# PERSISTENT HOMOLOGY ON MUSICAL BARS


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*Mathematics and Computation in Music, Atlanta, Georgia, 21-24 June 2022*


## PERSISTENT HOMOLOGY

### Filtration and Persistence

- **Filtered simplicial complex** (figure 1): $\emptyset = K^{-1} \subset K^0 \subset \ldots \subset K^N = K$.
- **Persistent Homology**: computing simplicial homology $H_*(K^i)$ over $\mathbb{F}_2$ for each time $i$.
- **Barcodes** (figure 6): graph where the horizontal axis is progress in the filtration and a bar that starts at time $s$ and ends at time $t$ is a generator of $H_*(K^s)$ that is still one for $H_*(K^{t-1})$ but not at time $t$.

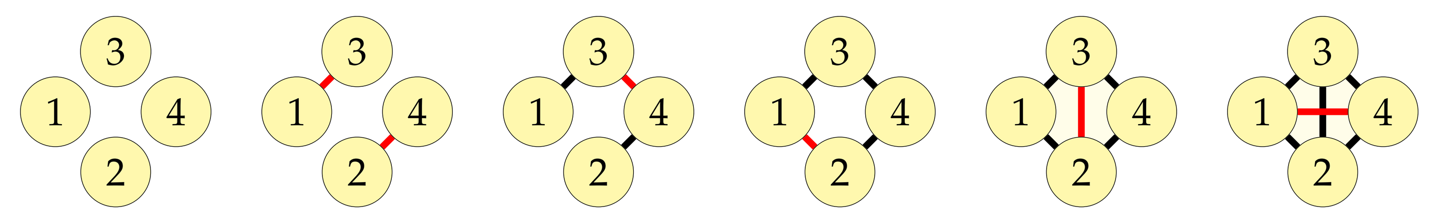
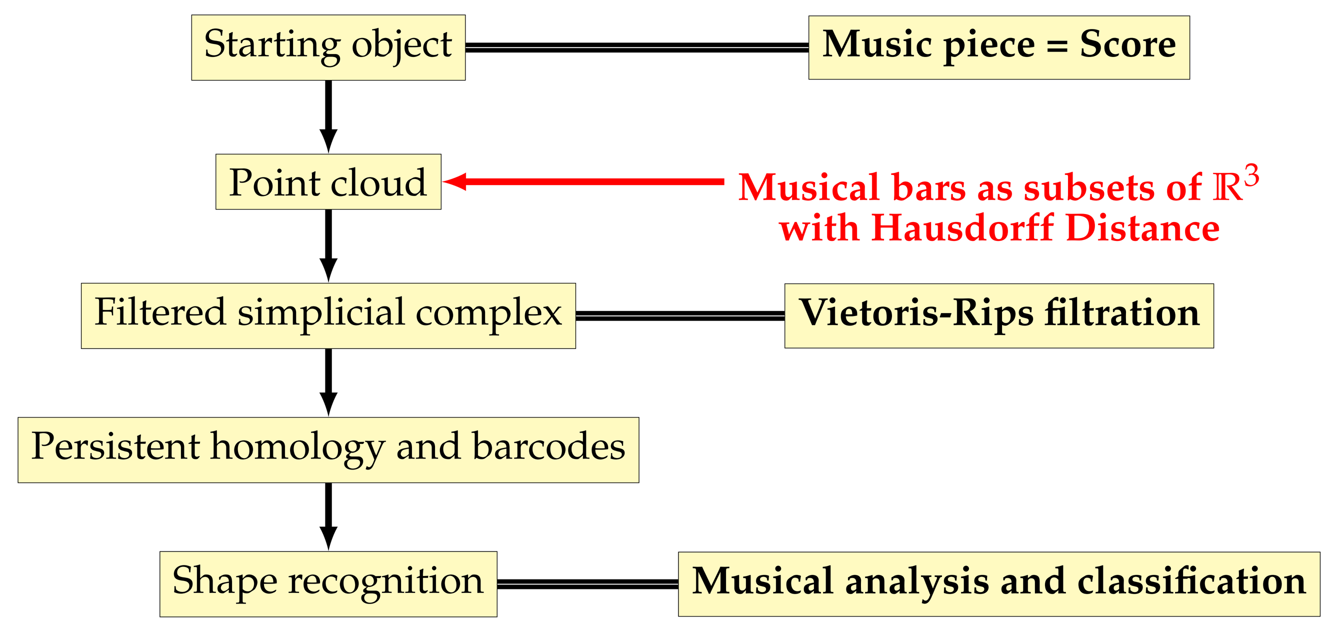


