Gluing cluster tilting objects on surfaces

Recollement des objets amas-basculants sur une surface

Rencontre de l'ANR Charms, Strasbourg

Merlin Christ, IMJ-PRG

May 30, 2024

This work was funded by the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101034255.

Contents

1. Recap: rank 2 cluster categories of unpunctured surfaces
2. Gluing cluster tilting subcategories
3. Examples: higher rank and perverse schobers

Disclaimer: we will skip a few technical points.

Higgs category

Let (Q, F, W) be an ice quiver ($F \subset Q$ frozen subquiver) with potential W. Let $A_{Q,F}$ be the corresponding cluster algebra with coefficients arising from F.

Theorem (Yilin Wu)

There is a Frobenius extriangulated category $\mathcal{H}$, called the Higgs category, with the following properties:

- *$\mathcal{H}$ has a canonical cluster-tilting object with endomorphism algebra kQ. The extriangulated category $\mathcal{H}$ thus categorifies $A_{Q,F}$.*
- *The 'usual' triangulated 2-CY cluster category is the stable category (=localization) of $\mathcal{H}_S$.*

For marked surfaces

Let S be an oriented surface with nonempty boundary and markings $M \subset \partial S$. For each triangulation of S have an ice quiver with potential (Q, F, W). We denote the corresponding Higgs category by $\mathcal{H}_S$.

Theorem (C.)

- *The Higgs category $\mathcal{H}_S$ is equivalent as an extriangulated category to the 1-periodic topological Fukaya category of S, hence also*

$$\mathcal{H}_S \simeq \mathcal{D}^{\mathrm{perf}}(\mathrm{gentle}) \otimes^{\mathrm{dg}} \underbrace{\mathcal{D}^{\mathrm{perf}}(\Pi_2(A_1))/\mathcal{D}^{\mathrm{fin}}(\Pi_2(A_1))}_{=:\mathcal{C}_1}$$

  Here $\mathcal{C}_1$ is the 1-CY cluster category of type A_1 and equivalent to the 1-periodic derived ∞-category of A_1.

- *$\mathcal{H}_S$ admits a relative right 2-Calabi–Yau structure which induces the extriangulated structure.*

More details on the relative Calabi–Yau structure

The restriction to boundary functor

$$F \colon \mathcal{H}_{\mathsf{S}} \longrightarrow \prod_{\pi_0(\partial \mathsf{S} \backslash M)} \mathcal{C}_1$$

is right 2-Calabi–Yau.

We call an extension $\alpha \colon X \to Y[1]$ in $\mathcal{H}_{\mathsf{S}}$ exact if $F(\alpha) = 0$. This determines the exact/extriangulated structure.

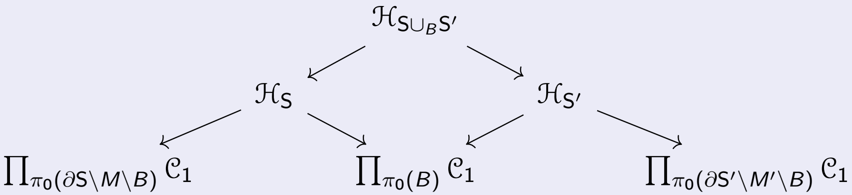
The image of the right adjoint of F gives the subcategory of injective-projective objects. Geometrically, these correspond to boundary arcs.

Gluing properties of $\mathcal{H}_{\mathsf{S}}$

Let S, S' be two marked surfaces, $B \subset \partial \mathsf{S} \backslash M, \partial \mathsf{S}' \backslash M'$ a common boundary component.

Corollary (C.)

There is a limit/pullback diagram of ∞-categories:



This is further a composition of right Calabi–Yau spans.

Note: if $\mathsf{S} \cup_B \mathsf{S}'$ has sufficient marked points, then the functors $\mathcal{H}_{\mathsf{S}\cup_B \mathsf{S}'} \to \mathcal{H}_{\mathsf{S}}, \mathcal{H}_{\mathsf{S}'}$ have fully faithful right adjoints.

Cluster tilting objects

Let $\mathcal{D}$ be an extriangulated category (typically Hom-finite, 2-CY).

Definition

An additive subcategory $\mathcal{T} \subset \mathcal{D}$ is called cluster tilting if

- $\mathcal{T}$ is rigid: $\mathbb{E}(T, T') \simeq 0$ for all $T, T' \in \mathcal{T}$.
- $\mathcal{T}$ has the 2-term resolution property: every object $X \in \mathcal{D}$ is part of an exact sequence $X \to T_0 \to T_1$ with $T_0, T_1 \in \mathcal{T}$.

Gluing context

Consider a pullback diagram

$$
\begin{array}{ccccc}
\mathcal{D} & \xrightarrow{j_1^L} & \mathcal{D}_1 & \longrightarrow & \mathcal{C}_1 \\
{\scriptstyle j_2^L}\downarrow & \lrcorner & \downarrow{\scriptstyle i_1^L} & & \\
\mathcal{D}_2 & \xrightarrow[i_2^L]{} & \mathcal{C}_2 & & \\
\downarrow & & & & \\
\mathcal{C}_3 & & & &
\end{array}
$$

describing a composition of right 2-Calabi–Yau spans and with the right adjoints j_1, j_2, i_1, i_2 fully faithful.

