q-analogues of Bernoulli numbers & zeta operators at negative integers

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Usual Bernoulli numbers

The **Bernoulli numbers** are given by the generating series

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x}{\exp(x)-1}.$$

This can be restated as

$$exp(x+Bx) - exp(Bx) = x$$

by using the **umbral** (symbolic) convention $B^n = B_n$.
By Taylor expansion, one finds

$$(B+1)^n - B^n = \begin{cases} 1 & \text{if} \quad n=1, \\ 0 & \text{else.} \end{cases}$$

Usual Bernoulli numbers

$$(B+1)^n - B^n = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{else.} \end{cases}$$

One can use this equation to compute the Bernoulli numbers :

$$1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66, 0, -691/2730, \\ 0, 7/6, 0, -3617/510, 0, 43867/798, 0, -174611/330, \ldots$$

The numbers B_{2n+1} vanish when $n \geq 1$.
Rational numbers, with important properties, well-known in number theory.
Used in the Euler–Maclaurin summation formula.
Related to values of the Riemann zeta function at negative integers.

Riemann ζ function

The **Riemann ζ function** is defined for $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - \frac{1}{p^s}},$$

where the product runs over the set P of prime numbers.
It can be extended to a meromorphic function on $\mathbb{C}$ with unique pole at $s = 1$.
Euler has computed the values at negative integers :

$$\zeta(1 - n) = \frac{-B_n}{n},$$

for $n \geq 2$.

Carlitz q-Bernoulli numbers

Leonard Carlitz has introduced (in 1948) q-analogues of Bernoulli numbers defined by the initial value $\beta_0 = 1$ and the formula

$$q(q\beta + 1)^n - \beta^n = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases}$$

with the convention that $\beta^n = \beta_n$. This gives the following fractions

$$\beta_0 = 1, \quad \beta_1 = -\frac{1}{\Phi_2}, \quad \beta_2 = \frac{q}{\Phi_2\Phi_3},$$
$$\beta_3 = \frac{q(1-q)}{\Phi_2\Phi_3\Phi_4}, \quad \beta_4 = \frac{q(q^4 - q^3 - 2q^2 - q + 1)}{\Phi_2\Phi_3\Phi_4\Phi_5},$$

where Φ_n are cyclotomic polynomials.

Carlitz q-Bernoulli numbers

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30, \dots$$

$$\beta_0 = 1, \quad \beta_1 = -\frac{1}{\Phi_2}, \quad \beta_2 = \frac{q}{\Phi_2\Phi_3}, \quad \beta_3 = \frac{q(1-q)}{\Phi_2\Phi_3\Phi_4},$$
$$\beta_4 = \frac{q(q^4 - q^3 - 2q^2 - q + 1)}{\Phi_2\Phi_3\Phi_4\Phi_5}, \dots$$

q-analogues : Bernoulli numbers are recovered by letting $q = 1$.
denominator : a product of cyclotomic polynomials of order between 2 and $n+1$, with multiplicity at most one. Multiplicity can be zero (starting with Φ_3 absent in β_7).
numerator : a factor q for $n \geq 2$, a factor $1-q$ when $n \geq 3$ is odd, and a big (irreducible ?) factor.

Zeroes and poles

Nice pattern, that needs to be explained : many zeros on the circle, some on the positive real line, a few others

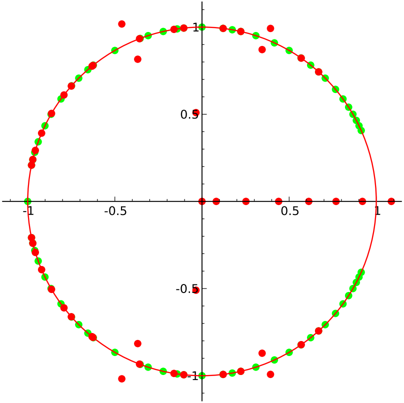

FIGURE: Roots ● and poles ● of the Carlitz q-Bernoulli number β_{14}

q-Bernoulli numbers are natural.

In the works of Carlitz, the q-Bernoulli numbers have been related to the q-Eulerian numbers.
They appear more recently in a completely different setting, involving Lie idempotents in the descent algebras of symmetric groups, dendriform algebras, pre-Lie algebras, etc.

As coefficients in a sum over rooted trees

$$
\begin{aligned}
\Omega_q = 1 - \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{q}{\Phi_2\Phi_3}\frac{1}{2} \\
- \frac{1}{\Phi_2\Phi_4} - \frac{q}{2\,\Phi_3\Phi_4} - \frac{q^2}{\Phi_2\Phi_3\Phi_4} - \frac{q(q-1)}{\Phi_2\Phi_3\Phi_4}\frac{1}{6} + \\
\frac{1}{\Phi_5} + \frac{q(1+q+q^2)}{2\,\Phi_2\Phi_4\Phi_5} + \frac{q^2}{\Phi_4\Phi_5} + \frac{q(q^3+q^2-1)}{6\,\Phi_3\Phi_4\Phi_5} + \frac{q^4}{2\,\Phi_3\Phi_4\Phi_5} + \\
\frac{q^3}{\Phi_2\Phi_4\Phi_5} + \frac{q^2(q^3+q^2-1)}{2\,\Phi_2\Phi_3\Phi_4\Phi_5} + \frac{q^2(q^3-q-1)}{2\,\Phi_2\Phi_3\Phi_4\Phi_5} + \\
\frac{q(q^4-q^3-2q^2-q+1)}{\Phi_2\Phi_3\Phi_4\Phi_5}\frac{1}{24} + \cdots
\end{aligned}
$$

CLAIM : The Carlitz q-Bernoulli numbers are **natural objects** !

