HABILITATION À DIRIGER DES RECHERCHES

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Par Frédéric Chapoton

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SUR LE SUJET

OPÉRADES COMBINATOIRES ET ALGÈBRES AMASSÉES

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Introduction

La théorie des représentations et la combinatoire algébrique sont des domaines proches, et dont l'interaction est riche de découvertes. Le thème le plus classique qui soit représentatif de cette relation est celui des diagrammes de Young, lié d'une part aux modules sur les groupes symétriques et d'autre part à l'étude combinatoire des partitions d'entiers. Mes recherches contribuent à cet échange entre algèbre et combinatoire, en reliant certaines opérades et des objets combinatoires classiques, ou bien en introduisant de la combinatoire dans certaines structures en théorie des représentations.

Parmi les objets étudiés, certains sont comme un leitmotiv : ce sont les polytopes de Stasheff ou associaèdres. Depuis leur introduction en topologie algébrique, ils ont été reliés à des domaines de plus en plus nombreux et leur étude s'est déplacée vers la combinatoire et l'algèbre. Une partie de mes recherches est motivée plus ou moins directement par la volonté de comprendre ces objets, et il semble que leur richesse ne soit encore qu'effleurée.

La raison-d'être initiale des polytopes de Stasheff est leur usage dans la description de l'associativité à homotopie près. Le cadre théorique convenable est celui de la théorie des opérades et la géométrie sous-jacente est celle des espaces de modules de courbes.

Un aspect entièrement nouveau des associaèdres est apparu avec leur intervention dans la théorie des algèbres amassées introduite récemment par S. Fomin et A. Zelevinsky. Ici, le paysage est fourni par la théorie des représentations, les bases canoniques, les catégories dérivées et les carquois. La géométrie apparaît également, en particulier via le spectre des algèbres amassées et certaines variétés toriques.

Par ailleurs, sans forcément impliquer les polytopes de Stasheff, les nombres de Catalan (qui comptent leurs sommets) sont depuis longtemps un thème de base de la combinatoire énumérative. On ne compte plus les familles d'objets dénombrés par les nombres de Catalan. Plus récemment, on a commencé à voir des structures algébriques sur des espaces vectoriels ayant ces nombres pour dimensions, notamment dans l'étude des fonctions quasi-symétriques et dans celle des polynômes de Macdonald via les schémas de Hilbert.

Il semble que le sujet approche d'une maturité suffisante pour permettre d'unifier de façon fructueuse ces différents aspects. Ainsi, pour la première fois, un rapport conceptuel a été obtenu entre une opérade liée aux polytopes de Stasheff et une famille de carquois liées aux algèbres amassées et aux modules basculants. Cette relation nouvelle a notamment suggéré une conjecture surprenante sur ces carquois et par généralisation une conjecture similaire pour les autres systèmes de racines, conjectures aujourd'hui démontrées par S. Ladkani.

Ce mémoire présente le cadre de mes travaux, en donne une description rapide et un aperçu de mes recherches actuelles.

Dans le premier chapitre, après une brève présentation de la théorie des opérades, on trouvera une description de mes travaux dans ce domaine et quelques unes des idées et des développements récents.

Le second chapitre est consacré aux algèbres amassées. Il commence par une esquisse générale de cette théorie et se poursuit par l'évocation de divers aspects plus spécialisés et par des perspectives de recherches dans cette direction.

Les chapitres suivants sont formés par une sélection de mes articles, certains publiés et d'autres encore inédits. Ce sont dans l'ordre les articles [Chac], [Cha05b] et [Chad] qui traitent d'opérades et les articles [CCS06] et [Chaa] sur des sujets liés aux algèbres amassées.

Le chapitre 3 ([Chac]) contient un article inédit qui associe un groupe de séries formelles à chaque opérade et étudie le cas de l'opérade PreLie.

Le chapitre 4 ([Cha05b]) reproduit un article paru, dans lequel on montre que certaines opérades ont une structure plus riche.

Le chapitre 5 ([Chad]) est formé par un article paru, qui introduit une structure d'opérade sur les fractions rationnelles.

Le chapitre 6 ([CCS06]) est la reproduction d'un article paru, fruit d'une collaboration avec P. Caldero et R. Schiffler, qui considère les carquois avec relations associés à certaines algèbres amassées.

Enfin, le chapitre 7 ([Chaa]) est un article inédit sur la cohomologie de certaines variétés toriques associées aux algèbres amassées.

Chapitre 1

Opérades combinatoires

1.1 Quelques aspects de la définition

On peut définir les opérades dans toute catégorie monoïdale symétrique. Pour fixer les idées, choisissons la catégorie des espaces vectoriels sur un corps $\mathbb{K}$ de caractéristique nulle.

On peut donner des définitions équivalentes variées de la notion d'opérade. Chacune de ces présentations est plus ou moins agréable à utiliser selon le contexte.

La présentation la plus classique des opérades est en termes de modules sur les groupes symétriques. Pour chaque entier n, soit $\mathfrak{S}_n$ le groupe symétrique sur l'ensemble $\{1, \ldots, n\}$.

Une *opérade* $\mathcal{P}$ consiste en une collection $(\mathcal{P}(n))_{n \geq 1}$ d'espaces vectoriels avec sur $\mathcal{P}(n)$ une structure de module sur le groupe symétrique $\mathfrak{S}_n$, en des applications linéaires $\circ$:

$$\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \mathcal{P}(m_2) \otimes \cdots \otimes \mathcal{P}(m_n) \longrightarrow \mathcal{P}(m_1 + \cdots + m_n) \tag{1.1}$$

appelées compositions et en un élément $\mathbf{1}$ de $\mathcal{P}(1)$ appelé unité. Ces données doivent vérifier des axiomes naturels d'équivariance, d'associativité et d'unité modelés sur les propriétés des applications multilinéaires. En particulier, pour tout espace vectoriel V, la collection des espaces $\hom(V^{\otimes n}, V)$ avec les actions naturelles des groupes symétriques et les compositions naturelles forment une opérade notée $\mathcal{E}nd(V)$.

En utilisant l'existence d'une unité $\mathbf{1}$, on peut modifier cette présentation en se donnant des compositions partielles

$$\circ_i \,:\, \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n + m - 1) \tag{1.2}$$

au lieu des morphismes de compositions. On obtient ces compositions partielles en spécialisant en l'unité presque tous les arguments des compositions $\circ$. Réciproquement, on retrouve $\circ$ en itérant des compositions partielles $\circ_i$.

Une autre approche de la notion d'opérade est possible, dans l'esprit de la théorie des espèces de structure utilisée en combinatoire [Joy86, BLL98]. L'idée est de remplacer la suite des groupes symétriques par la catégorie ou plus précisément le groupoïde ayant comme objets les ensembles finis et comme morphismes les bijections. L'avantage est d'obtenir de façon transparente les axiomes d'équivariance. Une opérade dans ce cadre est un foncteur de ce groupoïde des ensembles finis dans la catégorie des espaces vectoriels, muni de morphismes de composition vérifiant des axiomes similaires.

Il existe aussi une variante simplifiée de la notion d'opérade, sans les actions des groupes symétriques. Plus précisément, une *opérade non-symétrique* $\mathcal{P}$ consiste en une collection $(\mathcal{P}(n))_{n\geq 1}$ d'espaces vectoriels, en des applications linéaires

$$\circ_i \,:\, \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1) \tag{1.3}$$

satisfaisant des axiomes similaires à ceux des opérades, sauf l'axiome d'équivariance qui n'a plus de sens en l'absence des actions des groupes symétriques.

La terminologie peut prêter à confusion ; en effet une opérade non-symétrique n'est pas une opérade. Toutefois, toute opérade est en particulier une opérade non-symétrique et il existe une construction naturelle associant une opérade à chaque opérade non-symétrique. Elle consiste simplement à remplacer chaque $\mathcal{P}(n)$ par le module libre sur $\mathfrak{S}_n$ engendré par $\mathcal{P}(n)$. Il y a alors une unique extension possible des opérations de composition qui définisse une structure d'opérade.

1.2 Opérades avec des structures supplémentaires

On rencontre assez souvent des opérades ayant une structure plus riche.

Par exemple, une *opérade de Hopf* peut être définie comme une opérade dans la catégorie monoïdale symétrique des cogèbres sur le corps $\mathbb{K}$. C'est donc une opérade $\mathcal{P}$ dont chaque $\mathcal{P}(n)$ est une cogèbre, avec des relations de compatibilité entre ces coproduits coassociatifs et les compositions $\circ_i$. Ce genre de structure se rencontre notamment dans le cas des opérades définies comme l'homologie d'une autre opérade dans la catégorie des variétés ou des espaces topologiques. En effet, tout espace est muni de la diagonale, qui est un coproduit coassociatif dans la catégorie des espaces. Ceci est similaire au fait que l'homologie d'un groupe topologique est une algèbre de Hopf.

Un autre exemple important est la structure d'*opérade cyclique*. On peut voir les opérades comme une théorie des opérations ayant plusieurs arguments (entrées) et un résultat (sortie). Les opérades cycliques sont alors des opérades où l'on ne distingue plus entre entrées et sortie. Plus formellement, on demande que l'action du groupe $\mathfrak{S}_n$ (sur les entrées) soit étendue à une action de $\mathfrak{S}_{n+1}$ et que certaines relations entre cette action et les compositions soient satisfaites. Il existe une variante appelée *opérade anticyclique* où les axiomes pour l'action de $\mathfrak{S}_{n+1}$ sont modifiés par l'introduction de signes appropriés.

De façon similaire, on définit une notion d'opérade non-symétrique cyclique ou anticyclique en demandant l'existence d'une action d'un groupe cyclique d'ordre $n+1$ sur l'espace vectoriel $\mathcal{P}(n)$. Cette action doit satisfaire certaines compatibilités avec les compositions.

Comme précédemment pour les opérades, toute opérade cyclique est une opérade non-symétrique cyclique par restriction de l'action de $\mathfrak{S}_{n+1}$ au sous-groupe cyclique engendré par le cycle $(1, 2, 3, \dots, n, n+1)$. Ce foncteur d'oubli possède un adjoint donné par l'induction du groupe cyclique au groupe symétrique.

1.3 Opérades et groupes de Coxeter

Il y a clairement une relation superficielle entre les opérades et les groupes symétriques : une opérade est une collection de modules sur les groupes symétriques, avec des structures supplémentaires. Il semble possible que cette relation soit plus profonde, au moins dans le cas de certaines opérades “sympathiques”. L’idée fondamentale est de chercher à comprendre les modules sous-jacents à une opérade comme cas particuliers d’une construction générale associée aux groupes de Weyl finis ou bien aux groupes de Coxeter finis.

Le cas des opérades Comm, Ass et Lie est représentatif de ce qui est recherché en général. Le module Comm(n) est le module trivial pour le groupe symétrique $\mathfrak{S}_n$. L’analogue pour les groupes de Coxeter finis est bien sûr le module trivial. De même, l’opérade Det correspond au module sgn donné par la signature, à une suspension près.

Le module Ass(n) est la représentation régulière du groupe symétrique $\mathfrak{S}_n$. L’analogue pour les groupes de Coxeter finis est alors la représentation régulière.

Pour le module Lie(n), il faut utiliser l’arrangement d’hyperplans complexes associé au groupe symétrique $\mathfrak{S}_n$. On peut alors décrire Lie(n) $\otimes$ sgn comme le module donné par l’homologie en degré maximal de cet arrangement d’hyperplans. Ceci se généralise donc aux groupes de Coxeter finis.

Ce dernier exemple est mieux compris par le biais de l’opérade de Gerstenhaber. Le module gradué Gerst(n) est en effet exactement identifiable à l’homologie de l’arrangement d’hyperplans et la composante de degré maximale de Gerst(n) est Lie(n).

Quelque chose de tout-à-fait similaire devrait exister pour l’opérade de Poisson. Il existe une filtration sur Ass dont le gradué associé est Poiss. Quelques indices ont été obtenus dans [Cha03] en faveur d’une interprétation de Poiss(n) par le biais de fractions rationnelles sur l’arrangement d’hyperplans associé au groupe symétrique $\mathfrak{S}_n$. Une interprétation alternative en termes de fonctions de Heaviside, selon la terminologie de Gelfand et Varchenko [VG87], est peut-être possible. Il s’agit en fait de définir une filtration naturelle sur la représentation régulière.

Deux exemples sont pour l’instant seulement définis dans les cas $\mathbb{A}$ et $\mathbb{B}$ de la classification de Killing-Cartan des systèmes de racines. Le premier est l’opérade Perm qui devrait correspondre à un module de dimension donnée par le nombre de Coxeter. Dans les cas $\mathbb{A}$ et $\mathbb{B}$, ce module est juste la représentation standard comme groupe de permutations ou de permutations signées. Ce type de représentation ensembliste n’existe sans doute pas pour les autres cas, où il faut chercher plutôt une représentation linéaire.

L’autre exemple est l’opérade PreLie qui est définie sur les arbres enracinés dans [CL01] en collaboration avec M. Livernet. Dans un travail avec B. Vallette, nous avons introduit le module analogue pour le type $\mathbb{B}$ à l’aide de la construction d’un poset et de sa cohomologie. L’extension aux autres cas ne semble pas possible par la même stratégie, pour la même raison que pour Perm.

Notons aussi qu’il pourrait exister une réalisation de PreLie en termes de formes différentielles et de fractions rationnelles comme celle entrevue pour l’opérade Poiss(n).

Une idée vague derrière cette volonté d’étendre aux groupes de Coxeter les opérades les plus simples est que ces opérades doivent être fortement liées aux arrangements d’hyperplans de type $\mathbb{A}$. La stratégie est donc d’essayer d’interpréter autant d'opérades que possible en termes de la géométrie des arrangements d'hyperplans, puis de généraliser en utilisant ceci comme une définition pour les autres groupes de Coxeter. Le fruit de cet effort devrait être la définition de nouveaux objets associés aux arrangements de Coxeter et peut-être de nouvelles opérades.

1.4 Opérades cycliques et anticycliques

Il est bien connu que les opérades Poiss, Lie et Comm sont des opérades cycliques et que Ass est une opérade non-symétrique cyclique. La description des actions de $\mathfrak{S}_{n+1}$ sur les composantes de ces opérades est immédiate pour Comm et connue également dans les autres cas. On obtient ainsi des modules intéressants sur les groupes symétriques, en particulier les modules de Whitehouse dans le cas Lie.

Dans l'article [Cha05b], j'ai montré que quelques opérades, dont notamment certaines des opérades introduites par Loday, ont une structure naturelle d'opérade anticyclique. Parmi ces six opérades, les trois les plus intéressantes sont certainement les opérades PreLie, Dias et Dend. J'ai obtenu une formule pour le caractère des modules sur $\mathfrak{S}_{n+1}$ ainsi définis, et une conjecture dans le cas PreLie. Dans le cas PreLie, qui est de loin le plus complexe, la conjecture a été démontrée depuis, comme conséquence immédiate d'un résultat obtenu dans la publication [Cha07a].

La structure d'opérade non-symétrique anticyclique des opérades Dend et Dias semble avoir des relations intéressantes avec les carquois et la théorie des représentations. On en reparlera plus loin (voir §2.7).

La structure d'opérade anticyclique de PreLie est encore mal comprise, et sera peut-être la source de développements intéressants. Une première conjecture surprenante dans ce contexte est liée aux hyperarbres. Un hypergraphe sur un ensemble fini I est un ensemble de parties non vides de I qui recouvrent I. Un chemin dans un hypergraphe est une suite formée alternativement de parties dans cet hypergraphe et d'éléments de I telle qu'un élément i de I soit dans chaque partie P qui lui est adjacente dans la suite. Un cycle est un chemin qui commence et finit par la même partie P. Un hyperarbre est alors un hypergraphe sans cycle.

L'ensemble des hypergraphes sur I peut être muni d'un ordre partiel, dont les relations de couvertures consistent à remplacer deux parties de I par leur réunion. J. McCammond et J. Meier ont montré que ce poset est Cohen-Macaulay [MM04]. Par conséquent, l'homologie de l'espace simplicial associé est concentrée en degré maximal et munie d'une action du groupe des permutations de I. Je conjecture dans [Cha07a] que ce module (ou son tordu par le module signature) est isomorphe au module associé à la structure anticyclique de PreLie. On peut aisément vérifier que la dimension correspond et que la conjecture marche pour les petits cardinaux.

Un autre problème, a priori sans rapport, est le suivant. Il est connu que les algèbres pré-Lie libres sont aussi libres comme algèbres de Lie [Foi02]. On a obtenu dans [Chab] une description des générateurs, ou plus précisément du foncteur qui associe à un espace vectoriel V l'espace des générateurs de l'algèbre pré-Lie libre sur V en tant qu'algèbre de Lie. Ce foncteur est analytique, *i.e.*

admet une description de la forme

$$V \mapsto \oplus_{n\geq 1} M(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}, \tag{1.4}$$

pour une certaine collection de modules $M(n)$ sur $\mathfrak{S}_n$. Il apparaît alors que les modules $M(n)$ pour $\mathfrak{S}_n$ s'obtiennent facilement à partir des modules anticycliques $\mathrm{PreLie}(n-1)$ pour $\mathfrak{S}_n$. Il reste à comprendre les raisons conceptuelles de cette relation.

Enfin, on s'est aperçu récemment que certains aspects de la combinatoire des opérades pré-Lie et dendriforme évoquées plus haut apparaissent dans le cadre de la théorie des moules d'Ecalle, utilisée notamment dans l'étude des nombres polyzetas. Comprendre cette relation nouvelle est certainement un objectif digne d'intérêt. Une première étape dans cette direction est la définition d'une opérade anticyclique Mould sur les moules, contenant l'opérade Dend, dans la publication [Chad]. Depuis cette publication, ce travail a été étendu, avec notamment l'étude d'un plongement de l'opérade Zinbiel dans Mould, dans l'article [CHNT08].

