Root separation for irreducible integer polynomials

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1 Introduction

The height $H(P)$ of an integer polynomial $P(x)$ is the maximum of the absolute values of its coefficients. For a separable integer polynomial $P(x)$ of degree $d \geq 2$ and with distinct roots $\alpha_1, \ldots, \alpha_d$, we set

$$\mathrm{sep}(P) = \min_{1 \leq i < j \leq d} |\alpha_i - \alpha_j|$$

and define $e(P)$ by

$$\mathrm{sep}(P) = H(P)^{-e(P)}.$$

Following the notation from [8], for $d \geq 2$, we set

$$e(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P)$$

and

$$e_{\mathrm{irr}}(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P),$$

where the latter limsup is taken over the irreducible integer polynomials $P(x)$ of degree d. A classical result of Mahler [10] asserts that $e(d) \leq d-1$ for all d, and it is easy to check that $e_{\mathrm{irr}}(2) = e(2) = 1$. There is only one other value of d for which $e(d)$ or $e_{\mathrm{irr}}(d)$ is known, namely $d = 3$, and we have $e_{\mathrm{irr}}(3) = e(3) = 2$, as proved, independently, by Evertse [9] and Schönhage [11]. For larger values of d, the following lower bounds have been established by Bugeaud and Mignotte in [7]:

$$e_{\mathrm{irr}}(d) \geq d/2, \quad \text{for even } d \geq 4,$$

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$$e(d) \geq (d+1)/2, \quad \text{for odd } d \geq 5,$$
$$e_{\text{irr}}(d) \geq (d+2)/4, \quad \text{for odd } d \geq 5,$$

while Beresnevich, Bernik, and Götze [2] proved that

$$e_{\text{irr}}(d) \geq (d+1)/3, \quad \text{for every } d \geq 2.$$

Except those from [2], the above results are obtained by presenting explicit families of (irreducible) polynomials of degree d whose roots are close enough. The ingenious proof in [2] does not give any explicit example of such polynomials, but shows that algebraic numbers of degree d with a close conjugate form a 'highly dense' subset in the real line.

The aim of the present note is to improve all known lower bounds for $e_{\text{irr}}(d)$ when $d \geq 4$.

Theorem 1 *For any integer $d \geq 4$, we have*

$$e_{\text{irr}}(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)}.$$

To prove Theorem 1, we construct explicitly, for any given degree $d \geq 4$, a one-parametric family of irreducible integer polynomials $P_{d,a}(x)$ of degree d. We postpone to Section 3 our general construction and give below some numerical examples in small degree.

For $a \geq 1$, the roots of the polynomial

$$P_{4,a}(x) = (20a^4 - 2)x^4 + (16a^5 + 4a)x^3 + (16a^6 + 4a^2)x^2 + 8a^3x + 1,$$

are approximately equal to:

$$\begin{aligned}
r_1 &= -1/4a^{-3} - 1/32a^{-7} - 1/256a^{-13} + \ldots, \\
r_2 &= -1/4a^{-3} - 1/32a^{-7} + 1/256a^{-13} + \ldots, \\
r_3 &= -2/5a + 11/100a^{-3} + 69/4000a^{-7} + 4/5a\, i + \ldots, \\
r_4 &= -2/5a + 11/100a^{-3} + 69/4000a^{-7} - 4/5a\, i + \ldots.
\end{aligned}$$

Since $H(P_{4,a}) = O(a^6)$ and $\text{sep}(P_{4,a}) = |r_1 - r_2| = O(a^{-13})$, we obtain by letting a tend to infinity that $e_{\text{irr}}(4) \geq 13/6$.

Similar construction for degree five gives the family of polynomials

$$P_{5,a}(x) = (56a^5-2)x^5+(56a^6+4a)x^4+(80a^7+4a^2)x^3+(100a^8+8a^3)x^2+20a^4x+1$$

with two close roots

$$1/10a^{-4}+1/250a^{-9}+3/25000a^{-14}-3/250000a^{-19}\pm\sqrt{10}/500000a^{-43/2}+\ldots,$$

and we obtain that $e_{\mathrm{irr}}(5) \geq 43/16$.