QUESTION

Are they related to some kind of q-analogue of Riemann ζ function ?

Previous attempts of q-zeta function

One can find articles by many authors on various q-analogues of the Riemann ζ-function :

- Ivan Cherednik,
- Taekyun Kim
- Neal Koblitz,
- M. Kaneko, N. Kurokawa and M. Wakayama,
- Junya Satoh.

(not an exhaustive list)
They proposed many different functions as q-analogues of ζ.
BUT : They did not find any **simple relationship** with Carlitz q-Bernoulli numbers.
These functions do not have an Eulerian product.

q-analogue is a linear operator

Main Idea

The correct q-analogue of the value $\zeta_q(s)$ is not a complex number, but a linear operator on the vector space of formal power series in q.

Consider the space $\mathbb{C}[[q]]$ of formal power series in q.
For every integer $n \geq 1$, define a linear operator F_n by

$$F_n(f(q)) = f(q^n).$$

This is some kind of "Frobenius operator".

Key lemma

Now introduce the q-numbers :

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

Let s be any complex number.

CRUCIAL LEMMA

For every integers m and n, one has

$$\left(\frac{1}{[m]_q^s} F_m\right)\left(\frac{1}{[n]_q^s} F_n\right) = \frac{1}{[mn]_q^s} F_{mn} = \left(\frac{1}{[n]_q^s} F_n\right)\left(\frac{1}{[m]_q^s} F_m\right).$$

This is a q-analogue of the obvious fact that

$$\frac{1}{m^s}\frac{1}{n^s} = \frac{1}{(mn)^s} = \frac{1}{n^s}\frac{1}{m^s}.$$

Definition of q-zeta operators

One can now introduce the linear operator $\zeta_q(s)$:

$$\zeta_q(s) = \sum_{n \geq 1} \frac{1}{[n]_q^s} F_n,$$

for every $s \in \mathbb{C}$.
To ensure convergence, one has to restrict the domain to the space $q\mathbb{C}[[q]]$ of formal power series without constant term.
This operator can be factorised (by using the key lemma) :

$$\zeta_q(s) = \prod_{p \in P} (\mathrm{Id} - \frac{1}{[p]_q^s} F_p)^{-1},$$

which is the q-analogue of the Eulerian product for $\zeta(s)$.

Rationality at negative integers

For example, consider $\zeta_q(0)$ acting on q :

$$\zeta_q(0)q = \sum_{n \geq 1} \frac{1}{[n]_q^0} F_n q = \sum_{n \geq 1} q^n = q/(1-q).$$

Proposition

For every integer $j > 0$, and every integer $n \geq 0$, the formal power series $\zeta_q(-n)q^j$ is a rational fraction, *i.e.* belongs to $\mathbb{Q}(q)$.

This is obvious for $n = 0$, where one gets $q^j/(1-q^j)$.

q-analogue of Euler result

Proposition

For every integer $j > 0$, and every integer $n \geq 0$, the formal power series $\zeta_q(-n)q^i$ is a rational fraction with a pole at $q = 1$.

Theorem

For every every integer $n \geq 2$, there holds

$$\zeta_q(1-n)(q-(n+1)q^2) = \beta(n).$$

This formula is a q-analogue of the Euler formula

$$\zeta(1-n)(-n) = B_n,$$

relating Bernoulli numbers and values of ζ at negative integers.

Higher q-analogues

Taekyun Kim has considered some other q-analogues of Bernoulli numbers, similar to Carlitz q-Bernoulli numbers. Fix an integer $k \geq 1$. The k^{th} higher q-analogue is defined by $\beta_0 = \frac{k}{[k]_q}$ and

$$q^k(q\beta + 1)^n - \beta^n = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases}$$

For $k = 1$, they are Carlitz q-Bernoulli numbers.
One can show that they satisfy

$$\zeta_q(1-n)(kq^k - (n+k)q^{k+1}) = \beta(n).$$

q-zeta functions from q-zeta operator

One can interpret the q-zeta functions considered by several authors as

$$\zeta_q(s)q, \quad \zeta_q(s)q^t, \quad \zeta_q(s)q^s, \quad \zeta_q(s)q^{s/2}, \quad \zeta_q(s)q^{s-m}, \quad \zeta_q(s)q^{s-1}.$$

This does not quite fit in our framework of formal power series, unless the power of q is an integer.

A second variable enters.