Chapitre 2

Algèbres amassées

La notion d'algèbre amassée[1] est due à S. Fomin et A. Zelevinsky. Leur motivation pour introduire ces nouveaux objets provenait essentiellement de la théorie des représentations. Un des objectifs était de mieux comprendre les bases canoniques des groupes quantiques et plus précisément le comportement des éléments de la base canonique duale vis à vis de la multiplication. Il fallait comprendre sous quelles conditions le produit de deux éléments de cette base était encore dans cette base.

Dans cette optique, S. Fomin et A. Zelevinsky ont introduit une famille d'algèbres commutatives. Chacune de ces algèbres amassées est construite, à partir d'une donnée combinatoire, comme sous-algèbre d'un corps de fractions rationnelles.

Commençons par une description grossière et simplifiée. La donnée combinatoire initiale est une matrice antisymétrique. Son nombre de colonnes est appelé le *rang*. On associe aux colonnes de cette matrice un ensemble de variables algébriquement indépendantes. La paire (matrice,variables) est appelée une graine. Ensuite on définit un processus de *mutation* qui associe à chaque graine et chaque colonne de cette graine un nouvelle graine de même rang. En effectuant de façon répétée toutes les mutations possibles, on engendre un ensemble de graines (potentiellement infini). Les variables qui apparaissent dans une des graines sont toutes des fractions rationnelles en les variables de la graine initiale (en fait des polynômes de Laurent, mais ceci n'est pas immédiat). On les appelle les variables d'amas. A chaque graine est associée son amas, qui est l'ensemble des variables d'amas qui apparaissent dans cette graine.

Les graines obtenues ainsi forment un graphe régulier, nommé graphe de mutation. On dit que deux graines de ce graphe sont équivalentes par mutation. Deux amas adjacents, qui correspondent à deux graines reliés par une mutation, diffèrent exactement par une variable. Ces deux variables x' et x'' sont alors reliées par une relation du type

$$x'x'' = M_1 + M_2, \tag{2.1}$$

où M_1 et M_2 sont des monômes, premiers entre eux, en les variables communes aux deux amas adjacents. Ces relations sont nommées *relations d'échange.*

[1] C'est la terminologie française d'usage pour le terme anglo-saxon "cluster algebra".

En général, on travaille sur un anneau de base, dit anneau de coefficients, souvent un anneau de polynômes ou de polynômes de Laurent en des variables dites coefficients. Il ne s'agit pas d'une simple extension des scalaires, car les coefficients interviennent dans le processus de mutation, par le biais de l'introduction de lignes supplémentaires dans la matrice, qui n'est alors plus carrée mais rectangulaire.

2.1 Exemples et motivations

Considérons l'algèbre des fonctions sur le groupe SL_2. Elle est engendrée par les éléments a, b, c, d de la matrice

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.2}$$

soumis à la relation

$$ad = 1 + bc. \tag{2.3}$$

Cette algèbre est une algèbre amassée de rang 1 dont les variables d'amas sont a et d et les coefficients sont b et c. Il y a exactement deux amas $\{a\}$ et $\{d\}$. La relation (2.3) ci-dessus est la relation d'échange entre ces deux amas.

Une famille d'exemples agréable, à la fois simple et suffisamment générale pour illustrer la notion d'algèbre amassée, est fournie par les algèbres des fonctions sur les grassmanniennes $G(2, n)$ paramétrant les sous-espaces de dimension 2 dans l'espace vectoriel $\mathbb{C}^n$.

L'espace ambiant $\mathbb{C}^n$ est muni de sa base canonique $\{e_1, \ldots, e_n\}$. Un sous-espace de dimension 2 peut alors être décrit, en en choisissant une base arbitraire, comme une matrice à 2 lignes et n colonnes. Soit $\Delta_{i,j}$ le mineur de taille 2 de cette matrice associé aux colonnes i et j.

On obtient ainsi des coordonnées homogènes $\Delta_{i,j}$ sur $G(2, n)$ pour chaque paire (i, j) dans $\{1, \ldots, n\}$ (coordonnées de Plücker). Un résultat classique dit que l'algèbre des fonctions homogènes sur $G(2, n)$ est engendrée par ces coordonnées, sujettes aux relations suivantes, également dites de Plücker :

$$\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}, \tag{2.4}$$

pour tous les entiers $1 \leq i < j < k < l \leq n$.

On a une structure d'algèbre amassée sur l'algèbre $\mathbb{C}[G(2, n)]$. Les variables d'amas sont les mineurs Δ_{ij} pour deux indices i et j non adjacents modulo n et les coefficients sont les autres mineurs. On peut représenter ces mineurs graphiquement dans un polygone régulier à n sommets numérotés dans un ordre cyclique de 1 à n. Le mineur Δ_{ij} est associé au segment de droite ij. Dans ce modèle, les coefficients sont les bords du polygone et les variables d'amas sont les diagonales. Les amas sont alors les triangulations du polygone. Les relations d'échange correspondent aux paires de triangulations qui diffèrent par une seule diagonale et sont données par les relations de Plücker.

Un des objectifs des algèbres amassées est d'obtenir une description simple d'une base de certaines algèbres de fonctions. Ces bases devraient correspondre à la spécialisation classique de bases canoniques ou semi-canoniques d'origine quantique, et la théorie des algèbres amassées devrait permettre de définir ces bases sans passer par l'introduction d'un paramètre q de déformation quantique, en restant entièrement dans le cadre classique.

Un autre résultat attendu de la théorie est la définition d'une structure positive, c'est à dire une présentation de l'algèbre des fonctions sur la variété par des relations ne faisant pas intervenir le signe $-$. L'existence d'une telle structure permet en particulier de définir la variété tropicale associée en remplaçant, partout dans la présentation positive, la somme par le maximum et le produit par la somme.

Une des méthodes pour trouver une structure d'algèbre amassée sur une algèbre de fonctions consiste, lorsqu'on dispose d'un analogue quantique de cette algèbre, à construire un ensemble maximal (pour l'inclusion) de coordonnées qui q-commutent deux à deux, *i.e.* dont les produits dans les deux ordres possibles diffèrent seulement par une puissance de q en facteur. Ces coordonnées devraient former un amas. On tente alors de reconstruire la matrice antisymétrique en cherchant les amas adjacents. Une variante quasi-classique de cette stratégie consiste à utiliser une structure de Poisson et des coordonnées log-canoniques.

A ce jour, on connaît notamment des structures d'algèbres amassées sur les algèbres de fonctions des grassmanniennes [Sco06], des doubles cellules de Bruhat [BFZ05] et des variétés de drapeaux généralisées [GLS08].

2.2 Relation aux systèmes de racines et polytopes

La question se pose naturellement de comprendre quelles sont les graines (ou plutôt les matrices antisymétriques) telles que l'ensemble des amas obtenus par le processus de mutation soit fini. On dit que ce sont les graines de type fini. On a vu des exemple précédemment avec les algèbres de fonctions sur les grassmanniennes de plans. La réponse est due à S. Fomin et A. Zelevinsky.

Théorème 2.2.1 *Si une graine Σ est de type fini, alors il existe une graine Σ' équivalente par mutation à Σ et dont la matrice est obtenue par antisymétrisation d'une matrice de Cartan de type fini. Si on antisymétrise arbitrairement une matrice de Cartan de type fini, on obtient une graine de type fini et toutes ces graines sont équivalentes par mutation. Deux graines associées à deux matrices de Cartan différentes ne sont pas équivalentes par mutation. Chacune de ces graines apparaît dans une algèbre amassée de type fini.*

Dans cet énoncé, antisymétriser consiste à remplacer les 2 sur la diagonale par des 0 et à multiplier de façon antisymétrique par un signe chaque coefficient hors-diagonale.

Il y a donc une bijection entre les types d'algèbres amassées de type fini et les diagrammes de Dynkin de type fini (liste de Killing-Cartan : $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, $\mathbb{F}_4, \mathbb{G}_2$) qui classifient déjà bien d'autres choses, notamment les algèbres de Lie semi-simples et les groupes de Coxeter finis cristallographiques.

On peut décrire assez explicitement le graphe régulier formé par les amas et les mutations dans le cas des algèbres amassées de type fini. Pour cela, on utilise les systèmes de racines.

Soit donc Φ un système de racines cristallographiques et $\Phi_{>0}$ un choix de racines positives. Soit Π l'ensemble des racines simples positives. On définit l'ensemble $\Phi_{\geq -1}$ des racines presque positives comme l'union de $\Phi_{>0}$ et de $-\Pi$.

En utilisant le fait que les diagrammes de Dynkin sont des arbres, donc des graphes bipartites, on définit deux involutions τ_+ et τ_- agissant sur l'ensemble $\Phi_{\geq -1}$. Leur définition est une modification linéaire par morceau des deux produits de réflexions simples associés à la bipartition. Par conséquent le produit de τ_+ et τ_- est un analogue linéaire par morceau d'une transformation de Coxeter.

Proposition 2.2.2 *Les deux involutions τ_+ et τ_- engendrent un groupe diédral fini. Chaque orbite de ce groupe dans $\Phi_{\geq -1}$ rencontre $-\Pi$ en exactement une ou deux racines.*

Cette propriété permet de définir une notion de compatibilité entre racines presque positives en imposant que cette notion soit invariante sous l'action de τ_+ et τ_- et que la compatibilité avec un élément de $-\Pi$ soit donnée par une certaine condition explicite, précisément que la racine simple positive correspondante n'intervienne pas dans l'écriture de l'autre racine dans la base des racines simples.

On peut alors montrer que les amas dans une algèbre amassée de type Φ sont en bijection avec les parties de $\Phi_{\geq -1}$, maximales pour l'inclusion, formées d'éléments deux à deux compatibles.

Dans mon premier travail sur les algèbres amassées, en collaboration avec S. Fomin et A. Zelevinsky [CFZ02], nous avons montré l'existence de polytopes associés aux systèmes de racines, nommés les associaèdres généralisés.

Le problème était le suivant : la relation de compatibilité définit une éventail simplicial dont les cônes de dimension 1 sont indexés par $\Phi_{\geq -1}$. Par la théorie des variétés toriques, cet éventail correspond à une variété torique lisse. Savoir si il existe un polytope réalisant la structure simpliciale de l'éventail est équivalent à savoir si la variété est projective. Il faut donc construire un diviseur ample, ce qui se traduit en termes de fonctions linéaires par morceaux sur l'éventail. Nous avons montré que le cône ample est non vide pour chacun des types finis. La méthode utilisée est de considérer la partie de ce cône ample invariante par l'action de τ_+ et τ_-.

En particulier, pour les types $\mathbb{A}$ et $\mathbb{B}$, on obtient de nouvelles preuves de l'existence des familles de polytopes nommés associaèdres et cycloèdres, qui étaient déjà connus auparavant. Les autres polytopes sont nouveaux et sont nommés associaèdres généralisés.

Par les travaux de Marsh, Reineke et Zelevinsky [MRZ03], on peut plus généralement associer un éventail à chaque carquois sur un diagramme de Dynkin, de telle sorte que l'éventail associé au carquois alterné soit celui étudié précédemment. Il paraît intéressant de montrer que tous ces éventails sont projectifs. Des travaux récents de Hohlweg, Lange et Thomas [HLT] montrent (dans un cadre un peu plus général, celui des treillis cambriens) l'existence de polytopes, ce qui est essentiellement équivalent.

Il reste cependant à décrire explicitement le cône des diviseurs amples sur la variété torique correspondante. En particulier, cette question devrait être reliée au problème de trouver une base naturelle des anneaux de cohomologie de ces variétés toriques. Un premier pas dans cette direction a été accompli dans [Chaa], reproduit plus loin. Les cônes amples associés à ces éventails semblent avoir des propriétés remarquables, dont certaines rappellent les cônes de Lusztig.

Par ailleurs, il semble en fait possible de généraliser encore cette construction d'un éventail en partant non plus d'un carquois, mais d'une graine arbitraire. En effet, les carquois sur une graphe de Dynkin correspondent à certaines graines dans l'algèbre amassée de ce type, mais il y a en général beaucoup d'autres graines. On peut alors remplacer la paramétrisation des variables d'amas par leur dénominateur (d-vecteur), qui ne fonctionne pas en général, par une autre paramétrisation (g-vecteur) obtenue en utilisant des coefficients bien choisis (coefficients principaux).

2.3 Systèmes Y et dilogarithme de Rogers

Un autre aspect de la théorie des algèbres amassées est celui des systèmes Y associés [FZ03]. Il s'agit d'une famille de polynômes de Laurent "duale" en un certain sens de la famille des variables d'amas, et qui correspond en un sens à la mutation des matrices. Cette notion de système Y a ses racines en physique théorique, dans le domaine des théories des champs conformes ou intégrables en dimension 2.

On considère un carquois alterné sur un diagramme de Dynkin de type $\mathbb{A}$, $\mathbb{D}$ ou $\mathbb{E}$. Soit I l'ensemble des sommets du diagramme de Dynkin. Soit $A = (A_{i,j})$ la matrice antisymétrique associée. Comme le diagramme est bipartite, on peut choisir un signe $\varepsilon(i) \in \{-,+\}$ pour chaque sommet i de telle façon que deux sommets voisins soient de signes différents.

On définit alors deux involutions τ_+ et τ_- sur le corps des fractions rationnelles $\mathbb{C}((u_i)_{i\in I})$ comme suit :

$$\tau_\pm(u_j) = \begin{cases} \frac{\prod_{i-j}(1+u_i)}{u_j} \text{ si } \varepsilon(j) = \pm, \\ u_j \text{ sinon.} \end{cases} \tag{2.5}$$

S. Fomin et A. Zelevinsky ont obtenu la description suivante [FZ03].

Théorème 2.3.1 *Les orbites des monômes u_i pour l'action de τ_+ et τ_- sur $\mathbb{C}((u_i)_{i\in I})$ sont finies. Il existe une unique bijection entre l'union des ces orbites et l'ensemble $\Phi_{\geq -1}$des racines presque positives qui respecte les actions de τ_+ et τ_- et envoie u_i sur la racine négative $-\alpha_i$.*

Soit L la fonction dilogarithme de Rogers définie sur $[0,1]$ par

$$L(x) = -\int_0^x \frac{\log(1-z)}{z}dz + \frac{1}{2}\log(x)\log(1-x). \tag{2.6}$$

Soit $\mathbb{L}$ la fonction définie sur $\mathbb{R}_+$ par

$$\mathbb{L}(x) = L\left(\frac{1}{1+x}\right). \tag{2.7}$$

J'ai obtenu dans [Cha05a] le résultat suivant.

Théorème 2.3.2 *Soit $\mathbf{Y}$ l'union des orbites. Alors on a*

$$\sum_{y\in\mathbf{Y}} \mathbb{L}(y) = n\frac{\pi^2}{6}, \tag{2.8}$$

où n est le nombre de sommets du diagramme de Dynkin.

Il semble possible que ce genre de propriétés soit vrai dans un cadre plus général que celui des diagrammes de Dynkin de type fini, en particulier, pour les graines données par le produit cartésien de deux diagrammes de Dynkin alternés. Soient donc X et X' deux diagrammes de Dynkin de type $\mathbb{A}$, $\mathbb{D}$ ou $\mathbb{E}$. Soit $X \times X'$ leur produit cartésien comme graphe, muni d'une orientation telle que chaque carré formé par le produit cartésien d'une arête de X par une arête de X' soit cycliquement orienté. Ce graphe est bien sûr bipartite et on peut alors définir comme précédemment des involutions τ_+ et τ_- agissant sur les fractions rationnelles.

On conjecture alors aussi que le produit $\tau_+\tau_-$ est encore périodique dans ce contexte et que

$$\sum_{y \in \mathbf{Y}} \mathbb{L}(y) = nm\frac{\pi^2}{6}, \tag{2.9}$$

où $\mathbf{Y}$ désigne l'union des orbites des variables u_i.

Le cas $\mathbb{A} \times \mathbb{A}$, qui est le plus simple du point de vue combinatoire, semble pouvoir être approché par l´étude des grassmaniennes, qui possèdent des graines de ce type et sur lesquelles agit naturellement un groupe diédral qui pourrait correspondre à l'action de $\tau_+\tau_-$. Deux preuves indépendantes ont été données pour la périodicité du système Y pour le type $\mathbb{A} \times \mathbb{A}$ par A. Szenes [Sze06] et par A. Volkov [Vol07]. Elles ne semblent pas faire ce lien avec les grassmanniennes. Des graines similaires apparaissent aussi dans les travaux récents de B. Leclerc reliant les algèbres amassées aux représentations des algèbres affines quantiques.

Enfin, B. Keller a démontré récemment la périodicité pour les graines produits $X \times X'$ en général, non pas pour les systèmes Y mais pour les algèbres amassées, en utilisant la notion de catégorie amassée, dont on va parler maintenant.

2.4 Carquois et catégories dérivées

Une nouvelle approche des algèbres amassées est apparue depuis leur introduction par S. Fomin et A. Zelevinsky. Elle fait appel à la théorie des carquois et des catégories dérivées.

Des relations classiques existent entre les systèmes de racines de type $\mathbb{A}$, $\mathbb{D}$ ou $\mathbb{E}$ et les carquois. Pour un tel système de racines, le choix d'une orientation pour chaque arête du diagramme de Dynkin définit un carquois. Par un théorème de Gabriel, les objets indécomposables dans la catégorie des modules sur ce carquois sont alors en bijection avec les racines positives du système de racines.

Il est possible d'obtenir une relation semblable entre les variables d'amas d'une algèbre amassée de type fini et les racines presque positives du système de racines. La catégorie qui joue un rôle central ici est définie comme quotient d'une catégorie dérivée par une auto-équivalence.