**Figure 1** – A filtered complex with 6 times of filtration.


**Figure 2** – Barcodes for filtration of figure 1 in degree 0 (left) and degree 1 (right).

### Context

**Topological Data Analysis:**




**How should we associate a filtered complex with a given musical piece?**

## APPLICATION TO THE MUSICAL BARS OF A SCORE

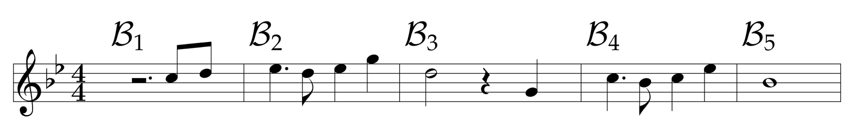
### Musical Bars of a Score

**Definition.** A **musical bar** is a finite subset $\mathcal{B}$ of $\mathbb{R}^3$ where an element of $\mathcal{B}$ is called a **note** characterized by three coordinates:

- the **position**, which refers to its place in the bar
- the **duration**, expressed in beats
- the **pitch**, which is the value of the note in term of its fundamental frequency

**Definition.** A **score** $\mathcal{S}$ is a finite set of **distinct bars**.

**Example.** A score $\mathcal{S} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\}$ with 5 distinct bars and its description:




$$\mathcal{B}_1 = \{(3, 1/2, C_5),\ (7/2, 1/2, D_5)\}$$
$$\mathcal{B}_2 = \{(0, 3/2, E_5\flat),\ (3/2, 1/2, D_5),\ (2, 1, E_5\flat),\ (3, 1, G_5)\}$$
$$\mathcal{B}_3 = \{(0, 4, D_5)\}$$
$$\mathcal{B}_4 = \{(0, 3/2, C_5),\ (3/2, 1/2, B_4\flat),\ (2, 1, C_5),\ (3, 1, E_5\flat)\}$$
$$\mathcal{B}_5 = \{(0, 4, B_4\flat)\}$$

### Filtration: the Vietoris-Rips Complex

**Definition.** Let $X = \{x_1, \ldots, x_n\}$ be a **point cloud** and $\epsilon \geq 0$ be a fixed **parameter**. The **Vietoris-Rips complex** $\mathcal{R}_\epsilon(X)$ is the simplicial complex where

- the set of vertices is $X = \{x_1, \ldots, x_n\}$
- $\sigma = \{x_1, \ldots, x_k\}$ is a $k$-simplex *iff* $d(x_i, x_j) \leq \epsilon \ \ \forall (x_i, x_j) \in \sigma^2$.

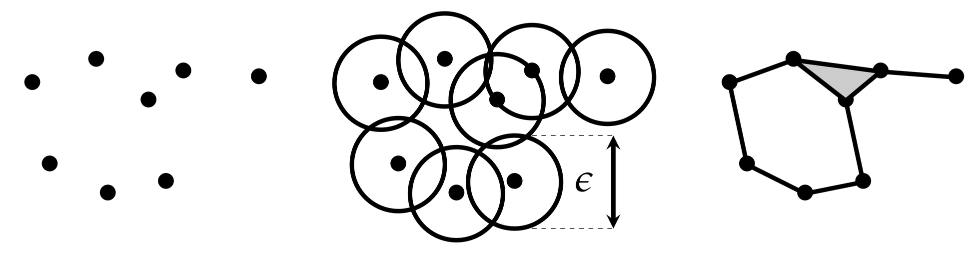



**Figure 3** – The Vietoris-Rips complex $R_\epsilon(X)$.

→ For two given parameters $\epsilon$ and $\epsilon'$ with $\epsilon \leq \epsilon'$, there is an inclusion

$$R_\epsilon(X) \hookrightarrow R_{\epsilon'}(X)$$

and we get a filtered complex by **increasing** the paramater $\epsilon$.

→ The time is **discretized** by setting $\epsilon = t\rho$, with $t \in \{0, 1, \ldots, 100\}$ and $\rho$ is a fixed constant. We call $\epsilon$ the **error margin (e.m.)**.

### The Score as a Point Cloud

**Definition.** Let $\mathcal{B}_i$ and $\mathcal{B}_j$ be two musical bars. The Hausdorff distance $d_H$ between $\mathcal{B}_i$ and $\mathcal{B}_j$ is defined by

$$d_H(\mathcal{B}_i, \mathcal{B}_j) = \max \left\{ \max_{n_i \in \mathcal{B}_i} \min_{n_j \in \mathcal{B}_j} d_1(n_i, n_j);\ \max_{n_j \in \mathcal{B}_j} \min_{n_i \in \mathcal{B}_i} d_1(n_i, n_j) \right\}$$

where $d_1(x, y) = \|x - y\|_1 = \sum_i |x_i - y_i|$.