Let $\mathcal{T}_1 \subset \mathcal{D}_1$, $\mathcal{T}_2 \subset \mathcal{D}_2$ additive subcategories such that $i_1(\mathcal{C}_2) \subset \mathcal{T}_1$ and $i_2(\mathcal{C}_2) \subset \mathcal{T}_2$. Define $\mathcal{T} \coloneqq j_1(\mathcal{T}_1) \cup j_2(\mathcal{T}_2) \subset \mathcal{D}$

Gluing cluster tilting subcategories

$$
\begin{array}{ccccc}
\mathcal{D} & \xrightarrow{j_1^L} & \mathcal{D}_1 & \longrightarrow & \mathcal{C}_1 \\
{\scriptstyle j_2^L}\downarrow & \lrcorner & \downarrow{\scriptstyle i_1^L} & & \\
\mathcal{D}_2 & \xrightarrow[i_2^L]{} & \mathcal{C}_2 & & \\
\downarrow & & & & \\
\mathcal{C}_3 & & & &
\end{array}
$$

Let $\mathcal{T}_1 \subset \mathcal{D}_1$, $\mathcal{T}_2 \subset \mathcal{D}_2$, $\mathcal{T} := j_1(\mathcal{T}_1) \cup j_2(\mathcal{T}_2) \subset \mathcal{D}$ as before.

Proposition (In preparation, C.)

If $\mathcal{T}_1 \subset \mathcal{D}_1$, $\mathcal{T}_2 \subset \mathcal{D}_2$ have the 2-term resolution property, then so does $\mathcal{T} \subset \mathcal{D}$.

Can also show rigidity of $\mathcal{T}$ given rigidity of $\mathcal{T}_1, \mathcal{T}_2$ and further assumptions, thus showing $\mathcal{T}$ is cluster tilting.

Proof idea

Abstract nonsense: for every $X \in \mathcal{D}$, there is a pullback diagram in $\mathcal{D}$:

$$
\begin{array}{ccccc}
X & \xrightarrow{\text{unit}} & j_1 j_1^L(X) & & \\
{\scriptstyle \text{unit}}\downarrow & \square & \downarrow{\scriptstyle \text{unit}} & & \\
j_2 j_2^L(X) & \xrightarrow{\text{unit}} & j_1 i_1 i_1^L j_1^L(X) & = & j_2 i_2 i_2^L j_2^L(X)
\end{array}
$$

We choose 2-term resolutions of $j_i^L(X)$ by $\mathcal{T}_i$ in $\mathcal{D}_i$, $i = 1, 2$. Applying j_i, we obtain resolutions of $j_i j_i^L(X)$ by $\mathcal{T}$ in $\mathcal{D}$. These combine to a 2-term resolution of X.

Examples from the theory of perverse schobers on surfaces

Perverse schobers are perverse sheaves of higher categories (due to Kapranov-Schechtman). Given a spanning ribbon graph G of marked surface S, can define a perverse schober by the assignment

$$\begin{aligned} \textit{trivalent vertex } v &\mapsto \mathit{Fun}(A_2, \mathcal{C}_1) \simeq \mathcal{D}^{\mathrm{perf}}(A_2) \otimes^{\mathrm{dg}} \mathcal{C}_1 \\ \textit{edge } e &\mapsto \mathcal{C}_1 \\ \textit{edge, vertex intersect} &\mapsto (\mathcal{C}_1 \hookrightarrow \mathit{Fun}(A_2, \mathcal{C}_1)) \end{aligned}$$

This produces gluing contexts for cluster tilting objects as above, reproving the existence of cluster tilting objects in the Higgs category $\mathcal{H}_S$.

Marked surfaces with punctures

Consider ribbon graph with 1-valent vertices at the punctures.
Associate to 1-valent vertex of ribbon graph the 2-periodic derived category $\mathcal{D}^{\mathrm{perf}}(k[t_2^{\pm}])$, with $|t_2| = 2$ and $k[t_2^{\pm}]$ the graded Laurent algebra.
There is a right 2-Calabi–Yau functor (right adjoint not fully faithful)

$$F : \mathcal{D}^{\mathrm{perf}}(k[t_2^{\pm}]) \longrightarrow \mathcal{C}_1 \,.$$

To obtain a fully faithful right adjoint, can replace 1-valent vertex by 2-valent vertex with value $\mathcal{D}^{\mathrm{perf}}(k[t_2^{\pm}]) \times_F^{\rightarrow} \mathcal{C}_1$ (recollement/gluing of dg-categories along functor).

Upshot: obtain cluster tilting objects on categories for punctures surfaces (conjecturally equivalent to Higgs categories).

Higher rank cluster categories of surfaces

We replace the 1-CY cluster category $\mathcal{C}_1$ of type A_1 by the 1-CY cluster category of type ADE.

Miantao Liu (c.f. talk tomorrow): there exists a cluster tilting object if S is the marked triangle. By gluing, we can thus obtain cluster categories categorifying the higher rank cluster algebras of surface in the sense of Fock–Goncharov and Goncharov–Shen.

Interesting direction: describe rigid objects/cluster tilting objects in terms of webs in the surface. Relate this with higher rank Skein algebras.

Further question: do gluing techniques apply to study the cluster categories of closed surfaces? (In progress.)