Plus précisément, soit Q un carquois de type Dynkin. Alors la catégorie $\operatorname{mod} Q$ des modules sur Q est héréditaire. Ceci entraîne que la catégorie dérivée bornée de $\operatorname{mod} Q$ admet une description simple : en particulier, les objets indécomposables de $\operatorname{Dmod} Q$ sont exactement les décalés $M[i]$ pour $i \in \mathbb{Z}$ des objets indécomposables M de $\operatorname{mod} Q$. La catégorie $\operatorname{Dmod} Q$ est une catégorie triangulée, munie de deux auto-équivalences, le décalage [1] et la translation d'Auslander-Reiten τ. A équivalence triangulée près, elle ne dépend pas du carquois, mais seulement du diagramme de Dynkin sous-jacent.

On peut alors considérer la catégorie quotient de $\operatorname{Dmod} Q$ par l'auto-équivalence $\tau \circ [-1]$. B. Keller a montré que la catégorie quotient a une structure naturelle de catégorie triangulée avec une auto-équivalence induite [Kel05]. Cette catégorie triangulée quotient est appelée la catégorie des amas, et notée C. Une propriété fondamentale de la catégorie C est la symétrie suivante :

$$\dim \operatorname{Ext}(M, N) = \dim \operatorname{Ext}(N, M). \tag{2.10}$$

On peut alors considérer les objets amas-basculants dans la catégorie C, *i.e.* les objets M qui sont sommes directes maximales sans multiplicité d'indécomposables et vérifient $\operatorname{Ext}(M, M) = 0$. Ces objets basculants doivent correspondre bijectivement aux amas. Les variables d'amas correspondent aux objets indécomposables de C. La relation de compatibilité entre objets indécomposables est donnée par l'annulation de Ext. Ceci a été montré en général par A. Buan, R. Marsh, M. Reineke, I. Reiten et G. Todorov dans [BMR+06].

Avec P. Caldero et R. Schiffler, nous avons obtenu dans [CCS06] une description combinatoire de la catégorie C pour le type $\mathbb{A}$ en s'appuyant sur la description des amas de type $\mathbb{A}$ donnée par S. Fomin et A. Zelevinsky en termes de triangulations d'un polygone régulier (voir la description de la structure d'algèbre amassée des grassmanniennes de plans au §2.1). Les objets indécomposables de C sont les diagonales et les morphismes sont des compositions de rotations élémentaires d'une diagonale dans le sens positif autour d'une de ses extrémités. Ce type de description a depuis été obtenu pour le type $\mathbb{D}$ par R. Schiffler.

Un autre aspect de ce travail est la définition pour chaque amas de type $\mathbb{A}$ d'un carquois avec relations, dont les représentations contrôlent les dénominateurs des polynômes de Laurent exprimant les variables d'amas dans cet amas.

Dans une suite de ce travail [CC06], avec P. Caldero, nous avons obtenu une formule explicite pour les variables d'amas dans un amas de type Dynkin. Cette formule est du type de celles qui définissent les algèbres de Hall, et fait intervenir des grassmanniennes de sous-modules et leurs caractéristiques d'Euler. Elle a ouvert la voie à la preuve d'une des conjectures majeures du domaine, qui est la positivité de tous les polynômes de Laurent qui expriment les variables d'amas dans un amas fixée, obtenue depuis (sous certaines hypothèses) dans une série de travaux de Caldero et Keller [CK08, CK06].

Cette formule directe pour les variables d'amas est certainement très utile du point de vue théorique et de nombreux travaux ultérieurs s'appuient sur ce résultat.

2.5 Aspects énumératifs

Soit W un groupe de Coxeter fini, h son nombre de Coxeter et $e_1, \dots, e_n$ ses exposants. On peut remarquer que ces nombres dépendent seulement du complément de l'arrangement d'hyperplans associé au groupe W.

On note

$$\operatorname{Cat}_W = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1} \tag{2.11}$$

et on appelle ce nombre le nombre de Catalan généralisé de type W.

Voici les valeurs de ces nombres :

A_n	B_n	D_n	E_6	E_7	E_8	F_4	$I_2(h)$	H_3	H_4
$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	$h+2$	32	280

Dans le cas A_n, on obtient le nombre de Catalan c_{n+1} habituel, qui est très classique en combinatoire, notamment dans le domaine des arbres.

On observe au cas par cas le résultat suivant.

Proposition 2.5.1 *Le nombre d'amas dans une algèbre amassée de type fini associée à un système de racines est donné par* Cat_W*, où* W *est le groupe de Weyl du système de racines.*

Autrement dit, le nombre d'amas dans une algèbre amassée de type fini admet une expression simple remarquable en termes des invariants du groupe de Weyl associé. Ces nombres de Catalan généralisés apparaissent également dans d'autres contextes reliés aux groupes de Coxeter finis. On peut notamment associer à chaque groupe de Coxeter fini un ensemble de partitions non-croisées, dont voici une description. Cette construction est due à T. Brady et C. Watt et à D. Bessis indépendamment.

Soit donc W un groupe de Coxeter fini. On regarde W comme groupe engendré par l'ensemble T de toutes ses réflexions plutôt que par l'ensemble S des réflexions simples. Pour la fonction longueur ℓ définie par cet ensemble T de générateurs, les éléments de longueur maximale sont exactement les éléments de Coxeter. On définit un ordre partiel sur les éléments de W comme la clôture transitive de la relation suivante : $x \leq y$ si il existe t dans T tel que $y = tx$ et $\ell(y) = 1 + \ell(x)$. Le poset des partitions non-croisées est alors formé par les éléments de W qui sont inférieurs pour cet ordre à un élément de Coxeter c fixé. A isomorphisme près, ce poset ne dépend pas du choix de c. Il possède la propriété non-évidente d'être un treillis.

Proposition 2.5.2 *Le nombre de partitions non-croisées associées à un groupe de Coxeter fini* W *est donné par* Cat_W *.*

Il y a donc une relation énumérative directe entre les amas et les partitions non-croisées. J'ai obtenu dans [Cha04] un raffinement conjectural de cette relation sous la forme d'un changement de variables qui devrait transformer un polynôme en deux variables énumérant les amas selon deux paramètres en un autre polynôme en deux variables énumérant les partitions non-croisées de même type selon deux paramètres. Cette conjecture a été démontrée depuis par C. A. Athanasiadis. Une généralisation par D. Armstrong a aussi été démontrée par la suite.

Par ailleurs, une explication convaincante de la relation entre le nombre d'amas et le nombre de partitions non-croisées a été obtenue indépendamment par Reading [Rea07] d'une part et Athanasiadis, Brady et Watt d'autre part [ABMW06]. Les deux preuves s'appuient sur des constructions (treillis cambriens) qui généralisent la combinatoire des amas à tous les groupes de Coxeter finis.

Un autre contexte où apparaissent les nombres de Catalan généralisés est celui des antichaines dans le poset des racines positives. Soit Φ un système de racines cristallographique et $\Phi_{>0}$ l'ensemble des racines positives. On définit une relation d'ordre partiel sur $\Phi_{>0}$ comme suit : $\alpha \leq \beta$ si $\beta - \alpha$ est une combinaison linéaire positive de racines simples. Les antichaines dans cet ordre partiel sont les parties de $\Phi_{>0}$ formées d'éléments deux-à-deux non-comparables pour la relation $\leq$. Ces antichaines sont en bijection naturelle avec les idéaux supérieurs pour l'ordre $\leq$ sur $\Phi_{>0}$.

Proposition 2.5.3 *Le nombre d'antichaines dans le poset des racines positives d'un système de racines est donné par* Cat_W *où* W *est le groupe de Weyl correspondant.*

A ma connaissance, il n'y a pas encore d'explication satisfaisante à cette relation entre cette occurence des nombre de Catalan généralisés et les deux précédentes. Deux pistes explicatives semblent se dégager : d'une part via la cohomologie des variétés toriques (voir §2.2), d'autre part via une équivalence de catégories dérivées entre certains carquois (voir §2.6).

Comme fruit inattendu de cette recherche énumérative, j'ai découvert une formule qui compte le nombre de réflexions pleines dans les groupes de Coxeter finis. Une réflexion pleine est une réflexion dont toutes les décompositions réduites font intervenir toutes les réflexions simples. Par exemple, dans les groupes symétriques (type $\mathbb{A}$), seule la réflexion par rapport à la racine maximale est pleine. Plus généralement, si W est un groupe de Weyl, les réflexions pleines correspondent aux racines positives dont l'expression dans la base des racines simples n'a pas de coefficient nul.

Théorème 2.5.4 *Le nombre de réflexions pleines dans un groupe de Coxeter fini* W *est*

$$f_W = \frac{1}{|W|}(nh) \prod_{i=2}^{n} (e_i - 1). \tag{2.12}$$

Pour l'instant, la preuve de cette formule est au cas par cas. Dans l'article [Cha06] où elle est introduite, je propose un scénario susceptible d'expliquer cette relation et j'effectue quelques vérifications concernant la plausibilité de cette hypothétique explication.

Cette formule est apparue ensuite dans un article de S. Fomin et N. Reading [FR], avec une preuve uniforme mais qui dépend de l'existence d'une preuve uniforme que le nombre d'amas est bien donné par la formule attendue, *i.e.* (2.11).

2.6 Carquois cambriens et équivalences dérivées

N. Reading et H. Thomas ont introduit récemment une famille de treillis, nommés les treillis cambriens, qui permettent de généraliser la combinatoire des algèbres amassées de type fini à tous les groupes de Coxeter finis.

Un treillis cambrien est associé à chaque élément de Coxeter dans un groupe de Coxeter fini. Dans le cas des groupes cristallographiques, choisir un Coxeter revient à choisir une orientation du diagramme de Dynkin, *i.e.* un carquois. Dans cette situation, le treillis admet une définition équivalente, en termes de la combinatoire des amas associés aux carquois. Ceci permet en particulier d'identifier les ensembles sous-jacents aux treillis cambriens et les ensembles d'amas.

On peut considérer ces treillis cambriens comme des carquois avec relations, et donc définir la catégorie dérivée bornée des modules sur ces carquois. Ces catégories cambriennes sont de dimension homologique finie et munies d'une auto-équivalence canonique, nommée translation d'Auslander-Reiten.

On a alors la conjecture suivante, dont l'origine est l'étude de l'opérade dendriforme (voir §2.7).

Conjecture 2.6.1 *La catégorie dérivée cambrienne ne dépend (à équivalence triangulée près) que du diagramme de Dynkin, et pas du choix du carquois. La translation d'Auslander-Reiten est d'ordre $h+1$ ou $2h+2$ au niveau du groupe de Grothendieck de cette catégorie.*

Cette conjecture doit être rapprochée du résultat bien connu (Gelfand et Ponomarev) selon lequel la catégorie dérivée des modules sur un carquois de Dynkin ne dépend que du diagramme et pas du carquois, et la translation d'Auslander-Reiten au niveau du groupe de Grothendieck est donnée par un élément de Coxeter du groupe de Weyl associé, donc d'ordre h.

La conjecture 2.6.1 incite à penser qu'il doit exister une construction de cette catégorie dérivée commune qui ne dépende que du système de racine et pas du carquois. La conjecture suivante est une réponse possible.

Conjecture 2.6.2 *La catégorie dérivée cambrienne est équivalente à la catégorie dérivée des modules sur les treillis de idéaux supérieurs (ou antichaines) du poset des racines positives.*

Ici, la relation d'ordre sur les idéaux supérieurs est simplement l'inclusion.

En particulier, la conjecture 2.6.2 impliquerait la relation d'égalité entre le nombre d'amas et le nombre d'antichaines, qui reste pour l'instant sans explication.

Plus récemment, des conjectures similaires (parallèles) aux conjectures 2.6.1 et 2.6.2 sont apparues. Elles relient des posets sur les modules basculants entre eux (analogue de la conjecture 2.6.1) et avec le poset des antichaines ne contenant pas de racines simples (analogue de la conjecture 2.6.2).

La conjecture 2.6.1 et son analogue pour les modules basculants viennent d'être en partie démontrées par S. Ladkani [Lada, Ladb]. Il a obtenu, par construction explicite de certains foncteurs, le résultat attendu d'équivalence dérivée entre les catégories associés à différents carquois sans cycle de même graphe sous-jacent. La question de l'ordre de la translation d'Auslander-Reiten (au niveau du groupe de Grothendieck) reste ouverte (sauf en type $\mathbb{A}$, voir ci-dessous), mais semble accessible par les mêmes méthodes.

2.7 Catégorification des opérades

Dans l'article [Cha07b], j'ai montré la périodicité conjecturée ci-dessus pour les treillis cambriens dans le cas du carquois equiorienté de type $\mathbb{A}$, qui correspond aux treillis de Tamari, objets très classiques en combinatoire. La méthode de preuve utilise la théorie des opérades de façon essentielle. Plus précisément, on relie la structure anticyclique de l'opérade non-symétrique dendriforme et la translation d'Auslander-Reiten sur le groupe de Grothendieck de la catégorie des modules sur les treillis de Tamari. On a donc une relation au niveau des groupes de Grothendieck, mais il semble possible de la relever au niveau des catégories. Ceci consisterait à catégorifier l'opérade dendriforme. C'est sans nul doute un des projets que je vais poursuivre.

Une relation similaire (après passage au dual linéaire) existe entre la structure anticyclique de l'opérade diassociative et les carquois de Dynkin de type $\mathbb{A}$. Soit Q_n le carquois equiorienté (de 1 vers n) de type $\mathbb{A}_n$. On peut alors définir des foncteurs

$$\Delta_i : \quad \text{mod } Q_{n+m-1} \longrightarrow \quad \text{mod } Q_m \otimes Q_n \tag{2.13}$$

qui correspondent moralement aux applications transposées des compositions $\circ_i$.

Tous ces foncteurs vérifient les axiomes duaux de ceux satisfaits par les opérations $\circ_i$ dans les opérades non-symétriques. On obtient donc, par passage aux groupes de Grothendieck, une coopérade non-symétrique qui se trouve être le dual de l'opérade diassociative. Ces résultats sont dans l'article [Cha08]. Un corollaire notable est la préservation des formes d'Euler par les applications duales des $\circ_i$.

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Chapitre 3

On a group of series indexed by rooted trees

This paper introduces a group of generalized power series associated to any augmented operad, then focuses on the case of the PreLie operad, where the series are indexed by rooted trees. The solution of flow equations using the pre-Lie structure on vector fields on an affine space gives rise to an interesting element of this group.

3.0 Introduction

This paper is composed of three main ingredients : the first is the definition of a group of power series $\mathsf{G}_{\mathcal{P}}$ for any augmented operad $\mathcal{P}$. A related Lie algebra has been introduced before, from another point of view, by Kapranov and Manin [KM01].

The rest of the paper focuses on the case of the PreLie operad [CL01]. The group $\mathsf{G}_{\mathrm{PreLie}}$ is a group of series indexed by rooted trees. The second component of the paper is given by two quotient operads of PreLie corresponding respectively to linear trees and corollas. The operad on linear trees is just the associative operad Ass, whereas the operad Mu on corollas seems to be new. By functoriality, one gets group maps $\mathsf{G}_{\mathrm{PreLie}} \to \mathsf{G}_{\mathrm{Ass}}$ and $\mathsf{G}_{\mathrm{PreLie}} \to \mathsf{G}_{\mathrm{Mu}}$. Whereas $\mathsf{G}_{\mathrm{Ass}}$ is isomorphic to the group of formal power series for the composition product, G_{Mu} is related to the group of formal power series for the pointwise multiplication product.

The final ingredient comes from flow equations for vector fields on the affine space A^n. We recall the formula which gives the Taylor expansion of the solution of a flow equation using the pre-Lie structure on the vector space of vector fields on A^n. We have not traced the precise origin of this formula, although it is certainly not new, see [Bro00] and references therein. This formula can be interpreted as a distinguished element of the group $\mathsf{G}_{\mathrm{PreLie}}$, which we choose to denote by $\exp^*$, for its image in $\mathsf{G}_{\mathrm{Ass}}$ is $\exp - 1$, where exp is the usual exponential function.

3.1 A group associated to an augmented operad

Let $\mathcal{P}$ be an operad in the category $\mathrm{Vect}_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ and assume that $\mathcal{P}(0) = \{0\}$ and that $\mathcal{P}(1) = \mathbb{Q}e$ where e is the unit of $\mathcal{P}$. Such an operad is called augmented.

Let $\mathsf{F}\mathcal{P} = \oplus_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be the direct sum of the coinvariant spaces, which can be identified with the underlying vector space of the free $\mathcal{P}$-algebra on a generator v, and $\widehat{\mathcal{P}} = \prod_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be its completion.

Let $x = \sum_m x_m$, $y = \sum_n y_n$ be two elements of $\widehat{\mathcal{P}}$. Choose any representatives $\bar{x}_m$ of x_m (resp. $\bar{y}_n$ of y_n) in the operad $\mathcal{P}$. Then one can check that the following formula defines a product on $\widehat{\mathcal{P}}$.

$$x \times y = \sum_{m \geq 1} \sum_{n_1,\ldots,n_m \geq 1} \langle \gamma(\bar{x}_m, \bar{y}_{n_1}, \ldots, \bar{y}_{n_m}) \rangle, \tag{3.1}$$

where $\langle\,\rangle$ is the quotient map to the coinvariants and γ is the composition map of the operad $\mathcal{P}$.