One can turn $\zeta_q(s)$ into an operator on formal power series in two variables q and z by extending the "Frobenius operator" by

$$F_n(f(q,z)) = f(q^n, z^n).$$

Then $\zeta_q(s)$ makes sense as an operator on formal power series in q and z without constant term.

Proposition

For every integer $n \geq 0$, the formal power series $\zeta_q(-n)z$ is a rational fraction of q and z, *i.e.* belongs to $\mathbb{Q}(q,z)$.

For example,

$$\zeta_q(0)z = z/(1-z),$$
$$\zeta_q(-1)z = \frac{z}{(1-z)(1-qz)}.$$

The proof is by induction on n using the difference operator

$$\Delta(f(q,z)) = \frac{f(q,qz) - f(q,z)}{q-1},$$

which satisfies

$$\Delta(z^n) = [n]_q z^n$$

and therefore sends

$$\zeta_q(-n)z \mapsto \zeta_q(-n-1)z.$$

As Δ maps fractions to fractions, one gets that every $\zeta_q(-n)z$ is in $\mathbb{Q}(z,q)$.

These fractions have been considered before in the study of the symmetric groups. This is closely related to the original viewpoint of Carlitz.

Proposition

One has

$$\zeta_q(-n)z = \frac{\sum_{\sigma \in S_n} q^{\operatorname{maj}\sigma} z^{\operatorname{des}\sigma}}{\prod_{i=0}^{n} 1 - q^i z}$$

where $\operatorname{maj}, \operatorname{des}$ are the Major index and descent number of permutations.

The fraction $\zeta_q(-n)z$ is therefore a generating function for two parameters on the symmetric group S_n.

General Dirichlet series

The formalism above for the Riemann zeta function can be applied to any Dirichlet series.

$$L(s) = \sum_{n\geq 1} \frac{a_n}{n^s} \quad \longleftrightarrow \quad L_q(s) = \sum_{n\geq 1} \frac{a_n}{[n]_q^s} F_n.$$

If the Dirichlet series is multiplicative, $L_q(s)$ will have a factorisation, over the set P of prime numbers, as an operator.
This allows for example to define incomplete operators by removing a finite number of primes.
Also, for any two Dirichlet series L and L', the operators $L_q(s)$ and $L'_q(s)$ **commute** (by the key lemma).
But this is not true in general for $L_q(s)$ and $L'_q(t)$ with $s \neq t$.

One can show for L-series associated with **Dirichlet characters** that $L_q(-n)z$ is a rational fraction of q and z for every $n \geq 0$.

Generating series for these values $L_q(-n)z$ for $n \geq 0$ satisfy simple functional equations.

In a few cases, one can describe the numerator in a combinatorial way.
For example, in the case of the primitive Dirichlet character of conductor 4, the fractions $L_q(-n)z$ are related to the hyperoctahedral groups (Coxeter groups of type B/C)

Eisenstein series

There is also another q-zeta function, considered by Rivoal, Zudilin, Jouhet & Mosaki and others in transcendence theory :

$$\zeta_{q=1}(-k+1)\frac{z}{1-z} = \sum_{n\geq 1} n^{k-1}\frac{z^n}{1-z^n},$$

where q is taken to be 1.
This is related to the classical Eisenstein series (modular form) Ei_k whose associated Dirichlet series is

$$\zeta(s-k+1)\zeta(s)$$

This may suggest to consider

$$\zeta_q(-k+1)\zeta_q(0)z = \zeta_q(-k+1)\frac{z}{1-z}$$

as a q-analogue of the Eisenstein series.

Relation with Lambert series

A **Lambert series** is a sum of the following shape

$$\sum_{n\geq 1} a_n \frac{q^n}{1-q^n}.$$

This kind of series can be restated, using the associated operator

$$L_q(s) = \sum_{n\geq 1} \frac{a_n}{[n]_q^s} F_n,$$

as

$$L_q(0)\frac{q}{1-q} = L_q(0)\zeta_q(0)q.$$

q-analogues of polylogarithms

The usual polylogarithm function is defined by

$$\mathcal{L}_k(z) = \sum_{n \geq 1} \frac{z^n}{n^k}$$

This can be written as

$$\zeta_{q=1}(k)z$$

And therefore suggest the following (well-known) q-analogue

$$\zeta_q(k)z = \sum_{n \geq 1} \frac{z^n}{[n]_q^k}$$

The q-analogue of $\mathcal{L}_1$ has a nice functional equation, analogue of

$$\log(1 - x - y + xy) = \log(1 - x) + \log(1 - y)$$

Missing points, open directions

1 : back to q=1

How to deduce the classical results by letting q tends to 1 ?

2 : other explicit values

Find some other examples of closed evaluation (outside Dirichlet characters)

3 : functional equation, modularity, completed operator

the **functional equation** for the ζ operator
or the definition of a nice **Archimedean factor**
or some kind of q-analogue of modular forms

3 : zeta functions of orders

Understand the relation to genus zeta functions of orders (Louis Solomon, Marleen Denert)