Our construction is applicable as well for $d = 3$. It gives the family

$$P_{3,a}(x) = (8a^3 - 2)x^3 + (4a^4 + 4a)x^2 + 4a^2x + 1$$

with close roots $-1/2a^{-2} - 1/4a^{-5} \pm \sqrt{2}/8a^{-13/2}$, showing that $e_{\mathrm{irr}}(3) \geq 13/8$. This is weaker than the known result $e_{\mathrm{irr}}(3) = 2$, but it could be noted that in the examples showing that $e_{\mathrm{irr}}(3) = 2$ the coefficients of the polynomials involved have exponential growth, while in our example the coefficients have polynomial growth, only.

The constant term of every polynomial $P_{d,a}(x)$ constructed in Section 3 is equal to 1. This means that the reciprocal polynomial of $P_{d,a}(x)$ is monic. Therefore, Theorem 1 gives also a lower bound for the quantity

$$e^*_{\mathrm{irr}}(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P),$$

where the limsup is taken over the *monic* irreducible integer polynomials. Regarding this quantity, the following estimates have been established by Bugeaud and Mignotte in [8]:

$$e^*_{\mathrm{irr}}(2) = 0, \quad e^*_{\mathrm{irr}}(3) \geq 3/2,$$

$$e^*_{\mathrm{irr}}(d) \geq (d-1)/2, \quad \text{for even } d \geq 4,$$

$$e^*_{\mathrm{irr}}(d) \geq (d+2)/4, \quad \text{for odd } d \geq 5,$$

while Beresnevich, Bernik, and Götze [2] proved that

$$e^*_{\mathrm{irr}}(d) \geq d/3, \quad \text{for every } d \geq 3.$$

In particular, for $d = 5$, the current best estimate is $e^*_{\mathrm{irr}}(5) \geq 7/4$.

Our construction allows us to improve these results when d is odd and at least equal to 7.

Theorem 2 *For any odd integer $d \geq 7$, we have*

$$e^*_{\mathrm{irr}}(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)} - 1.$$

To prove Theorem 2, we simply observe that if α and β denote the two very close roots of a polynomial $P_{d,a}(x)$ constructed in Section 3, then α and β satisfy

$$|\alpha|^{-1}, |\beta|^{-1} = O(a^{d-1}) = O(H(P_{d,a})^{1/2}),$$

and

$$\left|\frac{1}{\alpha} - \frac{1}{\beta}\right| = \frac{|\alpha - \beta|}{\alpha\beta}$$

is very small, where, clearly, $1/\alpha$ and $1/\beta$ are roots of the reciprocal polynomial of $P_{d,a}(x)$.

2 Application to Mahler's and Koksma's classifications of numbers

The families of polynomials constructed for the proof of Theorem 1 can be used in the context of [3]. Let d be a positive integer. Mahler and, later, Koksma, introduced the functions w_d and w_d^* in order to measure the quality of approximation of real numbers by algebraic numbers of degree at most d. For a real number ξ, we denote by $w_d(\xi)$ the supremum of the exponents w for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(x)$ of degree at most d. Following Koksma, we denote by $w_d^*(\xi)$ the supremum of the exponents w^* for which

$$0 < |\xi - \alpha| < H(\alpha)^{-w^*-1}$$

has infinitely many solutions in real algebraic numbers α of degree at most d. Here, $H(\alpha)$ stands for the naïve height of α, that is, the naïve height of its minimal defining polynomial.

For an overview of results on w_d and w_d^*, the reader can consult [5], especially Chapter 3. Let us just mention that it is quite easy to establish the inequalities

$$w_d^*(\xi) \le w_d(\xi) \le w_d^*(\xi) + d - 1,$$

for any transcendental real number ξ, and that

$$w_d^*(\xi) = w_d(\xi) = d$$

holds for almost all real numbers ξ, with respect to the Lebesgue measure.

For $d \ge 2$, R. C. Baker [1] showed that the range of values of the function $w_d - w_d^*$ includes the interval $[0, (d-1)/d]$. This has been substantially improved in [3], where it is shown that the function $w_d - w_d^*$ can take any value in $[0, d/4]$. Further results are obtained in [4, 6], including that the function $w_2 - w_2^*$ (resp. $w_3 - w_3^*$) takes any value in $[0, 1)$ (resp. in $[0, 2)$). The proofs in [3, 4, 6] make use of families of polynomials with close roots.

In particular, the upper bound $d/4$ is obtained by means of the family of polynomials $x^d - 2(ax-1)^2$ of height $2a^2$ and having two roots separated by $O(a^{-(d+2)/2})$.