Proposition 3.1.1 *The product $\times$ defines the structure of an associative monoid on the vector space $\widehat{\mathcal{P}}$. Furthermore, this product is $\mathbb{Q}$-linear on its left argument.*

Proof. Let us first prove the associativity. On the one hand, one has

$$\begin{aligned}(x \times y) \times z &= \sum_m \sum_{p_1,\ldots,p_m} \langle \gamma(\overline{(x \times y)}_m, \bar{z}_{p_1}, \ldots, \bar{z}_{p_m}) \rangle \\ &= \sum_m \sum_{n_1,\ldots,n_m} \sum_{p_1,\ldots,p_{n_1+\cdots+n_m}} \langle \gamma(\gamma(\bar{x}_m, \bar{y}_{n_1}, \ldots, \bar{y}_{n_m}), \bar{z}_{p_1}, \ldots, \bar{z}_{p_{n_1+\cdots+n_m}}) \rangle.\end{aligned} \tag{3.2}$$

On the other hand, one has

$$\begin{aligned}x \times (y \times z) &= \sum_m \sum_{n_1,\ldots,n_m} \langle \gamma(\bar{x}_m, \overline{(y \times z)}_{n_1}, \ldots, \overline{(y \times z)}_{n_m}) \rangle \\ &= \sum_m \sum_{n_1,\ldots,n_m} \sum_{(q_{i,j})} \langle \gamma(\bar{x}_m, \gamma(\bar{y}_{n_1}, \bar{z}_{q_{1,1}}, \ldots, \bar{z}_{q_{1,n_1}}), \ldots, \gamma(\bar{y}_{n_m}, \bar{z}_{q_{m,1}}, \ldots, \bar{z}_{q_{m,n_m}})) \rangle.\end{aligned} \tag{3.3}$$

Using then the "associativity" of the operad, one gets the associativity of $\times$. It is easy to check that the image of the unit e of the operad $\mathcal{P}$ is a two-sided unit for the $\times$ product. The left $\mathbb{Q}$-linearity is clear from the formula (3.1). ⬟

Proposition 3.1.2 *An element y of $\widehat{\mathcal{P}}$ is invertible for $\times$ if and only if the first component y_1 of y is non-zero.*

Proof. The direct implication is trivial. The reverse one is proved by a very standard recursive argument. ⬟

Let us call $\mathsf{G}_{\mathcal{P}}$ the set of invertible elements of $\widehat{\mathcal{P}}$ for the $\times$ product.

Proposition 3.1.3 G *is a functor from the category of augmented operads to the category of groups.*

Proof. The functoriality follows from inspection of the definitions of $\widehat{\mathcal{P}}$ and $\times$. ⬟

In fact, one can see $\mathsf{G}_{\mathcal{P}}$ as the group of $\mathbb{Q}$-points of a pro-algebraic group. The Lie algebra of this pro-algebraic group is given by the usual linearization process on the tangent space $\widehat{\mathcal{P}}$, resulting in the formula

$$[x, y] = \sum_{m\geq 1}\sum_{n\geq 1}\langle \bar{x}_m \circ \bar{y}_n - \bar{y}_n \circ \bar{x}_m\rangle, \tag{3.4}$$

where

$$\bar{x}_m \circ \bar{y}_n = \sum_{i=1}^{m} \gamma(\bar{x}_m, \underbrace{e, \ldots, e}_{i-1 \text{ units}}, \bar{y}_n, e, \ldots, e). \tag{3.5}$$

The graded Lie algebra structure on $\mathsf{F}\mathcal{P}$ defined by the same formulas has already appeared in the work of Kapranov and Manin on the category of right modules over an operad [KM01, Th. 1.7.3]. They explained that it acts by polynomial vector fields on the underlying vector space of any $\mathcal{P}$-algebra. For the endomorphism operad $\mathrm{End}(L)$ of a vector space L, $\mathsf{G}_{\mathrm{End}(L)}$ is the group of formal diffeomorphisms of L preserving the origin. So $\mathsf{G}_{\mathcal{P}}$ acts by formal diffeomorphisms on any $\mathcal{P}$-algebra L.

3.2 The flow of a vector field

Let A^n be the affine space of dimension n over $\mathbb{R}$. It is well-know [Cha01, Mat68] that there is a structure of pre-Lie algebra on the vector space V_n of smooth vector fields on A^n (see [CL01] for the notion of pre-Lie algebra). More precisely, let $x_1, \ldots, x_n$ be coordinates on A^n. Given two vector fields $F = \sum F_i\partial_i$ and $F' = \sum F'_j\partial_j$, their pre-Lie product $F \leftarrow F'$ is given by

$$F \leftarrow F' := \sum\sum F'_j(\partial_j F_i)\partial_i. \tag{3.6}$$

This does not depend on the choice of affine coordinates.

Let $F \in \mathsf{V}_n$ be a vector field. The flow equation of F is the following equation for a smooth function $g : \mathsf{A}^1 \to \mathsf{A}^n$ of the variable t :

$$\begin{cases} \frac{dg}{dt} = F(g), \\ g(0) = g_0, \end{cases} \tag{3.7}$$

where g_0 is any chosen point in A^n. Smoothness of F ensures uniqueness of the solution.

One can give a formal Taylor development at $t = 0$ of the solution of (3.7) using the following construction in the pre-Lie algebra of vector fields (see [Cha01, Proposition. 4]). Let $F^{\leftarrow k}$ denote the k^{th} right iterate of F for the pre-Lie product, *i.e.*

$$F^{\leftarrow k} = \underbrace{((\cdots((F \leftarrow F) \leftarrow F)\cdots \leftarrow F) \leftarrow F)}_{k \text{ factors}}. \tag{3.8}$$

Proposition 3.2.1 *The solution of the flow equation (3.7) has the following formal Taylor expansion :*

$$g(t) = g_0 + \sum_{k\geq 1} F^{\leftarrow k}(g_0)\frac{t^k}{k!}. \tag{3.9}$$

The proof is given in [Cha01].

In the special case where the vector field is linear, *i.e.* $F = \sum a_{i,j}x_i\partial_j$ with $a_{i,j} \in \mathbb{R}$, the formula (3.9) gives back the classical exponential formula :

$$g(t) = g_0 + (\exp(tA) - I)g_0 = \exp(tA)g_0. \tag{3.10}$$

where A is the matrix $(a_{i,j})$.

One can define an element $\exp^*(v)$ of the completed free pre-Lie algebra on a generator v over $\mathbb{Q}$ by

$$\exp^*(v) := \sum_{k\geq 1} \frac{v^{\leftarrow k}}{k!}. \tag{3.11}$$

Définition 1 *The image of* $\exp^*(v)$ *by the usual identification between the completed free pre-Lie algebra on* $\{v\}$ *and* $\widehat{\mathrm{PreLie}}$ *is an element of the group* $\mathsf{G}_{\mathrm{PreLie}}$*, which is denoted by* $\exp^*$*. Its inverse in this group is denoted by* $\log^*$*.*

The group $\mathsf{G}_{\mathrm{PreLie}}$ acts on the completed free pre-Lie algebra on v, and the action of $\exp^*$ on v is $\exp^*(v)$.

If F is a vector field on A^n, then the difference between the flow at time 0 and the flow at time 1 can be considered as another vector field G on A^n. In fact, formula (3.9) says that it is formally given by $\exp^* F$. So the meaning of $\log^*$ is the reverse operation : knowing the displacement G between time 0 and time 1, $\log^* G$ formally recovers the vector field F.

3.3 Linear trees and composition

As shown in [CL01], the PreLie operad can be described in terms of labeled rooted trees. By convention, edges are oriented towards the root. Let us call linear the trees that do no branch, that is to say all vertices have at most one incoming edge.

We recall here briefly (see [CL01] for more details) the definition of the composition of two labeled rooted trees T and S on the vertex sets I and J respectively. Let $i \in I$; the composition of S at the vertex i of T is given by

$$T \circ_i S = \sum_f T \circ_i^f S, \tag{3.12}$$

where the sum runs over all maps f from the set of incoming edges of the vertex i of T to the set of vertices of S, and $T \circ_i^f S$ can be described as follows : replace the vertex i by the tree S, grafting back the subtrees of T previously attached to i, according to the map f.

Proposition 3.3.1 *The subspace of* PreLie *spanned by non-linear labeled trees is an ideal. The quotient map* ϕ *coincides with the usual map from* PreLie *to the associative operad* Ass*.*

Proof. Using the description above of the composition map of the operad PreLie, it is clear that the composition of two labeled trees, at least one of which is non-linear, is again non-linear. The quotient operad, spanned by labeled linear trees, has dimension $n!$ in rank n. Its composition can be easily identified with the associative operad Ass. The quotient map is then checked on generators of PreLie to be the same as the usual map. ⬟

By functoriality of the group construction, there is a map, still denoted by ϕ, from $\mathsf{G}_{\text{PreLie}}$ to G_{Ass}.

Proposition 3.3.2 *The group* G_{Ass} *is isomorphic to the group of invertible formal power series in* $x\mathbb{Q}[[x]]$ *for the composition product.*

Proof. It is more convenient here to work at the monoid level with $\widehat{\text{Ass}}$ and $x\mathbb{Q}[[x]]$. The vector space $\text{Ass}(n)_{\mathfrak{S}_n}$ is one dimensional for all n, with a basis given by the image by ϕ of the linear tree with n nodes. Let us denote this basis element by θ_n. By left linearity of both monoids, it is sufficient to check the product rule for θ_m and $f = \sum_{n\geq 1} f_n \theta_n$. One finds that

$$\theta_m \times f = \sum_{n_1,\dots,n_m \geq 1} f_{n_1} \dots f_{n_m} \theta_{n_1+\dots+n_m}, \tag{3.13}$$

which proves that the linear map defined by $x^n \mapsto \theta_n$ is an isomorphism between the monoids $\widehat{\text{Ass}}$ and $x\mathbb{Q}[[x]]$. The propositionosition follows by taking invertible elements. ⬟

Proposition 3.3.3 *The image of* $\exp^*$ *by* ϕ *is* $\exp x - 1$ *and the image of* $\log^*$ *is* $\log(1 + x)$*.*

Proof. One has to prove that the coefficient of the linear tree with n nodes in $\exp^*$ is $1/n!$ for all n. This can be done by recursion, using (3.11). ⬟

This propositionosition explains what happens for linear vector fields. The linearity of a vector field F on A^n implies that the pre-Lie algebra generated by F inside V_n is associative, so that $\exp^*$ reduces to the standard exponential minus one.

3.4 Corollas and pointwise multiplication

There is another interesting kind of trees, opposite to linear trees. Let us call corollas the trees of depth no greater than two, where the depth is the maximum number of vertices in a chain of adjacent vertices starting from the root.

Proposition 3.4.1 *The subspace of* PreLie *spanned by labeled non-corollas is an ideal.*

Proof. Using the description above of the composition of PreLie, one shows that the depth of the composition of two labeled trees is greater or equal than the maximum of the depths of these labeled trees. Therefore, if one of the labeled trees has depth greater or equal to three, so does the composition. ⬟

One can give a simple description of the quotient operad Mu. It has dimension n in rank n with basis given by the image of the labeled corollas with n nodes. Let us call μ_i^n the image of the corolla with n nodes and with root labeled by i for $i = 1, \ldots, n$.

Then μ_1^1 is the unit of Mu and the composition is given by

$$\begin{cases} \mu_i^n \circ_i \mu_j^\ell = \mu_{i+j-1}^{n+\ell-1}, \\ \mu_i^n \circ_h \mu_j^\ell = 0 \text{ for } h \neq i \text{ and } \ell \geq 2. \end{cases} \tag{3.14}$$

Let G_1 be the group of formal power series of the form $1 + x\mathbb{Q}[[x]]$ for the pointwise multiplication product and G_2 be the multiplicative group $\mathbb{Q}^*$. There is an action of G_2 on G_1 by substitution : $\lambda \cdot f(x) = f(\lambda x)$. A group similar to the semi-direct product group $G_2 \ltimes G_1$ has been considered in [BF03, §2].

From the description of Mu above, one deduces that

Proposition 3.4.2 *The group* G_{Mu} *is isomorphic to* $G_2 \ltimes G_1$.

Proof. The vector space $\mathrm{Mu}(n)_{\mathfrak{S}_n}$ is one-dimensional for all n, with basis given by the image of the corolla with n nodes. Let us denote this basis element by ν_{n-1}. Any element of G_{Mu} can be uniquely written as the product $\lambda(\sum_{m\geq 0} f_m \nu_m)$ of $\lambda \in \mathbb{Q}^*$ and $f = \sum_{m \geq 0} f_m \nu_m$ with $f_0 = 1$. Let us compute the product of $\lambda f = \lambda(\sum_{m\geq 0} f_m \nu_m)$ and $\theta g = \theta(\sum_{n \geq 0} g_n \nu_n)$ with the conventions $f_0 = 1$ and $g_0 = 1$. One finds that

$$\lambda f \times \theta g = \sum_{m\geq 0} \sum_{n \geq 0} \lambda f_m \theta^m (\theta g_n) \nu_{n+m} = \lambda\theta \sum_{m\geq 0} \sum_{n \geq 0} \theta^m f_m g_n \nu_{n+m}. \tag{3.15}$$

One defines a map from G_{Mu} to $G_2 \ltimes G_1$ by $\lambda(\sum_m f_m \nu_m) \mapsto (\lambda, f(x))$ with $f(x) = \sum_m f_m x^m$. The product in $G_2 \ltimes G_1$ is given by

$$(\lambda, f(x))(\theta, g(x)) = (\lambda\theta, f(\theta x) g(x)). \tag{3.16}$$

Hence the map is an isomorphism. ⬟

Let us denote by ψ the quotient map from PreLie to Mu.

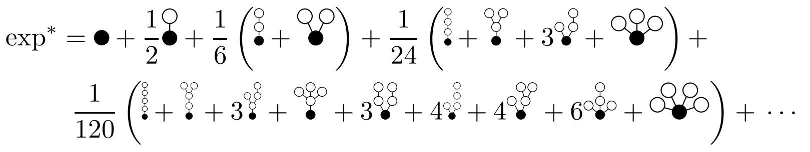
Proposition 3.4.3 *The image of* $\exp^*$ *by* ψ *is* $(\exp x - 1)/x$ *and the image of* $\log^*$ *is* $x/(\exp x - 1)$.

Proof. One must prove that the coefficient of the corolla with n nodes in $\exp^*$ is $1/n!$ for all n. The argument is a simple recursion using (3.11). ⬟

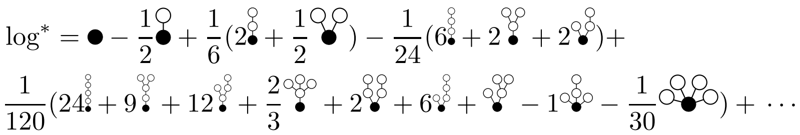
Therefore the coefficients of the corollas in $\log^*$ are related to the Bernoulli numbers, whose exponential generating function is precisely $x/(\exp x - 1)$.

3.5 Expansion

The coefficients of $\exp^*$, easily computed by Formula (3.11), are sometimes called the Connes-Moscovici coefficients [Kre99]. There exists a direct procedure to compute the coefficient of any rooted tree in $\exp^*$, see [Bro00, §2.2]. It may be interesting to find a way, other than inversion in the group $\mathsf{G}_{\mathrm{PreLie}}$, to compute the coefficients of $\log^*$. Here are the first terms of $\exp^*$ and $\log^*$ in the rooted tree basis of $\widehat{\mathrm{PreLie}}$.



(3.17)



(3.18)

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Chapitre 4

On some anticyclic operads

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Some binary quadratic operads are endowed with anticyclic structures and their characteristic functions as anticyclic operads are determined, or conjectured in one case.

4.0 Introduction

The most classical operads are probably the three operads describing commutative and associative algebras, associative algebras and Lie algebras. They fit in the following diagram :

$$\mathrm{Comm} \longleftarrow \mathrm{Ass} \longleftarrow \mathrm{Lie}\,. \tag{4.1}$$

These operads have very nice properties. First of all, they are binary quadratic and Koszul in the sense of the work of Ginzburg and Kapranov on the Koszul duality of operads [GK94]. More precisely Ass is self-dual and Lie and Comm are dual to each other. Moreover they are basic examples of cyclic operads, a notion introduced in [GK95] and related to the moduli spaces of curves with marked points. The developments of Koszul duality of operads and of the theory of cyclic operads were both partly motivated by the work of Kontsevich on non-commutative symplectic geometry [Kon93], where three parallel constructions are made for these three operads.

Our aim is to explain that most of the properties of this classical sequence of operads also hold for two other diagrams involving some binary quadratic operads introduced by Loday [Lod01] and others [Cha01, CL01]. The first of these diagrams is

$$\mathrm{Perm} \longleftarrow \mathrm{Dias} \longleftarrow \mathrm{Leib} \tag{4.2}$$

and the other one is

$$\mathrm{Zinb} \longleftarrow \mathrm{Dend} \longleftarrow \mathrm{PreLie}\,. \tag{4.3}$$

All these operads are already known to be binary quadratic and Koszul. More precisely, $(\mathrm{Perm}, \mathrm{PreLie})$, $(\mathrm{Dias}, \mathrm{Dend})$ and $(\mathrm{Leib}, \mathrm{Zinb})$ are Koszul dual pairs of operads. The main objective of this article is to show that they are anticyclic operads and that the maps in the two diagrams above are maps of anticyclic operads.

One interesting point with cyclic and anticyclic operads is that they allow to build Lie algebras and graph complexes [GK98, CV03, Mar99]. There has been a lot of work on the graph complexes for the three classical cyclic operads Comm, Ass and Lie. It may be worth considering the analogous structures for the six new anticyclic operads introduced here.

Let us also remark that there is a fourth classical cyclic operad, namely the Poisson operad, which can be obtained as the graded cyclic operad associated to a filtration of the cyclic operad Ass. Similar objects exists in the diagrams above, namely filtrations of the anticyclic operads Dias and Dend and associated graded anticyclic operads, one of which is related to the pre-Poisson algebras studied in [Agu00].

4.1 Anticyclic operads

We briefly state some general facts on operads and anticyclic operads. A convenient reference on this subject is [Mar99], see also [GK94, GK95]. Most of the operads considered here will be in the monoidal category of vector spaces over the field $\mathbb{Q}$, but the true ambient category is the category of chain complexes of vector spaces over $\mathbb{Q}$.

