Examples of semidistributive lattices

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Definition (Dickson 1962)

A torsion pair for $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ consisting of two subcategories of $\mathcal{A}$ such that :

1. $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for every $T \in \mathcal{A}$ et $F \in \mathcal{F}$.
2. For every $X \in \mathcal{A}$, there is $t(X) \in T$ and $f(X) \in \mathcal{F}$ and a short exact sequence

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Example : $\mathcal{A} = Ab$, $\mathcal{T}$ is the class of torsion groups and $\mathcal{F}$ the class of torsionfree groups. There are (a lot) more torsion pairs on this category !

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If $(\mathcal{T}, \mathcal{F})$ is a torsion pair for $\mathcal{A}$, it follows :

- $\mathcal{F} = \mathcal{T}^{\perp} = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(T, X) = 0 \ \forall T \in \mathcal{T}\}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$.

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If $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$ are two torsion pairs for $\mathcal{A}$, we set $(\mathcal{T}, \mathcal{F}) \leq (\mathcal{T}', \mathcal{F}')$ if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{F}' \subseteq \mathcal{F}$.

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We also have :

- $\operatorname{Tors}(\mathcal{A})$ the poset of torsion classes of $\mathcal{A}$.
- $\operatorname{Torsf}(\mathcal{A})$ the poset of torsionfree classes of $\mathcal{A}$.

In representation theory

A be a finite dimensional algebra over a field.

Definition (BGP, APR,BB,H,...)

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$T \in \operatorname{mod} A$ is a τ-tilting module if :

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Theorem (DIRRT 2017)

The poset of torsion classes of $\text{mod } A$ is a semidistributive lattice.

Distributive lattices

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Definition

A lattice $(L, \leqslant)$ is distributive if for every $a, b, c \in L$

1. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and
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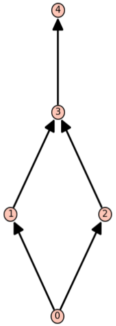

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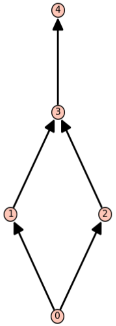

$(P, \leqslant)$ is a poset. $I \subseteq P$ is an ideal of P if

$$x \in I, \quad y \leqslant x \Rightarrow y \in I.$$

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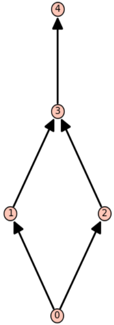

$(P, \leqslant)$ is a poset. $I \subseteq P$ is an **ideal** of P if

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$(\mathrm{Ideal}(P), \subseteq)$ is a **distributive lattice**.

Theorem (Birkhoff 1947)

A finite lattice is distributive if and only if it is isomorphic to $\mathrm{Ideal}(P)$ *for a finite poset* P.

Semidistributive lattices

Definition (Jònsson 1961)

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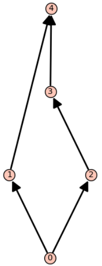

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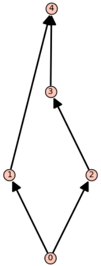

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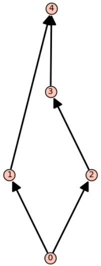

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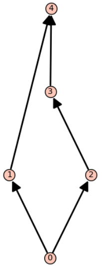

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- What about Birkhoff theorem ?

Fondamental thm of semidistributive lattices

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- New relations $\twoheadrightarrow$ by $x \twoheadrightarrow y$ if $\forall y \to z$, we have $x \to z$ and $\hookrightarrow$ dually. $(\twoheadrightarrow, \hookrightarrow) = \text{Fac}(\to)$.

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- $\mathrm{Mult}(\twoheadrightarrow, \hookrightarrow)$ is the relation R given by xRy if $\exists z$ with $x \twoheadrightarrow z \hookrightarrow y$.
- Then (Ш$, \to, \twoheadrightarrow, \hookrightarrow)$ is a factorization system if $\to\; = \mathrm{Mult}(\twoheadrightarrow, \hookrightarrow)$ and $(\twoheadrightarrow, \hookrightarrow) = \mathrm{Fac}(\to)$.

