositions 1 and 2, we conclude that for all $k \geq 3$ we have $m = n = 0$. Now, $v_0 = w_0 = 1$ gives the trivial extension $d = 0$. For $(z_0, z_1) = (t, s)$, $v_1 = w_1 = 32k^4 - 16k^2 + 1$ gives $d = 64k^5 - 48k^3 + 8k = c_3$, while for $(z_0, z_1) = (-t, -s)$, $v_1 = w_1 = 8k^2 - 1$ gives $d = 4k = c_1$. This completes the proof of Theorem 1.

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