Recall that an operad $\mathcal{P}$ is a collection of modules $\mathcal{P}(n)$ over the symmetric groups $\mathfrak{S}_n$ together with composition maps satisfying some axioms modelled after the composition of multi-linear maps. A non-symmetric operad $\mathcal{P}$ is a similar structure without the actions of the symmetric groups. If $\mathcal{P}$ is a non-symmetric operad then the collection $\mathcal{P}(n) \otimes \mathbb{Q}\mathfrak{S}_n$ is naturally an operad.

An anticyclic non-symmetric operad is a non-symmetric operad $\mathcal{P}$ together with an action of a cyclic group of order $n+1$ on $\mathcal{P}(n)$ satisfying some axioms. In particular, the action of the cyclic group is determined by the action on the generators of the non-symmetric operad.

Similarly, an anticyclic operad is an operad $\mathcal{P}$ together with an action of a symmetric group $\mathfrak{S}_{n+1}$ on $\mathcal{P}(n)$ extending the action of $\mathfrak{S}_n$ and satisfying similar axioms. Here the group $\mathfrak{S}_n$ is embedded in $\mathfrak{S}_{n+1}$ as the stabilizer of $n+1$. In this case too, the action of the big symmetric group is determined by the action on the generators of the operad.

The main difference with the notion of cyclic operad is that the unit 1 of the operad is mapped to -1 in an anticyclic case and to 1 in the cyclic case. There is also some change of sign in the axioms.

The Hadamard product of two operads $\mathcal{P}_1$ and $\mathcal{P}_2$ is an operad with underlying modules $\mathcal{P}_1(n) \otimes \mathcal{P}_2(n)$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are cyclic or anticyclic operads, then the Hadamard product is an anticyclic or cyclic operad for the tensor product action of the symmetric groups $\mathfrak{S}_{n+1}$. Whether the Hadamard product is cyclic or anticyclic is determined by the action on the unit 1, *i.e.* is given by a sign rule where "cyclic" is $+1$ and "anticyclic" is -1.

Let $\mathbb{Z}/(n+1)\mathbb{Z}$ be the subgroup of $\mathfrak{S}_{n+1}$ generated by the longest cycle $(n+1, n, \dots, 2, 1)$. If $\mathcal{P}$ is an anticyclic non-symmetric operad, then the collection of induced modules $\mathrm{Ind}_{\mathbb{Z}/(n+1)\mathbb{Z}}^{\mathfrak{S}_{n+1}} \mathcal{P}(n)$ has a natural structure of anticyclic operad.

There exists a differential graded anticyclic operad, called the determinant operad and denoted by Det, such that $\mathrm{Det}(n)$ is a chain complex of dimension

1 concentrated in degree $n-1$. The Hadamard product by the operad Det corresponds to the suspension of operads. It maps anticyclic operads to cyclic operads and vice-versa.

4.2 Koszul duality and Legendre transform

The theory of Koszul duality for operads has been introduced in [GK94] for binary quadratic operads.

For each binary quadratic operad $\mathcal{P}$, one can define its dual operad $\mathcal{P}^!$ by elementary linear algebra using the quadratic presentation of $\mathcal{P}$. In particular, the space $\mathcal{P}^!(2)$ of generators of $\mathcal{P}^!$ is the tensor product of the dual space of $\mathcal{P}(2)$ by the sign representation of $\mathfrak{S}_2$.

A binary quadratic operad $\mathcal{P}$ is Koszul if the natural morphisms of cooperads from the bar complex of $\mathcal{P}$ to the dual cooperad of the suspension of $\mathcal{P}^!$ is a quasi-isomorphism.

The Koszul dual operad $\mathcal{P}^!$ of a Koszul operad $\mathcal{P}$ has a natural structure of cyclic (resp. anticyclic) operad if $\mathcal{P}$ is anticyclic (resp. anticyclic). When $\mathcal{P}$ is cyclic or anticyclic, the action of the cycle (321) on $\mathcal{P}^!(2)$ is given by the transpose of the inverse of the action on the space $\mathcal{P}(2)$.

When $\mathcal{P}$ is cyclic (resp. anticyclic), both the bar complex and the dual cooperad of the suspension of $\mathcal{P}^!$ are anticyclic (resp. cyclic) and the natural map between them is automatically a morphism of anticyclic (resp. cyclic) cooperads.

It is known that generating series of dual Koszul operads are related by inversion for the plethysm. More precisely, let the characteristic function of an operad $\mathcal{P}$ be

$$\mathrm{ch}(\mathcal{P}) = \sum_{n\geq 1} \mathrm{ch}_n(\mathcal{P}(n)), \tag{4.4}$$

where $\mathrm{ch}_n(\mathcal{P}(n))$ is the symmetric function for the $\mathfrak{S}_n$-module $\mathcal{P}(n)$. The suspension ΣF of a symmetric function F is defined as follows :

$$\Sigma F = -F(-p_1, -p_2, -p_3, \dots), \tag{4.5}$$

where the p_k are power sums symmetric functions. Then one has the following relation for a Koszul operad $\mathcal{P}$:

$$\mathrm{ch}(\mathcal{P}) \circ \Sigma\, \mathrm{ch}(\mathcal{P}^!) = p_1, \tag{4.6}$$

where $\circ$ is the plethysm of symmetric functions.

A similar relation exists at the level of cyclic or anticyclic Koszul operads, where the generating functions are related by the Legendre transform of symmetric functions introduced in [GK95]. Let us give the precise statement.

The characteristic function of a cyclic or anticyclic operad $\mathcal{P}$ is defined by

$$\mathbf{Ch}(\mathcal{P}) = \sum_{n\geq 1} \mathbf{Ch}_{n+1}(\mathcal{P}(n)), \tag{4.7}$$

where $\mathbf{Ch}_{n+1}(\mathcal{P}(n))$ is the symmetric function for the $\mathfrak{S}_{n+1}$-module $\mathcal{P}(n)$.

Let F be a symmetric function with no term of degree 0 and 1 and such that the term of degree 1 of $\partial_{p_1} F$ does not vanish. The Legendre transform $G = \mathcal{L}F$ of F is defined by

$$F \circ \frac{\partial G}{\partial p_1} + G = p_1 \frac{\partial G}{\partial p_1}. \tag{4.8}$$

The Legendre transform is an involution, with the property that the derived symmetric functions satisfy

$$\frac{\partial F}{\partial p_1} \circ \frac{\partial G}{\partial p_1} = p_1. \tag{4.9}$$

Then if $\mathcal{P}$ is an anticyclic (resp. cyclic) operad which is Koszul as an operad, one has

$$\mathcal{L}\,\mathbf{Ch}(\mathcal{P}) = -\Sigma\,\mathbf{Ch}(\mathcal{P}^!), \tag{4.10}$$

where the anticyclic (resp. cyclic) structure on $\mathcal{P}^!$ is the one induced by the Koszul duality.

4.3 The diassociative operad

In this section and the next one, we will consider two non-symmetric operads.

First let us recall some known facts about the non-symmetric operad Dias of diassociative algebras, see [Lod01].

First, it has a quadratic presentation. The generators are two binary operations $x \dashv y$ and $x \vdash y$. These generators satisfy the following relations :

$$x \dashv (y \vdash z) = x \dashv (y \dashv z) = (x \dashv y) \dashv z, \tag{4.11}$$

$$x \vdash (y \dashv z) = (x \vdash y) \dashv z, \tag{4.12}$$

$$x \vdash (y \vdash z) = (x \vdash y) \vdash z = (x \dashv y) \vdash z. \tag{4.13}$$

Next, it is known that the space Dias(n) has dimension n with a base $(e^n_m)_{m=1,\dots,n}$ such that the composition in the operad is given by

$$e^n_m \circ_i e^\ell_k = \begin{cases} e^{n+\ell-1}_{m+k-1} & \text{if } i = m, \\ e^{n+\ell-1}_{m+\ell-1} & \text{if } i < m, \\ e^{n+\ell-1}_{m} & \text{if } i > m. \end{cases} \tag{4.14}$$

In the presentation above, $x_1 \dashv x_2$ is e^2_1 and $x_1 \vdash x_2$ is e^2_2. More generally, the element e^n_m is mapped to

$$x_1 \vdash x_2 \vdash \cdots \vdash x_m \dashv \cdots \dashv x_n, \tag{4.15}$$

with arbitrary parentheses. Conversely, any iterated product of the variables $x_1, x_2, \dots, x_n$ in this order corresponds to some e^n_m by the following recursive procedure. At each step, choose the sub-expression which is not on the bar side of $\dashv$ or $\vdash$, until there remains only one variable x_m. For example,

$$((x_1 \dashv x_2) \vdash x_3) \dashv (x_4 \vdash x_5) \tag{4.16}$$

is mapped to e^5_3.

Let us now introduce a notion of invariant bilinear map on a diassociative algebra. It is an antisymmetric map with value in some vector space :

$$\langle x\,,\, y\rangle = -\langle y\,,\, x\rangle, \tag{4.17}$$

such that

$$\langle x \dashv y\,,\, z\rangle = -\langle y \vdash z\,,\, x\rangle \text{ and } \langle x \vdash y\,,\, z\rangle = \langle y \dashv z\,,\, x\rangle - \langle y \vdash z\,,\, x\rangle. \tag{4.18}$$

Let us define a map τ_1 as multiplication by -1 on Dias(1) and a map τ_2 on Dias(2) by the following relation

$$\langle E(x,y)\,,\,z\rangle = \langle \tau_2(E)(y,z)\,,\,x\rangle \tag{4.19}$$

for each element E in $\{e_1^2, e_2^2\}$, using the previous conditions on $\langle\,,\,\rangle$.

Théorème 4.3.1 *There exists a unique collection τ of endomorphisms τ_n of the space* Dias(n) *extending τ_1 and τ_2 and endowing the operad* Dias *with a structure of non-symmetric anticyclic operad.*

Proof. Clearly τ_1 is of order 2 and τ_2 is of order 3. As Dias is generated by Dias(2), the structure map τ is unique if it exists. To check that τ_n can be defined for $n \geq 3$, it is enough to check that the notion of invariant form is compatible with the relations defining the operad Dias. Let us check one of these compatibility conditions, for the relation (4.12) :

$$\langle x \vdash (y \dashv z) - (x \vdash y) \dashv z\,,\,t\rangle \tag{4.20}$$
$$= \langle (y \dashv z) \dashv t - (y \dashv z) \vdash t\,,\,x\rangle + \langle z \vdash t\,,\,x \vdash y\rangle \tag{4.21}$$
$$= \langle (y \dashv z) \dashv t - (y \dashv z) \vdash t\,,\,x\rangle - \langle x \vdash y\,,\,z \vdash t\rangle \tag{4.22}$$
$$= \langle (y \dashv z) \dashv t - (y \dashv z) \vdash t\,,\,x\rangle - \langle y \dashv (z \vdash t) - y \vdash (z \vdash t)\,,\,x\rangle = 0, \tag{4.23}$$

where one has used the antisymmetry and the invariance to obtain an expression with the x variable only in the right slot. The remaining checks are quite similar and are left to the reader. ⬟

In fact, τ_n has a very simple expression in the base e^n.

Lemme 4.3.2 *The action of τ_n is given by*

$$\tau_n(e_m^n) = \begin{cases} -e_n^n & \textit{if } m = 1, \\ -e_n^n + e_{m-1}^n & \textit{else.} \end{cases} \tag{4.24}$$

Proof. It is readily true for $n = 1$ and 2. Let us first prove that $\tau_{n+1}(e_1^{n+1}) = e_{n+1}^{n+1}$. This follows from the equality :

$$\langle x_1 \dashv x_2 \dashv \cdots \dashv x_{n+1}\,,\,x_{n+2}\rangle = -\langle (x_2 \dashv \cdots \dashv x_{n+1}) \vdash x_{n+2}\,,\,x_1\rangle \tag{4.25}$$
$$= -\langle x_2 \vdash \cdots \vdash x_{n+2}\,,\,x_1\rangle. \tag{4.26}$$

Let us now compute $\tau_{n+1}(e_m^{n+1})$ for $m \geq 2$. Indeed, one has

$$\begin{aligned} &\langle x_1 \vdash x_2 \vdash \cdots \vdash x_m \dashv \cdots \dashv x_{n+1}\,,\,x_{n+2}\rangle = \\ &\langle (x_2 \vdash \cdots \vdash x_m \dashv \cdots \dashv x_{n+1}) \dashv x_{n+2} - (x_2 \vdash \cdots \vdash x_m \dashv \cdots \dashv x_{n+1}) \vdash x_{n+2}\,,\,x_1\rangle \\ &= \langle x_2 \vdash \cdots \vdash x_m \dashv \cdots \dashv x_{n+2} - x_2 \vdash \cdots \vdash x_{n+2}\,,\,x_1\rangle. \end{aligned} \tag{4.27}$$

This concludes the proof. ⬟

One can note that the matrix of τ_n in the base e^n is a companion matrix for the polynomial $1 + x + \cdots + x^n$ and can also be described as $-({}^tL)^{-1}L$ where L is the lower triangular matrix with 1 everywhere below the diagonal.

4.4 The dendriform operad

Let us now recall some facts about the non-symmetric operad Dend describing dendriform algebras, see [Lod01].

First, it also has a quadratic presentation. The generators are two binary operations $x \prec y$ and $x \succ y$. These generators satisfy the following relations :

$$(x \prec y) \prec z) = x \prec (y \prec z) + x \prec (y \succ z), \tag{4.28}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{4.29}$$

$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z. \tag{4.30}$$

Next, it is known that the dimension of the space $\mathrm{Dend}(n)$ is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. There is a base Y^n of $\mathrm{Dend}(n)$ indexed by planar binary trees with $n+1$ leaves, in which the composition of the operad has a simple expression. The base Y^2 is made precisely of $x_1 \prec x_2$ and $x_1 \succ x_2$.

Let us introduce a notion of invariant bilinear map on a dendriform algebra. It is an antisymmetric map :

$$\langle x \, , \, y \rangle = -\langle y \, , \, x \rangle \tag{4.31}$$

such that

$$\langle x \succ y \, , \, z \rangle = \langle y \prec z \, , \, x \rangle \text{ and } \langle x \prec y \, , \, z \rangle = -\langle y \prec z \, , \, x \rangle - \langle y \succ z \, , \, x \rangle. \tag{4.32}$$

As before, let us define a map τ_1 as multiplication by -1 on $\mathrm{Dend}(1)$ and a map τ_2 on $\mathrm{Dend}(2)$ by the following relation

$$\langle E(x,y) \, , \, z \rangle = \langle \tau_2(E)(y,z) \, , \, x \rangle \tag{4.33}$$

for each element E in the base Y^2 of $\mathrm{Dend}(2)$, using the previous conditions on $\langle \, , \, \rangle$.

Théorème 4.4.1 *There exists a unique collection τ of endomorphisms τ_n of the space* $\mathrm{Dend}(n)$ *extending τ_1 and τ_2 and endowing the operad* Dend *with a structure of non-symmetric anticyclic operad.*

Proof. Clearly τ_1 is of order 2 and τ_2 is of order 3. As Dend is generated by $\mathrm{Dend}(2)$, the maps τ are unique if they exist. To check that τ_n can be defined for $n \geq 3$, it is enough to check that the notion of invariant form is compatible with the three relations defining the operad Dend. Let us check that it works for the second compatibility condition :

$$\langle x \succ (y \prec z) - (x \succ y) \prec z \, , \, t \rangle \tag{4.34}$$

$$= \langle (y \prec z) \prec t \, , \, x \rangle + \langle z \prec t \, , \, x \succ y \rangle + \langle z \succ t \, , \, x \succ y \rangle \tag{4.35}$$

$$= \langle (y \prec z) \prec t \, , \, x \rangle - \langle x \succ y \, , \, z \prec t \rangle - \langle x \succ y \, , \, z \succ t \rangle \tag{4.36}$$

$$= \langle (y \prec z) \prec t \, , \, x \rangle - \langle y \prec (z \prec t) + y \prec (z \succ t) \, , \, x \rangle = 0, \tag{4.37}$$

where one has used the antisymmetry and the invariance to obtain an expression with the x variable alone on the right. The two remaining computations are just as simple and are left to the reader. ⬟

The matrix of τ in the base of trees certainly deserves more study. It seems to be related to the so-called Tamari lattices [HT72].

4.5 Four operads

Starting from this section, we consider operads in the usual sense, which means with actions of the symmetric groups. As explained in section 4.1, we can consider the two non-symmetric anticyclic operads just defined as anticyclic operads, still denoted Dias and Dend. We will show that some sub-operads and quotient operads of these inherit an anticyclic structure.

4.5.1 The Perm operad

The Perm operad, introduced in [Cha01], is a quotient operad of the diassociative operad Dias. The space Perm(n) has dimension n and the action of $\mathfrak{S}_n$ is the usual permutation representation.

The operad Perm is the quotient of Dias by the ideal generated by the element $x_1 \dashv x_2 - x_2 \vdash x_1$. The image of the product $x_1 \dashv x_2$ will be denoted $x_1 x_2$.