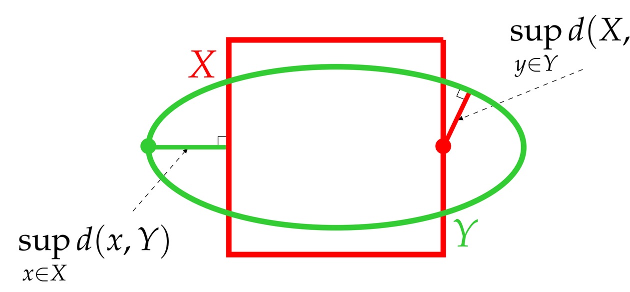



**Figure 4** – Hausdorff distance for two metric spaces $(X, d)$ and $(Y, d)$.

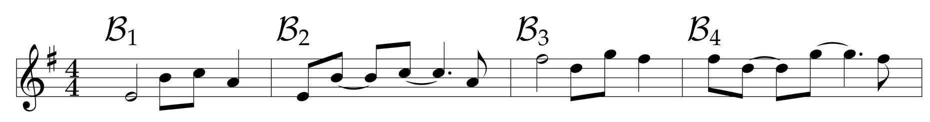



**Figure 5** – The score associated with the filtration and barcodes of figures 1 and 6.

## ANALYSIS OF A MUSICAL PIECE

### The Piece

→ The musical piece that is analysed is taken from the soundtrack of the French movie *Le Fabuleux Destin d'Amélie Poulain*, directed by Jean-Pierre Jeunet (2001).

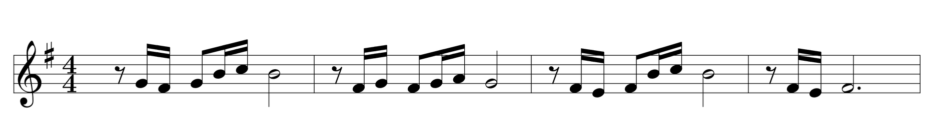
→ The music is ***Comptine d'un autre été: l'Après-midi*** for piano, composed and played by the minimalist composer Yann Tiersen (2001).

### Structure of the Score

The score has 53 musical bars but it only contains 39 **distinct bars**, and it is split in two parts. It is constructed in the following way:

- 3 different themes in the first part that are played again one octave higher in the second one.
- all the melody is constructed over 4 musical bars that are repeated and which constitute the musical accompaniment
- the piece ends with an *Em* chord played with whole notes.

**First Theme**

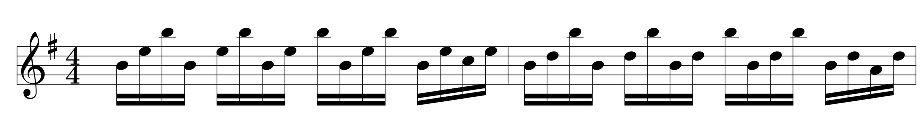


$\mathcal{B}_5$ to $\mathcal{B}_8$ and $\mathcal{B}_{22}$ to $\mathcal{B}_{25}$

**Second Theme**

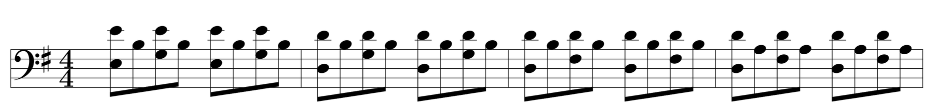
$\mathcal{B}_9$ to $\mathcal{B}_{16}$ and $\mathcal{B}_{26}$ to $\mathcal{B}_{33}$

**Third Theme**



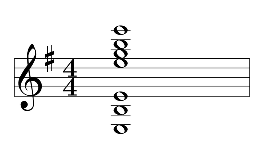
$\mathcal{B}_{17}$ to $\mathcal{B}_{21}$ and $\mathcal{B}_{34}$ to $\mathcal{B}_{38}$