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Theorem (Reading, Speyer, Thomas 2019)

A finite latice L is semidistributive if and only if it is isomorphic to $\text{Pairs}(\text{Ш})$ *for a 2-acyclic factorization system* $(\text{Ш}, \rightarrow, \twoheadrightarrow, \hookrightarrow)$.

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- The relations should be roughly be thought of as analogous to, ‘there is a nonzero map’ or ‘a sujerctive map’ or ‘an injective map’ in an abelian category

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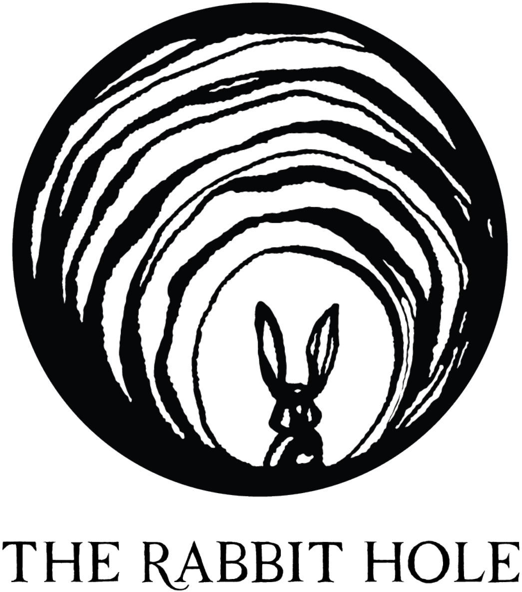

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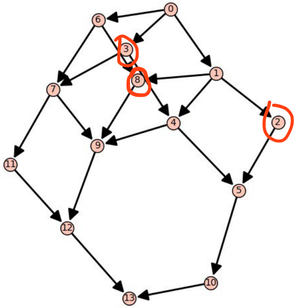

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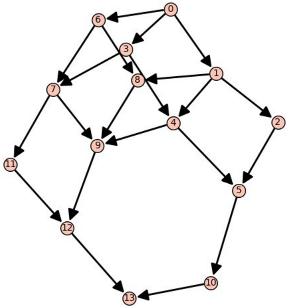

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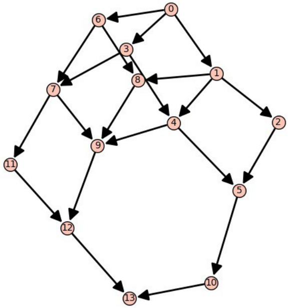

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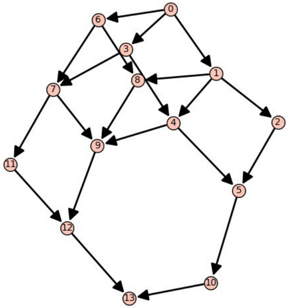
- Torsf$(\mathcal{A})$ wants to be join-semidistributive lattice.

Intuition or mathematical statement ?

- Problem 1 : there are abelian categories $\mathcal{A}$ such that $\mathrm{Torsp}(\mathcal{A})$ is not semidistributive.
- Example [IK 2021] $\mathrm{Torsp}(\mathcal{A})$ where $\mathcal{A}$ is the category of fg modules over a noetherian algebra.



- $\mathrm{Tors}(\mathcal{A})$: wants to be meet-semidistributive lattice
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- Idea : a non-split extension between two simple objects generates a pentagon in the lattice of torsion pairs.

Solution 1 : inspired by Adachi, Enomoto and Tsukamoto

Definition

A torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is an ω-torsion pair if $\mathrm{Ext}^1(\mathcal{T}, \mathcal{F}) = 0$.

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Theorem (AET 2021, R-)

Let A be an artin algebra and $(\mathcal{T}, \mathcal{F})$ be a torsion pair of mod A. *The following are equivalent :*

1. $(\mathcal{T}, \mathcal{F})$ *is an ω-torsion pair.*
2. $\mathcal{T}$ *and $\mathcal{F}$ are two Serre subcategories.*
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Corollary

- *The set of all ω-torsion pairs of $\operatorname{mod} A$ is a distributive sublattice of $\operatorname{Torsp}(\mathcal{A})$;*
- *Let $L \cong \operatorname{Ideal}(P)$ be a finite distributive lattice. Then L is isomorphic to the lattice of ω-torsion pairs of the incidence algebra of P.*

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So there are finite semidistributive lattices which are not sublattice of $\text{Torsp}(\mathcal{A})$ for $\mathcal{A}$ abelian length category !