The operad Perm is quadratic, generated by the binary product xy (regular representation of $\mathfrak{S}_2$) and with relations

$$(xy)z = x(yz) = x(zy). \tag{4.38}$$

Théorème 4.5.1 *The operad* Perm *has a unique structure of anticyclic operad such that the quotient map from* Dias *is a morphism of anticyclic operads.*

Proof. One has to check that the defining ideal is stable by the action of $\mathfrak{S}_{n+1}$ on Dias(n). It is enough to check this in Dias(2), which contains the generators of the ideal, where it is immediate. Hence Perm is a quotient anticyclic operad of Dias. ⬟

The resulting notion of invariant bilinear map is as follows. It is an antisymmetric map

$$\langle x\,,\, y\rangle = -\langle y\,,\, x\rangle \tag{4.39}$$

such that

$$\langle xy\,,\, z\rangle = -\langle zy\,,\, x\rangle \text{ and } \langle xy\,,\, z\rangle = \langle xz\,,\, y\rangle - \langle zx\,,\, y\rangle. \tag{4.40}$$

Théorème 4.5.2 *The action of* $\mathfrak{S}_{n+1}$ *on* Perm(n) *is isomorphic to the representation by reflections.*

Proof. Let us consider the action of $\mathfrak{S}_{n+1}$ by permutations on the module with base $(\varepsilon_i)_{i=1,\ldots,n+1}$. The reflection module is the submodule with base $b^n_m = \varepsilon_m - \varepsilon_{n+1}$ for $m = 1, \ldots, n$. The action of the subgroup $\mathfrak{S}_n$ is by permutations of the vectors b^n_m and the action of the cycle $\tau_n : (n+1, n, \ldots, 2, 1)$ is given by

$$\tau_n(b^n_m) = \begin{cases} -b^n_n & \text{if } m = 1, \\ -b^n_n + e^n_{m-1} & \text{else.} \end{cases} \tag{4.41}$$

On the other hand, the action of $\mathfrak{S}_n$ on the module Perm(n) with base $(e^n_m)_{m=1,\ldots,n}$ is by permutation of the vectors e^n_m. This base of Perm(n) is the image of the base $e^n_m \otimes 1$ of the operad Dend. Hence the action of the cycle τ_n is induced by the action of the cycle τ_n in the base e^n_m of the non-symmetric operad Dias which is given by Formula (4.24). This concludes the proof. ⬟

Corollaire 4.5.3 *The characteristic function of the anticyclic operad* Perm *is*

$$\mathbf{Ch}(\mathrm{Perm}) = (p_1 - 1)\exp(\sum_{k\geq 1} \frac{p_k}{k}) + 1. \tag{4.42}$$

Proof. This follows readily from the previous Theorem and the well-known fact that

$$\mathrm{ch}(\mathrm{Comm}) = \exp(\sum_{k\geq 1} \frac{p_k}{k}) - 1. \tag{4.43}$$

⬟

4.5.2 The Leibniz operad

The Leibniz operad is a sub-operad of Dias, also introduced in [Lod01]. It will be denoted by Leib.

It is the sub-operad of Dias generated by the element $[x_1, x_2] = x_1 \dashv x_2 - x_2 \vdash x_1$. Beware that this bracket is not antisymmetric. Leibniz algebras are "non-commutative Lie algebras" in some sense.

The quadratic presentation of Leib is the following. It is generated by the binary product $[x, y]$ (regular module of $\mathfrak{S}_2$) modulo the relation

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]. \tag{4.44}$$

It is known that the space $\mathrm{Leib}(n)$ is the regular representation of $\mathfrak{S}_n$, with base given by left-bracketed words in the variables $x_1, \ldots, x_n$.

Théorème 4.5.4 *The operad* Leib *has a unique structure of anticyclic operad such that the inclusion map into* Dias *is a morphism of anticyclic operads.*

Proof. It is enough to check that Leib(2) is indeed a submodule of Dias(2) for the action of $\mathfrak{S}_3$. This is immediate and implies that $\mathrm{Leib}(n)$ is stable by the action of the symmetric group $\mathfrak{S}_{n+1}$ on $\mathrm{Dias}(n)$. ⬟

The resulting notion of invariant bilinear map is as follows. It is an antisymmetric map

$$\langle x\,,\, y\rangle = -\langle y\,,\, x\rangle \tag{4.45}$$

such that

$$\langle [x, y]\,,\, z\rangle = \langle [z, y]\,,\, x\rangle \text{ and } \langle [x, y]\,,\, z\rangle = -\langle [z, x]\,,\, y\rangle - \langle [x, z]\,,\, y\rangle. \tag{4.46}$$

Théorème 4.5.5 *The action of* $\mathfrak{S}_{n+1}$ *on* $\mathrm{Leib}(n)$ *is isomorphic to the Lie module* $\mathrm{Lie}(n+1)$.

Proof. It was proved in [Cha01] that the operad Leib is the Hadamard product of the operads Perm and Lie. As explained in Section 4.1, the Hadamard product of a cyclic operad and an anticyclic operad is an anticyclic operad. One can check that the anticyclic structure obtained in this way on Leib coincides with the one introduced above. Hence the action of $\mathfrak{S}_{n+1}$ on $\mathrm{Leib}(n)$ is given by the tensor product of the $\mathfrak{S}_{n+1}$-modules $\mathrm{Lie}(n)$ and $\mathrm{Perm}(n)$. Then using Theorem 4.5.2 and [GK95, Corollary 6.8], this is known to be isomorphic to the module $\mathrm{Lie}(n+1)$. ⬟

4.5.3 The Pre-Lie operad

The PreLie operad has been introduced in [CL01]. It is the sub-operad of the dendriform operad generated by the operation $x_1 \curvearrowleft x_2 = x_2 \succ x_1 - x_1 \prec x_2$. The space PreLie($n$) has dimension n^{n-1}.

Let us recall the presentation of the operad PreLie. The product $x \curvearrowleft y$ is the regular module for $\mathfrak{S}_2$ and must satisfy the following relation :

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \quad (4.47)$$

Théorème 4.5.6 *The operad* PreLie *has a unique structure of anticyclic operad such that the inclusion map into* Dend *is a morphism of anticyclic operads.*

Proof. It is enough to check that PreLie(2) is indeed a submodule of Dend(2) for the $\mathfrak{S}_3$ action. This is immediate and implies that PreLie(n) is stable by the action of the symmetric group $\mathfrak{S}_{n+1}$ on Dend(n). ⬟

The resulting notion of invariant bilinear map is as follows. It is an antisymmetric map

$$\langle x \,,\, y\rangle = -\langle y \,,\, x\rangle \quad (4.48)$$

such that

$$\langle x \curvearrowleft y \,,\, z\rangle = -\langle x \curvearrowleft z \,,\, y\rangle \text{ and } \langle x \curvearrowleft y \,,\, z\rangle = -\langle y \curvearrowleft z \,,\, x\rangle + \langle z \curvearrowleft y \,,\, x\rangle. \quad (4.49)$$

Remark : As the space PreLie(n) is isomorphic as a $\mathfrak{S}_n$-module to the space with a base indexed by labelled rooted trees on n vertices, this implies the existence of a remarkable linear action of $\mathfrak{S}_{n+1}$ on this space.

Théorème 4.5.7 *One has the following equality of symmetric functions :*

$$\mathrm{ch}(\mathrm{PreLie})(1 + \mathbf{Ch}(\mathrm{PreLie})) = p_1(1 + \mathrm{ch}(\mathrm{PreLie}) + \mathrm{ch}(\mathrm{PreLie})^2). \quad (4.50)$$

Proof. One can check that PreLie is the Koszul dual anticyclic operad of Perm. Hence its characteristic function is obtained by a Legendre transform of the characteristic function of Perm.

But since Perm and PreLie are Koszul dual operads, it is known that

$$\mathrm{ch}(\mathrm{Perm}) \circ \Sigma\, \mathrm{ch}(\mathrm{PreLie}) = p_1. \quad (4.51)$$

This can be used to replace the equation defining the Legendre transform of **Ch**(Perm) by a relation no longer involving the plethysm. Applying the suspension Σ to this relation leads to Formula (4.50). ⬟

We propose here an explicit conjecture for the character of $\mathfrak{S}_{n+1}$ on PreLie(n) as a symmetric function.

Conjecture 4.5.8 *The characteristic function* **Ch**(PreLie) *of the anticyclic operad* PreLie *is*

$$\sum_{\lambda, |\lambda|\geq 1, \lambda_1 \neq 1} (\lambda_1 - 1)^{\lambda_1 - 2} \prod_{k \geq 2} \left((f_k(\lambda) - 1)^{\lambda_k} - k\lambda_k (f_k(\lambda) - 1)^{\lambda_k - 1}\right) \frac{p_\lambda}{z_\lambda}, \quad (4.52)$$

where the sums runs over non-empty partitions λ not having exactly one part of size 1*, λ_k denotes the number of parts of size k in the partition λ and $f_k(\lambda)$ denotes the number of fixed points of the k^{th} power of a permutation of cycle type λ. The notations p_λ and z_λ are classical for power sum symmetric functions and related constants.*

It is easy to check that the restriction to $\mathfrak{S}_n$ gives back the known formula for the action on rooted trees which can be found in [Lab86]. It has been checked by computer up to $n = 14$ that the expected characteristic function is indeed a positive linear combination of Schur functions and that Formula (4.50) holds.

4.5.4 The Zinbiel operad

The Zinbiel operad, denoted by Zinb, was introduced in [Lod01]. Maybe it would be more appealing to call it the shuffle operad. It is the quotient operad of Dend by the ideal generated by the following element

$$x_1 \prec x_2 - x_2 \succ x_1. \tag{4.53}$$

The image in Zinb of the product $x_1 \prec x_2$ will be denoted $x_1 x_2$.

The operad Zinb has a quadratic presentation. It is generated by the binary product xy (regular representation of $\mathfrak{S}_2$) subject to the relation :

$$(xy)z = x(yz) + x(zy). \tag{4.54}$$

The space Zinb(n) is isomorphic to the regular representation of $\mathfrak{S}_n$ and the composition of the operad can be described using shuffles of permutations.

Théorème 4.5.9 *The operad* Zinb *has a unique structure of anticyclic operad such that the quotient map from* Dend *is a morphism of anticyclic operads.*

Proof. Once again, it follows already from the check done for the PreLie operad that the ideal defining Zinb is indeed stable by the action of the symmetric group $\mathfrak{S}_{n+1}$ on Dend(n). Hence Zinb is a quotient anticyclic operad of Dend. ⬟

The resulting notion of invariant bilinear map is as follows. It is an antisymmetric map

$$\langle x\,,\, y\rangle = -\langle y\,,\, x\rangle \tag{4.55}$$

such that

$$\langle xy\,,\, z\rangle = \langle xz\,,\, y\rangle \text{ and } \langle xy\,,\, z\rangle = -\langle yz\,,\, x\rangle - \langle zy\,,\, x\rangle. \tag{4.56}$$

Théorème 4.5.10 *The action of $\mathfrak{S}_{n+1}$ on* Zinb(n) *is isomorphic to the Lie module* Lie($n+1$).

Proof. It is known that the operad Zinb is the Koszul dual of the operad Leib. One can check that the anticyclic structure of Leib is obtained from the anticyclic structure of Zinb by Koszul duality. Hence it follows that the characteristic of the anticyclic operad Zinb is related to the characteristic of the anticyclic operad Leib by a Legendre transform of symmetric functions. The characteristic of Leib is known by Theorem 4.5.5 to be

$$F = \sum_{n\geq 2} \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}. \tag{4.57}$$

One has to check that the Legendre transform of F is

$$G = \sum_{n\geq 2} \frac{1}{n} \sum_{d|n} \mu(d)(-1)^{n/d} p_d^{n/d}. \tag{4.58}$$

The derivative of G is

$$\partial_{p_1} G = \frac{p_1}{1+p_1}, \tag{4.59}$$

and one has

$$p_1 \partial_{p_1} G = \frac{p_1^2}{1+p_1}. \tag{4.60}$$

Let us compute $F \circ \partial_{p_1} G + G$. One finds

$$\sum_{n \geq 2} \frac{1}{n} \sum_{d|n} \mu(d) \left(\frac{1}{(1+p_d)^{n/d}} + (-1)^{n/d} \right) p_d^{n/d}. \tag{4.61}$$

Now fix $d \geq 2$ and look only at the extracted series in p_d :

$$\frac{\mu(d)}{d} \sum_{k \geq 1} \frac{1}{k} \left(\frac{1}{(1+p_d)^k} + (-1)^k \right) p_d^k. \tag{4.62}$$

This is easily expressed using logarithms and seen to vanish. Hence the expression $F \circ \partial_{p_1} G + G$ is given by

$$\sum_{n \geq 2} \frac{1}{n} \left(\frac{1}{(1+p_1)^n} + (-1)^n \right) p_1^n = \frac{p_1^2}{1+p_1} \tag{4.63}$$

as expected. ⬟

Let us remark that exactly the same computation with the Möbius function μ replaced by the Euler totient function φ corresponds to self-duality of the characteristic function for the cyclic Koszul operad Ass.

4.6 Characters of Dend and Dias

Théorème 4.6.1 *The characteristic function of the anticyclic operad* Dias *is*

$$\mathbf{Ch}(\mathrm{Dias}) = \sum_{n \geq 2} \left(p_1^n - \frac{1}{n} \sum_{d|n} \varphi(d) p_d^{n/d} \right). \tag{4.64}$$

Proof. It is easy to check that the anticyclic operad Dias is obtained as the Hadamard product of the anticyclic operad Perm by the cyclic operad Ass.

It is known that the characteristic function of the cyclic operad Ass is

$$\sum_{n \geq 2} \frac{1}{n} \sum_{d|n} \varphi(d) p_d^{n/d}, \tag{4.65}$$

where φ is the Euler totient function.

Using Theorem 4.5.2, one knows that the tensor product by the $\mathfrak{S}_{n+1}$ module Perm(n) is given by the operator $-\mathrm{Id} + \partial_{p_1}$. Then a simple computation proves the Theorem. ⬟

One can check that the anticyclic structure of Dend is the one obtained from the anticyclic structure of Dias by Koszul duality. Hence, by Legendre inversion, one gets that the characteristic function of Dend is related to the Legendre transform of the characteristic function of Dias.

Théorème 4.6.2 *The characteristic function of the anticyclic operad* Dend *is*

$$\mathbf{Ch}(\mathrm{Dend}) = 1 - p_1 - \sqrt{1-4p_1} - \sum_{n\geq 1}\left(\frac{1}{2n}\sum_{d|n}\varphi(d)\binom{2n/d}{n/d}p_d^{n/d}\right). \quad (4.66)$$

Proof. Let us check that the Legendre transform gives this result. Let F be the characteristic function (4.64) of Dias and let G be

$$1 + p_1 - \sqrt{1+4p_1} - \sum_{n\geq 1}\frac{1}{2n}\sum_{d|n}\varphi(d)\binom{2n/d}{n/d}(-p_d)^{n/d}. \quad (4.67)$$

One has to check that $F \circ \partial_{p_1} G + G = p_1 \partial_{p_1} G$.

Then one has

$$\partial_{p_1} G = \frac{1 + 2p_1 - \sqrt{1+4p_1}}{2p_1} \quad (4.68)$$

and

$$p_1 \partial_{p_1} G = \frac{1 + 2p_1 - \sqrt{1+4p_1}}{2}. \quad (4.69)$$

Let us compute $p_1\partial_{p_1} G - F \circ \partial_{p_1} G$. One finds

$$\frac{1 + 2p_1 - \sqrt{1+4p_1}}{2} - \left(\frac{1 + 2p_1 - \sqrt{1+4p_1}}{2p_1}\right)^2 \frac{1}{1 - \left(\frac{1+2p_1-\sqrt{1+4p_1}}{2p_1}\right)}$$
$$+ \sum_{n\geq 2}\frac{1}{n}\sum_{d|n}\varphi(d)\left(\frac{1 + 2p_d - \sqrt{1+4p_d}}{2p_d}\right)^{n/d}. \quad (4.70)$$

Let us split this sum into

$$\frac{1 + 2p_1 - \sqrt{1+4p_1}}{2} - \frac{\left(1 + 2p_1 - \sqrt{1+4p_1}\right)^2}{2p_1\left(\sqrt{1+4p_1} - 1\right)} - \frac{1 + 2p_1 - \sqrt{1+4p_1}}{2p_1} \quad (4.71)$$

and

$$\sum_{n\geq 1}\frac{1}{n}\sum_{d|n}\varphi(d)\left(\frac{1 + 2p_d - \sqrt{1+4p_d}}{2p_d}\right)^{n/d}. \quad (4.72)$$

Let us compute these separately. The first part is easily seen to be

$$1 + p_1 - \sqrt{1+4p_1}. \quad (4.73)$$

The second part becomes

$$\sum_{d\geq 1}\frac{\varphi(d)}{d}\sum_{k\geq 1}\frac{1}{k}\left(\frac{1 + 2p_d - \sqrt{1+4p_d}}{2p_d}\right)^{k}, \quad (4.74)$$

which is

$$\sum_{d\geq 1}\frac{\varphi(d)}{d}\log\left(\frac{1 + \sqrt{1-4p_d}}{2}\right). \quad (4.75)$$

Using then the Taylor expansion

$$-\log\left(\frac{1+\sqrt{1-4u}}{2}\right)=\sum_{n\geq 1}\frac{1}{2n}\binom{2n}{n}u^n, \tag{4.76}$$

one gets

$$-\sum_{d\geq 1}\frac{\varphi(d)}{d}\sum_{k\geq 1}\frac{1}{2k}\binom{2k}{k}(-p_d)^k, \tag{4.77}$$

which becomes

$$-\sum_{n\geq 1}\frac{1}{2n}\sum_{d|n}\varphi(d)\binom{2n/d}{n/d}(-p_d)^{n/d}. \tag{4.78}$$

Summing both parts gives, as expected, that $p_1\partial_{p_1}G - F\circ\partial_{p_1}G = G$. ⬟

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Chapitre 5

The anticyclic operad of moulds

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A new anticyclic operad Mould is introduced, on spaces of functions in several variables. It is proved that the Dendriform operad is an anticyclic suboperad of this operad. Many operations on the free Mould algebra on one generator are introduced and studied. Under some restrictions, a forgetful map from moulds to formal vector fields is then defined. A connection to the theory of tilting modules for quivers of type $\mathbb{A}$ is also described.