**Accompaniment**



$\mathcal{B}_1$ to $\mathcal{B}_4$

**Ending**



$\mathcal{B}_{39}$

### Barcodes in degrees 0 and 1

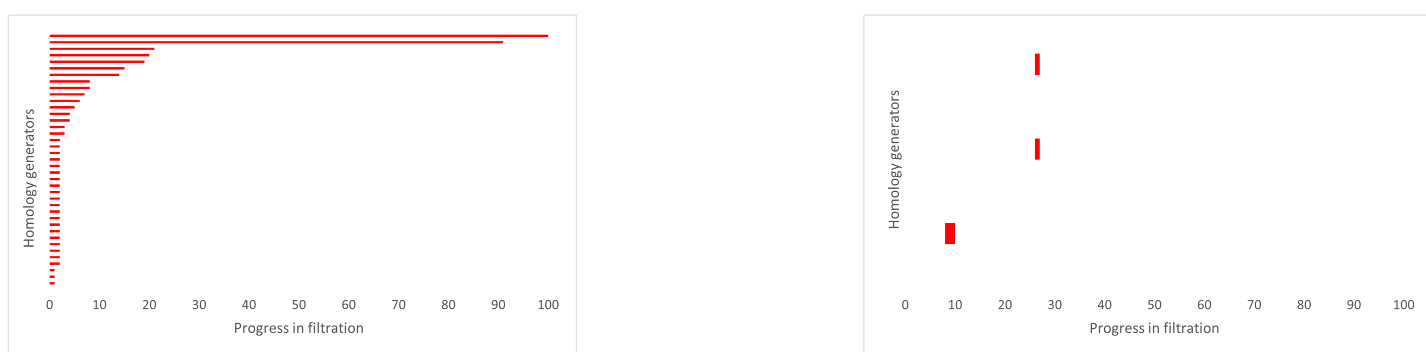



**Figure 6** – Barcodes for *Comptine d'un autre été: L'Après-midi* in degree 0 (left) and degree 1 (right).

### Associated Complex at 14%

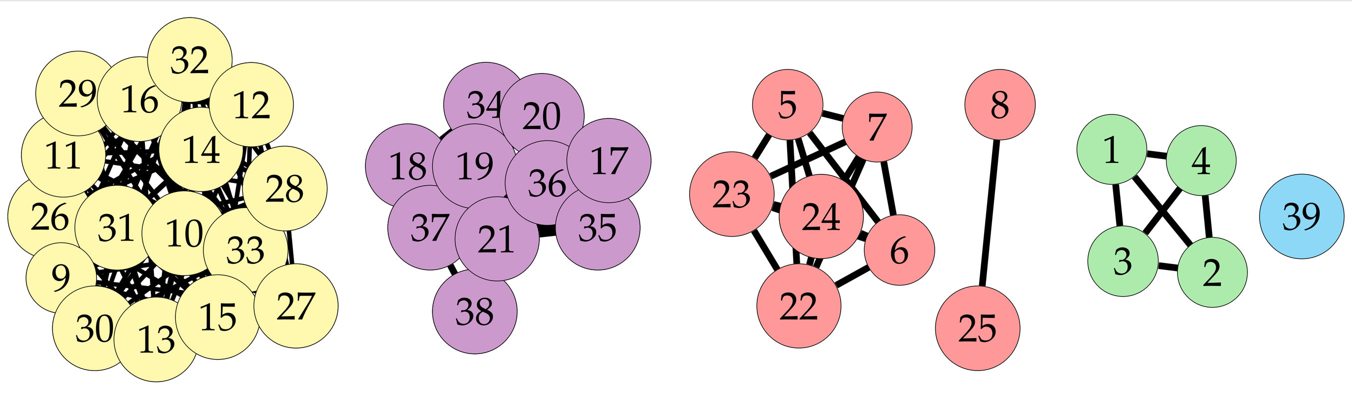


**Figure 7** – The associated complex with an **e.m.** of 14%: each component characterizes a theme of the piece.

### One-dimensional cycles

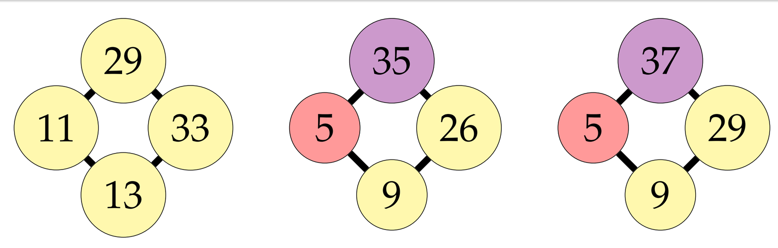


**Figure 8** – One-dimensional cycles from that appear with an **e.m.** of 8% (left) and 26% (middle and right).

### Analysis, interpretation and future work

**General idea for reading barcodes:**

- Several level of analysis depending of the error margin we choose to take.
- Focus on the largest bars which *persist* while the smallest ones are considered as *noise*.

**Barcodes in degree 0:**

- For $t \geq 21\%$: only 2 connected components, one corresponds to the last musical bar $\mathcal{B}_{39}$ and the other is a large dimensional complex where all musical bars are connected together.
- For $t \leq 8\%$: only small bars that we ignore as noise.
- For $8\% \leq t \leq 21\%$: there are 5, 6 or 7 classes. For $t = 14\%$, the associated complex from figure 7: 6 **connected components where each one corresponds to a theme of the song**.

**Barcodes in degree 1:**

- 3 different one-dimensional cycles: some edges linked musical bars of one given theme to the same one octave higher, but not systematically → not an obvious musical interpretation.

**Conclusion and prospect:**

- This approach can capture the global structure of the piece using barcode in degree 0.
- Highlight repeating patterns or musical loops in the score using degree 1.
- Apply this method to a more general and diverse corpus of music data to compare and see in which way it can capture the global structure of a music piece.
- Try new distances to analyze other characteristics of the score.