Corollary 1 *For any integer $d \geq 4$, the function $w_d - w_d^*$ takes every value in the interval*

$$\left[0, \frac{d}{2} + \frac{d-2}{4(d-1)}\right).$$

This corollary is established following the main lines of the proofs of similar results established in [3, 4, 6]. We omit the details.

3 Proof of Theorem 1: construction of families of integer polynomials

For each integer $d \geq 3$, we construct a sequence of integer polynomials $P_{d,a}(x)$ of degree d and arbitrarily large height having two roots very close to each other, and whose coefficients are polynomials in the parameter a.

For $i \geq 0$, let c_i denote the ith Catalan number defined by

$$c_i = \frac{1}{i+1}\binom{2i}{i}.$$

The sequence of Catalan numbers $(c_i)_{i\geq 0}$ begins as

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

and satisfies by the recurrence relation

$$c_{i+1} = \sum_{k=0}^{i} c_k c_{i-k}, \quad \text{for } i \geq 0. \tag{1}$$

For integers $d \geq 3$ and $a \geq 1$, consider the polynomial

$$\begin{aligned} P_{d,a}(x) \;=\; & (2c_0 a x^{d-1} + 2c_1 a^2 x^{d-2} + \ldots + 2c_{d-2} a^{d-1} x)^2 \\ & - (4c_1 a^2 x^{2d-2} + 4c_2 a^3 x^{2d-3} + \ldots + 4c_{d-2} a^{d-1} x^{d+1}) \\ & + (4c_1 a^2 x^{d-2} + 4c_2 a^3 x^{d-3} + \ldots + 4c_{d-2} a^{d-1} x) \\ & + 4a x^{d-1} - 2x^d + 1, \end{aligned}$$

which generalizes the polynomials $P_{3,a}(x)$, $P_{4,a}(x)$, $P_{5,a}(x)$ given in Section 1. It follows from the recurrence (1) that $P_{d,a}(x)$ has degree exactly d, and not $2d-2$, as it seems at a first look. Furthermore, we check that the height of $P_{d,a}(x)$ is given by the coefficient of x^2, that is,

$$H(P_{d,a}) = 4c_{d-2}^2 a^{2d-2} + 4c_{d-3}a^{d-2}.$$

By applying the Eisenstein criterion with the prime 2 on the reciprocal polynomial $x^d P_{d,a}(1/x)$, we see that the polynomial $P_{d,a}(x)$ is irreducible. Indeed, all the coefficients of $P_{d,a}(x)$ except the constant term are even, but its leading coefficient, which is equal to $4c_{d-1}a^d - 2$, is not divisible by 4.

Writing

$$g = g(a,x) = 2c_0 a x^{d-1} + 2c_1 a^2 x^{d-2} + \ldots + 2c_{d-2}a^{d-1}x,$$

we see that

$$P_{d,a}(x) = (1+g)^2 + x^d(4ax^{d-1} - 2(1+g)).$$

Rouché's theorem shows that $P_{d,a}(x)$ has exactly two roots in the disk centered at the origin and of radius $1/2$. Clearly, $(1+g)^2$ has a double root, say x_0, close to $-1/(2c_{d-2}a^{d-1})$; thus the polynomial $P_{d,a}(x)$ has two distinct roots close to this point. Indeed, the term $x^d(4ax^{d-1} - 2(1+g))$ is a small perturbation when x is near x_0. Evaluating $P_{d,a}(x)$ at $x_0 + \varepsilon$, we end up with the quadratic equation

$$(2c_{d-2}a^{d-1})^2\varepsilon^2 - 2x_0^d(2c_{d-2}a^{d-1})\varepsilon + 4ax_0^{2d-1} = 0,$$

and we get that ε, i.e. the root separation, is approximately

$$2 \cdot \frac{1}{4c^2a^{2d-2}} \cdot \frac{4c_{d-2}a^{d-1/2}}{(2c_{d-2}a^{d-1})^{d-1/2}} = \frac{1}{2^{d-3/2} \cdot c_{d-2}^{d+1/2} \cdot a^{d^2-d/2-1}}.$$

Hence, we have $H(P_{d,a}) = O(a^{2d-2})$, while $\operatorname{sep}(P_{d,a}) = O(a^{-(d^2-d/2-1)})$. This gives

$$e_{\mathrm{irr}}(d) \geq \frac{2d^2-d-2}{4(d-1)} = \frac{d}{2} + \frac{d-2}{4(d-1)},$$

and the proof of Theorem 1 is complete. ∎

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