5.0 Introduction

The aim of this article is to build and use a new connection between the theory of operads and the theory of moulds. Operads were introduced in algebraic topology in the 1960's. After being somewhat neglected for some decades, this notion has found a new impetus recently, in connection with mathematical physics, moduli spaces of curves and algebraic combinatorics.

Moulds have a rather different origin. They have been introduced in analysis by J. Ecalle, as a convenient tool to handle complicated singular functions, in relation with his theory of resurgence. Later, he developed around moulds a large apparatus which allowed him to make substantial progress in the theory of polyzetas [Eca02, Eca03, Eca04]. In this article, only the simplest case of moulds will be considered, not the more general case of bimoulds.

The first result of this article is the existence of a very simple structure of operad on moulds, denoted by Mould. In fact, one can define on moulds the finer structure of an anticyclic operad, involving in addition to composition maps some actions of cyclic groups.

This structure is then shown to contain as an anticyclic suboperad the so-called Dendriform operad introduced by J.-L. Loday [Lod01] and denoted by Dend, which has been much studied recently [LR98, LR02]. This provides a radically new point of view on the operad Dend and the key to some new results. The main result is the explicit description of the smallest subset of Dend containing its usual generators and closed under the anticyclic operad structure, by the mean of new combinatorial objects called non-crossing plants.

This article also contains the description of many different operations on moulds, coming either from the operad or from the mould viewpoint, and some of their properties. This is used to prove the existence of a morphism from the Lie algebra of moulds (for one of the Lie brackets and under some restrictions) to the Lie algebra of formal vector fields in one indeterminate. Interesting and natural examples of moulds are provided and their images by this map are computed.

In some sense, this article provides a reformulation of the basic results of Ecalle on moulds in a more classical algebraic language. This includes notably the so-called ARI bracket, for which we provide a very short definition using the operations obtained from the operad structure. We use this definition to prove some properties of this bracket. It should be said that our setting does not seem to extend to bimoulds, hence can only describe a small part of the theory of Ecalle.

In the last section, it is recalled that the Dendriform operad is strongly related to the theory of tilting modules for the equi-oriented quivers of type $\mathbb{A}$, and how the results of the present article fit very-well in this relationship. Some conjectural extension of the properties in type $\mathbb{A}$ to other Dynkin diagrams are proposed.

Many thanks to Jean Ecalle for discussions which have led to this research. The computer algebra system MuPAD has been used for some computations. This research has been supported by the ANR grant HOPFCOMBOP.

5.1 Notations and definitions

We recall here the terminology we need concerning moulds.

A *mould* is a sequence $(f_n)_{n\geq 1}$, where f_n is a function of the variables $\{u_1, \ldots, u_n\}$. A mould is said to have degree n if its only non-zero component is f_n. In this case, this unique component will be denoted f by a convenient abuse of notation.

A mould f of degree n is called *alternal* if it satisfies the following conditions, for $1 \leq i \leq n-1$:

$$\sum_{\sigma \in \mathrm{Sh}(i,n-i)} f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = 0, \tag{5.1}$$

where σ runs over the shuffle permutations of $\{1, \ldots, i\}$ and $\{i+1, \ldots, n\}$, i.e. permutations such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. One can then extend this definition : a mould is called alternal if each of its components is alternal.

A mould f of degree n is called *vegetal* if it satisfies the following equation :

$$u_1 \ldots u_n \sum_{\sigma \in \mathfrak{S}_n} f(t\, u_{\sigma(1)}, \ldots, t\, u_{\sigma(n)}) = n! f(t, \ldots, t), \tag{5.2}$$

where $\mathfrak{S}_n$ is the permutation group of $\{1, \ldots, n\}$. We will show that this property is preserved by many operations on moulds.

There is a natural associative product on moulds, defined for f of degree m and g of degree n by

$$\mathrm{MU}(f, g) = f(u_1, \ldots, u_m) g(u_{m+1}, \ldots, u_{m+n}). \tag{5.3}$$

The associated Lie bracket is

$$\mathrm{LIMU}(f,g) = \mathrm{MU}(f,g) - \mathrm{MU}(g,f). \tag{5.4}$$

These somewhat unusual notations are those used by Ecalle.

We will often use the following convenient shorthand notation :

$$u_{i..j} := \sum_{i \leq k \leq j} u_k. \tag{5.5}$$

At some places in the text, we will use the following shorthand notation. For a shuffle σ of two ordered sets S' and S'', let u_σ be the sequence u_s for $s \in S' \cup S''$ in the order specified by σ.

In the next section, we define what is usually called a non-symmetric operad. The reader which is not familiar with this notion should have a look at the definition for instance in [MSS02] or in [Lod01]. We recall here the definition only implicitly, by checking carefully that all the axioms are satisfied in our case.

There is also a notion of symmetric operad which is very similar with the additional data of actions of symmetric groups. There is a forgetful functor from symmetric operads to non-symmetric operads. We will use at some point the adjoint functor sending a non-symmetric operad to its symmetric version.

5.2 The Mould operad

For $n \geq 1$, let Mould(n) be the vector space of rational functions with rational coefficients in the variables $\{u_1, \dots, u_n\}$. We will show that the collection Mould $= (\mathrm{Mould}(n))_{n \geq 1}$ has the structure of an anticyclic non-symmetric operad. The reader is referred to [Mar99, MSS02] for the basics of the theory of operads and anticyclic operads.

First, let $\mathbf{1}$ be the function $1/u_1$ in Mould(1). This will be the unit of the operad.

Let us then introduce a map τ, which is called the *push*. It is defined on Mould(n) by

$$\tau(f)(u_1, \dots, u_n) = f(-u_{1\dots n}, u_1, \dots, u_{n-1}). \tag{5.6}$$

Note that τ has order $n+1$ on Mould(n). It will give the cyclic action of the operad. Let us note also that $\tau(\mathbf{1}) = -\mathbf{1}$. This is one of the axioms of an anticyclic operad.

Let us now introduce the composition maps $\circ_i$ from $\mathrm{Mould}(m) \otimes \mathrm{Mould}(n)$ to $\mathrm{Mould}(m+n-1)$, with $1 \leq i \leq m$. Let f be in Mould(m) and g be in Mould(n). The function $f \circ_i g$ is defined by

$$u_{i\dots i+n-1} f(u_1, \dots, u_{i-1}, u_{i\dots i+n-1}, u_{i+n}, \dots, u_{m+n-1}) g(u_i, \dots, u_{i+n-1}). \tag{5.7}$$

Théorème 5.2.1 *The push τ and composition maps $\circ_i$ define the structure of an anticyclic non-symmetric operad* Mould.

Proof. One has first to check that these composition maps do indeed define a non-symmetric operad. The unit $\mathbf{1}$ has clearly the expected properties : $\mathbf{1} \circ_1 f = f$ and $f \circ_i \mathbf{1} = f$ for all $f \in \mathrm{Mould}(m)$ and $1 \leq i \leq m$. One has also to check two “associativity” axioms.

Let f, g, h be in $\mathrm{Mould}(m), \mathrm{Mould}(n)$ and $\mathrm{Mould}(p)$.

Let i and j be such that $1 \leq i < j \leq m$. Then one has to check that

$$(f \circ_i g) \circ_{j+n-1} h = (f \circ_j h) \circ_i g. \tag{5.8}$$

Indeed, both sides are equal to

$$u_{i\ldots i+n-1}\, u_{j+n-1\ldots j+n+p-2}\, g(u_i, \ldots, u_{i+n-1}) h(u_{j+n-1}, \ldots, u_{j+n+p-2}) \\ f(u_1, \ldots, u_{i-1}, u_{i\ldots i+n-1}, u_{i+n}, \\ \ldots, u_{j+n-2}, u_{j+n-1\ldots j+n+p-2}, u_{j+p+n-1}, \ldots, u_{m+n+p-2}). \tag{5.9}$$

Let now i and j be such that $1 \leq i \leq m$ and $1 \leq j \leq n$. One has to check that

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h. \tag{5.10}$$

Indeed, both sides are equal to

$$f(u_1, \ldots, u_{i-1}, u_{i\ldots i+p+n-2}, u_{i+p+n-1}, \ldots, u_{m+n+p-2}) \\ g(u_i, \ldots, u_{j+i-2}, u_{j+i-1\ldots j+i+p-2}, u_{j+i+p-1}, \ldots, u_{i+p+n-2}) \\ u_{i\ldots i+p+n-2}\, u_{j+i-1\ldots j+i+p-2} h(u_{j+i-1}, \ldots, u_{j+i+p-2}). \tag{5.11}$$

This proves that Mould is a non-symmetric operad. Then one has to verify that τ gives furthermore an anticyclic structure on this operad.

For this, one has to check two identities. The first one is

$$\tau(f \circ_i g) = \tau(f) \circ_{i-1} g, \tag{5.12}$$

for $f \in \mathrm{Mould}(m)$, $g \in \mathrm{Mould}(n)$ and $2 \leq i \leq m$. Indeed, this holds true, as both sides are equal to

$$f(-u_{1\ldots m+n-1}, u_1, \ldots, u_{i-2}, u_{i-1\ldots n+i-2}, u_{n+i-1}, \ldots, u_{m+n-2}) \\ g(u_{i-1}, \ldots, u_{n+i-2}) u_{i-1\ldots n+i-2}. \tag{5.13}$$

The other identity that we have to check is

$$\tau(f \circ_1 g) = -\tau(g) \circ_n \tau(f), \tag{5.14}$$

for $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$. Again, this is true as both sides are equal to

$$-u_{n\ldots m+n-1} f(-u_{n\ldots m+n-1}, u_n, \ldots, u_{m+n-2}) g(-u_{1\ldots m+n-1}, u_1, \ldots, u_{n-1}). \tag{5.15}$$

⬟

Remarque 1 It follows from Eq. (5.7) that if f and g are homogeneous functions of weight d_f and d_g (where all variables u_i are taken with weight 1), then $f \circ_i g$ is also homogeneous of weight $d_f + d_g + 1$. Also, the action of τ clearly preserves the weight in that sense. Hence the collection of subspaces of homogeneous rational functions of weight $-n$ in $\mathrm{Mould}(n)$ is a anticyclic suboperad.

Remarque 2 Another consequence of Eq. (5.7) is the following. Let H_n be the product

$$H_n = \prod_{1 \leq i \leq j \leq n} u_{i \ldots j}. \tag{5.16}$$

One can see that, if $H_m f$ and $H_n g$ are polynomials, then so is $H_{m+n-1}(f \circ_i g)$. It is also true that $H_m \tau(f)$ is polynomial if $H_m f$ is, as one can easily check that τ preserves H_m up to sign.

Combining the two previous remarks, one gets that the subspace of Mould(n) made of homogeneous rational functions f of weight $-n$ such that $H_n f$ is a polynomial define a anticyclic suboperad, which is finite dimensional in each degree.

Remarque 3 By a similar argument, one can also note that the subspace of rational functions that have only poles of the shape $u_{i \ldots j}$ (at some power) for some $i \leq j$ is also stable for the composition and the cyclic action. Such functions will be said to have *nice poles*.

5.3 The Dendriform operad

The Dendriform operad was introduced by Loday [Lod01]. Later, it was shown to be an anticyclic operad [Cha05a]. We refer the reader to the book [Lod01] for more details on this operad.

Recall that the Dendriform operad is an operad in the category of vector spaces, generated by $\curlyvee_{\ell}$ and $\curlyvee_{r}$ of degree 2 with relations

$$\curlyvee_{\ell} \circ_1 (\curlyvee_{\ell} + \curlyvee_{r}) = \curlyvee_{\ell} \circ_2 \curlyvee_{\ell}, \tag{5.17}$$

$$\curlyvee_{r} \circ_1 \curlyvee_{\ell} = \curlyvee_{\ell} \circ_2 \curlyvee_{r}, \tag{5.18}$$

$$\curlyvee_{r} \circ_1 \curlyvee_{r} = \curlyvee_{r} \circ_2 (\curlyvee_{\ell} + \curlyvee_{r}). \tag{5.19}$$

The dimension of Dend(n) is the Catalan number

$$c_n = \frac{1}{n+1} \binom{2n}{n}. \tag{5.20}$$

There is a basis of Dend(n) indexed by the set $\mathcal{Y}(n)$ of rooted planar binary trees with $n+1$ leaves. In the presentation above, $\curlyvee_{\ell}$ and $\curlyvee_{r}$ correspond to the two planar binary trees in $\mathcal{Y}(2)$. We will sometimes denote by Y the unique planar binary tree of degree 1, which is the unit of the Dendriform operad.

The cyclic action is defined on the generators by

$$\tau(\curlyvee_{\ell}) = \curlyvee_{r}, \tag{5.21}$$

$$\tau(\curlyvee_{r}) = -(\curlyvee_{\ell} + \curlyvee_{r}). \tag{5.22}$$

Théorème 5.3.1 *There is a unique map ψ of anticyclic non-symmetric operads from* Dend *to* Mould *which maps $\curlyvee_{\ell}$ to $1/(u_1 u_{1\ldots 2})$ and $\curlyvee_{r}$ to $1/(u_{1\ldots 2} u_2)$.*

Proof. It is quite immediate to check, using the known quadratic binary presentation of Dend and the description of the cyclic action on these generators recalled above, that this indeed defines a morphism ψ of operads and that this morphism ψ is a morphism of anticyclic operads. ⬟

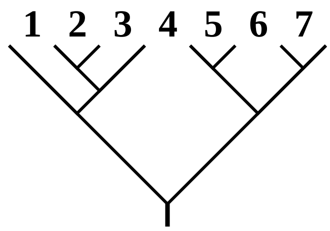

FIG. 5.1 – a planar binary tree T and the standard numbering

Let us now describe the image by ψ of a planar binary tree T in $\mathcal{Y}(n)$.

Let us define, for each inner vertex v of T, a linear function $\underline{\dim}(v)$ in the variables $u_1, \dots, u_n$. One can label from left to right the spaces between the leaves from 1 to n as in Fig. 5.1. Then the vertex v defines a pair of leaves (its leftmost and rightmost descendants), enclosing a subinterval $[i, j]$ of $[1, n]$. Let $\underline{\dim}(v)$ be $u_{i\dots j}$.

Proposition 5.3.2 *Let T be a planar binary tree. Then its image $\psi(T)$ is the inverse of the product of factors $\underline{\dim}(v)$ over all inner vertices v of T.*

For instance, the image of the tree T of Fig. 5.1 is

$$\psi(T) = 1/\left(u_{1\dots3}u_2u_{2\dots3}u_{1\dots7}u_5u_{5\dots7}u_7\right). \tag{5.23}$$

Proof. The proof is by induction on n. The proposition is true for $n = 1$ or 2.

Assume that the Proposition is true up to degree n. Let T be a planar binary tree in $\mathcal{Y}(n+1)$. By picking a top vertex of T (any inner vertex of maximal height), one can find a tree S in $\mathcal{Y}(n)$ and an index i such that $T = S \circ_i \curlyvee$ or $T = S \circ_i \curlyvee$.

Then one can check that the description given above has the correct behavior with respect to such compositions in Mould and in Dend. ⬟

Let us now introduce a classical map π from permutations of $\{1, \dots, n\}$ to planar binary trees in $\mathcal{Y}(n)$. First, note that one can use the standard numbering as in Fig. 5.1 to label the inner vertices of a planar binary tree from 1 to n from left to right. Then each tree T induces a natural partial order $\leq_T$ on $\{1, \dots, n\}$ by saying that $i \leq_T j$ if the inner vertex i is below (i.e. is an ancestor of) the inner vertex j. The map π is characterized by the property that $\pi(\sigma) = T$ if and only if the total order $\sigma(1) < \sigma(2) < \dots < \sigma(n)$ is an extension of the partial order $\leq_T$. In particular, $\sigma(1)$ must be the index of the bottom inner vertex of T. The map π is surjective and has a standard construction by induction, see for example [LR98]. For instance, the images by π of the permutations 4163527 and 4651372 are both the tree of Fig. 5.1.

Let us define the multi-residue of an element in Mould(n) with respect to a permutation $\sigma \in \mathfrak{S}_n$:

$$\oint_\sigma f = (2i\pi)^{-n} \oint_{u_{\sigma(1)}} \dots \oint_{u_{\sigma(n)}} f. \tag{5.24}$$

Proposition 5.3.3 *For a planar binary tree T in $\mathcal{Y}(n)$ and a permutation $\sigma \in \mathfrak{S}_n$, the multi-residue $\oint_\sigma \psi(T)$ does not vanish if and only if $\pi(\sigma) = T$.*

Proof. The proof is by induction. The statement is clear if $n = 1$. Let k be $\sigma(n)$.

Let us assume that $\pi(\sigma) = T$. By the discussion above on the properties of π, this implies that the vertex k is a top vertex of T (a maximal element for $\leq_T$). Then by computing the innermost residue with respect to u_k, the multi-residue $\oint_\sigma \psi(T)$ reduces to the multi-residue of the function obtained by replacing u_k by 0 in $u_k\psi(T)$, with respect to indices $\{1, \ldots, n\}$ but k, in some order. By renumbering the variables, this multi-residue is just $\oint_{\sigma'} \psi(T')$ where T' is obtained by removing the top vertex k from T and σ' is the induced permutation of $\{1, \ldots, n-1\}$. It is clear that $\pi(\sigma') = T'$, hence the residue $\oint_{\sigma'} \psi(T')$ is not zero by induction and therefore the residue $\oint_\sigma \psi(T)$ is not zero too.

Let us now assume that $\pi(\sigma) \neq T$. If the vertex numbered k is not a top vertex , then the residue with respect to u_k is zero, as u_k is not a pole of $\psi(T)$. If the vertex number k is a top vertex, then, with the same notations as above, one necessarily has that $\pi(\sigma') \neq T'$. Hence by induction $\oint_{\sigma'} \psi(T') = 0$ and therefore $\oint_\sigma \psi(T) = 0$.