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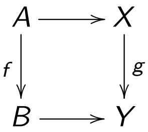

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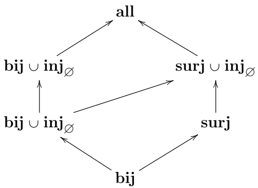

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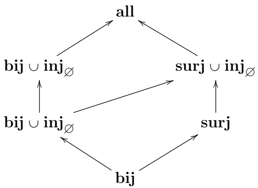

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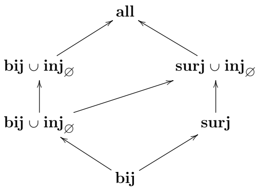

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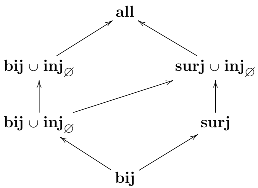

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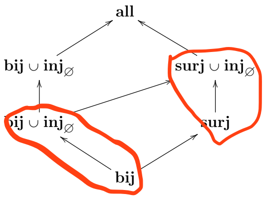

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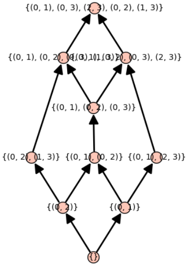
Example of the boolean lattice $\mathcal{P}([2])$.



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Let $(L, \leqslant)$ be a lattice. A transfer system $\lhd$ for L is a relation of partial ordering on L such that :

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Theorem (Quadrelli 2019)

The map sending a transfer system $\mathcal{R}$ to $({}^{\boxslash}\mathcal{R}, \mathcal{R})$ is an isomorphism between the poset of transfer systems and the poset of weak factorization systems.

Joint work with Yongle Luo

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Let Λ_n be the path algebra of an equioriented quiver of type A.

Theorem (Crawley-Boevey, Ma, R-,Sauter 2020)

- *There are* $[n]_2! := \prod_{i=1}^n (2^i - 1)$ *basic faithfully balanced modules for* Λ_n.
- *A fb-module has at least* n *indecomposable summands.*
- *The number of minimal fb-modules is* $n!$.

A poset of fb-modules

If M and N are two minimal fb-modules over Λ_n we define

$$M \leqslant N \iff (\mathrm{Fac}(M), \mathrm{Sub}(M)) \preceq (\mathrm{Fac}(N), \mathrm{Sub}(N)).$$

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Let $(fb(n), \leqslant)$ be the poset of minimal fb-modules over Λ_n. Then

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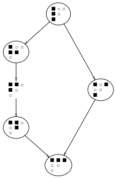

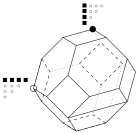

Joint work with Corteel and Jang

Theorem

The lattice $(fb(n), \leqslant)$ *is semidistributive*

Joint work with Corteel and Jang

Theorem

The lattice $(fb(n), \leqslant)$ is semidistributive and trim.

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good knowledge of join-irreducibles, covers, spine etc.

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The lattice $(fb(n), \leqslant)$ is congruence uniform.

Joint work with Corteel and Jang

Theorem

The lattice $(fb(n), \leqslant)$ *is* *semidistributive* *and* *trim.*

good knowledge of join-irreducibles, covers, spine etc.

Conjecture

The lattice $(fb(n), \leqslant)$ *is* *congruence uniform.*

$fb(3)$ is the poset of torsion pairs of a hereditary algebra of type B_2

Joint work with Corteel and Jang

Theorem

The lattice $(fb(n), \leqslant)$ is semidistributive and trim.

good knowledge of join-irreducibles, covers, spine etc.

Conjecture

The lattice $(fb(n), \leqslant)$ is congruence uniform.

$fb(3)$ is the poset of torsion pairs of a hereditary algebra of type B_2 but what about $fb(4)$?