⬟

Théorème 5.3.4 *The morphism ψ is injective.*

Proof. It is enough to prove that the the functions $\psi(T)$ for all planar binary trees in $\mathcal{Y}(n)$ are linearly independent. This follows from the previous proposition. ⬟

5.3.1 The Associative operad

It is known (see [Lod02]) that the Associative operad is the suboperad of the Dendriform operad generated by $\curlyvee + \curlyvee$. Furthermore, the basis of the one-dimensional space Ass(n) is mapped in Dend to the sum of all planar binary trees.

Inside the Mould operad, as the image of $\curlyvee + \curlyvee$ is $1/(u_1u_2)$, one can check by induction that the image of the basis of Ass(n) is the inverse of the product $u_1 \ldots u_n$.

Remarque 4 the Associative operad is not stable for the anticyclic structure, though. The smallest anticyclic suboperad of Mould containing Ass is Dend.

5.3.2 The graded Tridendriform operad

Several generalizations of the Dendriform operad have been introduced in [Cha02] and [LR04]. They all share the common point that they have in degree n a basis indexed by all planar trees instead of just planar binary trees.

Let us consider among them the operad *gr* TriDend which is the associated graded operad of the Tridendriform operad (which is a filtered operad). It has been considered both in [Cha02] and [LR04] and contains as a suboperad the Dendriform operad. The operad *gr* TriDend is generated by the dendriform generators together with another associative operation $\curlyvee$ in degree 2. One can deduce from the results of [LR04] a presentation by generators and relations of *gr* TriDend. It consists of the associativity relations for $\curlyvee$, of the 3 relations for the dendriform generators Y_{ℓ} and Y_{r} given at the beginning of §5.3, and of 3 more relations :

$$\Psi \circ_1 \mathrm{Y}_{r} = \Psi \circ_2 \mathrm{Y}_{\ell}, \tag{5.25}$$

$$\Psi \circ_1 \mathrm{Y}_{\ell} = \mathrm{Y}_{\ell} \circ_2 \Psi, \tag{5.26}$$

$$\mathrm{Y}_{r} \circ_1 \Psi = \Psi \circ_2 \mathrm{Y}_{r}. \tag{5.27}$$

Proposition 5.3.5 *By extending the morphism ψ by $\Psi \mapsto \frac{1}{u_1+u_2}$, one gets a morphism (still denoted by ψ) of operads from gr* TriDend *to* Mould.

Proof. This is an immediate verification. ⬟

We will not check here whether or not this morphism is injective. This may follow from the same kind of arguments as for the Dendriform operad. Indeed, the image of a planar tree by ψ has a simple description given by an obvious extension of Prop. 5.3.2.

Remarque 5 *The image of gr* TriDend *by ψ is not closed for the action of τ. It may be interesting to find a description of the smallest anticylic sub-operad of* Mould *containing gr* TriDend.

5.4 Free algebra on one generator

Let us consider the free algebra on one generator for the operad Mould. This can be identified with the direct sum of all spaces Mould(n), which will also be denoted by Mould.

5.4.1 Dendriform products

The inclusion of the operad Dend in Mould defines the structure of a dendriform algebra on the free Mould algebra on one generator : we have two binary operations $\succ$ and $\prec$ defined for $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$ by

$$f \succ g = \left(\frac{1}{u_1 u_{1\dots 2}} \circ_2 g\right) \circ_1 f = f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{m+1\dots m+n}}{u_{1\dots m+n}} \tag{5.28}$$

and

$$f \prec g = \left(\frac{1}{u_{1\dots 2} u_2} \circ_2 g\right) \circ_1 f = f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{1\dots m}}{u_{1\dots m+n}}. \tag{5.29}$$

Proposition 5.4.1 *If f and g are vegetal, then so are $f \succ g$ and $f \prec g$.*

Proof. Let us do the proof for $f \succ g$, the other case being similar. One has to compute

$$\frac{u_1 \dots u_{m+n}}{u_{1\dots m+n}} \sum_{\sigma \in \mathfrak{S}_{m+n}} f(t\, u_{\sigma(1)}, \dots, t\, u_{\sigma(m)}) g(t\, u_{\sigma(m+1)}, \dots, t\, u_{\sigma(m+n)}) \\ (u_{\sigma(m+1)} + \dots + u_{\sigma(m+n)}). \tag{5.30}$$

Let us introduce the set $E = \{\sigma(m+1), \dots, \sigma(m+n)\}$. Then one can rewrite the previous sum as

$$\frac{u_1 \dots u_{m+n}}{u_{1\dots m+n}} \sum_E \left(\sum_{i \in E} u_i \right) \sum_{\sigma} f(t\, u_{\sigma'(1)}, \dots, t\, u_{\sigma'(m)}) \sum_{\sigma''} g(t\, u_{\sigma''(m+1)}, \dots, t\, u_{\sigma''(m+n)}), \quad (5.31)$$

where E runs over the set of subsets of cardinal n of $\{1, \dots, m+n\}$, σ' is a bijection from $\{1, \dots, n\}$ to the complement of E and σ'' is a bijection from $\{m+1, \dots, m+n\}$ to E. Using the vegetal property of f and g, this reduces to

$$\frac{m!n!}{u_{1\dots m+n}} f(t, \dots, t) g(t, \dots, t) \sum_E \sum_{i \in E} u_i. \quad (5.32)$$

Reversing summations, this gives

$$(m+n-1)! n f(t \dots, t) g(t \dots, t), \quad (5.33)$$

which is $(m+n)!(f \succ g)(t, \dots, t)$. Hence $f \succ g$ is vegetal. ⬟

5.4.2 Associative product

The inclusion of the Associative operad in the Mould operad implies that the formula

$$\mathrm{MU}(f, g) = \left(\frac{1}{u_1 u_2} \circ_2 g \right) \circ_1 f \quad (5.34)$$

defines an associative product on the free Mould-algebra on one generator. One can check that this associative product is exactly the product called MU in the terminology of moulds, see (5.3). Hence the associated bracket is the so-called LIMU bracket. Note also that $f \succ g + f \prec g = \mathrm{MU}(f, g)$.

As a consequence of Prop. 5.4.1, one has

Corollaire 5.4.2 *If f and g are vegetal, then so are* $\mathrm{MU}(f, g)$ *and* $\mathrm{LIMU}(f, g)$.

5.4.3 Pre-Lie product

Recall that a pre-Lie product on a vector space V is a bilinear map $\curvearrowleft$ from V to V such that

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \quad (5.35)$$

This notion was introduced by Gerstenhaber in deformation theory [Ger64]. It is also related to manifolds with affine structures and to groups with left-invariant affine structures. As for associative algebras, the corresponding antisymmetric bracket $[x, y] = x \curvearrowleft y - y \curvearrowleft x$ is a Lie bracket. For a reference on pre-Lie algebras, the reader may consult [CL01].

As there is an injective morphism from the PreLie operad to the symmetric version of the Dendriform operad, hence also to the symmetric version of the Mould operad, one gets a pre-Lie product on the free Mould algebra on one generator and an injective map from the free Pre-Lie algebra on one generator to the free Mould algebra on one generator.

The pre-Lie product is defined by the formula

$$f \curvearrowleft g = \left(\frac{1}{u_1 u_{1\dots 2}} \circ_2 f\right) \circ_1 g - \left(\frac{1}{u_{1\dots 2} u_2} \circ_2 g\right) \circ_1 f. \tag{5.36}$$

or just as $g \succ f - f \prec g$. More explicitly, it is given by

$$f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{m+1\dots m+n}}{u_{1\dots m+n}} \\ - g(u_1, \dots, u_n) f(u_{n+1}, \dots, u_{m+n}) \frac{u_{1\dots n}}{u_{1\dots m+n}}. \tag{5.37}$$

Théorème 5.4.3 *The pre-Lie product $\curvearrowleft$ preserves alternality : if f and g are alternal, then so is $f \curvearrowleft g$.*

Proof. Let $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$. Let us fix $j \in \{1, \dots, m+n-1\}$. One has to check that

$$\sum_{\sigma \in \mathrm{Sh}(j, m+n-j)} (f \curvearrowleft g)(u_{\sigma(1)}, \dots, u_{\sigma(m+n)}) = 0. \tag{5.38}$$

Let us compute the first half of this sum :

$$\sum_{\sigma \in \mathrm{Sh}(j, m+n-j)} (g \succ f)(u_{\sigma(1)}, \dots, u_{\sigma(m+n)}). \tag{5.39}$$

This is

$$\frac{1}{u_{1\dots m+n}} \sum_{\sigma \in \mathrm{Sh}(j, m+n-j)} f(u_{\sigma(1)}, \dots, u_{\sigma(m)}) \\ g(u_{\sigma(m+1)}, \dots, u_{\sigma(m+n)})(u_{\sigma(m+1)} + \dots + u_{\sigma(m+n)}). \tag{5.40}$$

Let us introduce the set $E' = \{\sigma(1), \dots, \sigma(m)\}$ and let E'' be its complement. Then, by standard properties of shuffles, one can rewrite the previous sum as

$$\frac{1}{u_{1\dots m+n}} \sum_{E'} \sum_{\sigma'} f(u_{\sigma'}) \left(\sum_{k \in E''} u_k\right) \sum_{\sigma''} g(u_{\sigma''}), \tag{5.41}$$

where E' runs over the set of subsets of $\{1, \dots, m+n\}$ of cardinal m, σ' is a shuffle of $E' \cap \{1, \dots, j\}$ and $E' \cap \{j+1, \dots, m+n\}$ and σ'' is a shuffle of $E'' \cap \{1, \dots, j\}$ and $E'' \cap \{j+1, \dots, m+n\}$. Here we have used the shorthand notation introduced at the end of §5.1.

Using the alternality of f and g, this sum reduces to the terms where E' is either included in $\{1, \dots, j\}$ or in $\{j+1, \dots, m+n\}$ and E'' is either included in $\{1, \dots, j\}$ or in $\{j+1, \dots, m+n\}$. Hence, the sum reduces to

$$\frac{1}{u_{1\dots m+n}} (f(u_1, \dots, u_j) g(u_{j+1}, \dots, u_{m+n})(u_{j+1} + \dots + u_{m+n}) \\ + f(u_{j+1}, \dots, u_{m+n}) g(u_1, \dots, u_j)(u_1 + \dots + u_j)), \tag{5.42}$$

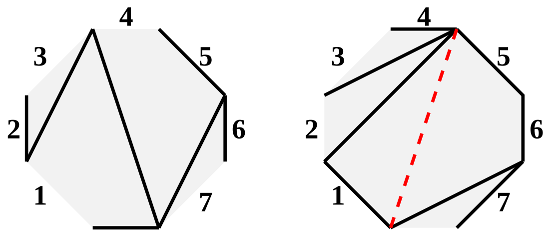

FIG. 5.2 – a non-crossing tree and a non-crossing plant

where the first term is present if $j = m$ and the second one if $j = n$.

One can compute in the same way the other half of the sum :

$$\sum_{\sigma \in \mathrm{Sh}(j,m+n-j)} (f \prec g)(u_{\sigma(1)}, \ldots, u_{\sigma(m+n)}) \tag{5.43}$$

and find exactly the same result. Hence the full sum vanishes as expected. ⬟

Corollaire 5.4.4 *The image of the free pre-Lie algebra on one generator is contained in the intersection of the free dendriform algebra with the set of alternal elements in* Mould*.*

It seems moreover that this inclusion could be an equality.

Remarque 6 It follows also from the definition of $\curvearrowleft$ given above that the Lie bracket LIMU associated to the associative product MU is also the bracket associated to the pre-Lie product $\curvearrowleft$.

5.5 Suboperads in the category of sets

The aim of this section is to describe two small suboperads of the image of Dend in Mould. The key point is that we work here in the category of sets rather than in the category of vector spaces as usual.

5.5.1 Combinatorics of non-crossing trees and plants

Let $n \geq 2$ be an integer. Consider the set of vertices of a regular polygon with $n + 1$ sides. One of these sides will be placed at the bottom and called the *base side*. The other sides are then numbered from 1 to n from left to right. A *diagonal* is a line segment drawn between any two vertices of the regular polygon. Two diagonals are *crossing* if they are distinct and meet at some point in the interior of the convex polygon.

An *non-crossing plant* consists of two disjoint subsets of the set of diagonals : the set of *numerator diagonals* (pictured dashed and red) and the set of *denominator diagonals* (pictured plain and black) with the following properties :

- any two diagonals in the union of these subsets are non-crossing,
- the simplicial complex made by the denominator diagonals is connected and contains all vertices,

– any numerator diagonal is contained in a closed cycle of denominator diagonals,
– any closed cycle of denominator diagonals contains exactly one numerator diagonal.

In the sequel, a diagonal will always mean implicitly a denominator diagonal, unless explicitly stated otherwise. Note that a side can only be a denominator diagonal.

Let us call *based non-crossing plant* a non-crossing plant that includes the base side of the regular polygon.

If there is no numerator diagonal, then a non-crossing plant is a *non-crossing tree*, i.e. a maximal set of pairwise non-crossing diagonals whose union is a connected and simply connected simplicial complex (i.e. a tree).

Non-crossing trees and based non-crossing trees are well-known combinatorial objects, see [Noy98] for example. Non-crossing plants seem not to have been considered before. Fig. 5.2 displays a based non-crossing tree on the left and a non-crossing plant on the right.

We need a precise recursive description of non-crossing plants. One has to distinguish three sorts of them, as depicted in Fig. 5.3.

The first kind (I) is when the plant is based and the base side is contained in a cycle of denominator diagonals, necessarily of length at least 4, as it must contain a numerator diagonal. For each other diagonal in this cycle, one can consider diagonals that are in the connected region (the one not containing the inner part of the cycle) between this diagonal and the boundary of the regular polygon. This defines a based non-crossing plant. Conversely, one can pick any list of length at least 3 of based non-crossing plants and put them on the sides of a closed cycle containing the base side and choose a numerator diagonal in this cycle.

The second kind (II) is when the plant is based, but the base side is not contained in a cycle of diagonals. Then there exists a unique side which is not in the plant and which bounds the same region as the base side. To the left and to the right of the square formed by this side and the base side, one can define two non-crossing plants. This also includes some degenerate cases, where one or both sides are empty. In these cases, the square becomes a triangle or just the base side and there is only one associated non-crossing plant (on the left or on the right) or none.

The third and last kind (III) is when the plant is not based. Consider the unique cycle (of length at least 3) that would be created by adding the base side. Just as in kind (I), one can define, by looking at the outer regions bounded by this cycle, a list of based non-crossing plants of length at least 2.

Let us now translate this trichotomy in terms of generating series and sketch the enumeration of non-crossing plants. We will use the generating series

$$P = \sum_{n\geq 1} p_n x^n = x + 3x^2 + 14x^3 + 80x^4 + \dots, \tag{5.44}$$

and

$$Q = \sum_{n\geq 1} q_n x^n = x + 2x^2 + 9x^3 + 51x^4 + \dots, \tag{5.45}$$

where p_n (resp. q_n) is the number of non-crossing plants (resp. based non-crossing plants) in the regular polygon with $n+1$ sides.

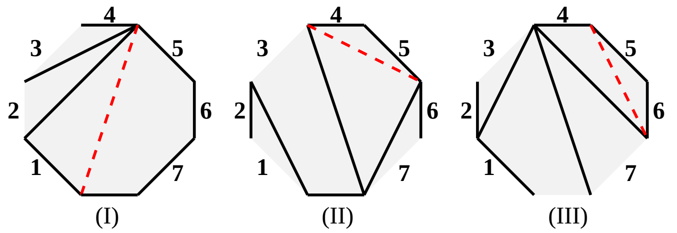

FIG. 5.3 – the 3 sorts of non-crossing plants

A non-crossing plant is either based or not. If it is not, it is of kind (III) and can be described using a list (of length greater than 2) of based non-crossing plants. Hence one has

$$P = Q + (Q^2/(1-Q)) = Q/(1-Q). \tag{5.46}$$

For a based non-crossing plant, either its base side is contained in a cycle or not. In the first case, it is of kind (I) and can be described using a list (of length k at least 3) of based non-crossing plants and the choice of an inner diagonal in a cycle of length $k+1$. In the second case, it is of type (II) and can be described by a pair of non-crossing plants or empty sides. Hence one has

$$Q = \sum_{k \geq 3} \frac{(k+1)(k-2)}{2} Q^k + x(1+P)^2. \tag{5.47}$$

From these equations, one gets that P satisfies the algebraic relation :

$$x - P + xP^2 + 2xP + P^2 + P^3 = 0. \tag{5.48}$$

Therefore P has a simple functional inverse :

$$x = \frac{P - P^2 - P^3}{(1+P)^2}. \tag{5.49}$$

One can remark that this series appear as example (g) in [Lod06].

Let us now introduce the following notions, used in the next section.

A *peeling point* of a non-crossing plant is a vertex (not in the base side) such that the only incident diagonals are sides. In Figure 5.2, the non-crossing plant on the right has 2 peeling points between sides 3 and 4 and between sides 5 and 6.

Lemme 5.5.1 *There is always at least one peeling point in a non-crossing plant in the $n+1$ polygon, for $n \geq 2$.*

Proof. By induction on $n \geq 2$. This is true for all 3 non-crossing plants , ,  in a triangle by inspection. Let us distinguish three cases, as before.

(I) The base side belongs to a cycle of diagonals.

If there is something else than the cycle, there is a peeling point by induction in one of the sub-non-crossing plants bounding the cycle. This gives a peeling point in the whole non-crossing plant.

If there is just a cycle, one can pick any vertex not in the base and not contained in the numerator diagonal. This vertex is a peeling point.

(II) The base side is a diagonal, but does not belong to a cycle of diagonals.

One can consider the left or right sub-non-crossing plant, which contains a peeling point by induction. This provides a peeling point in the full non-crossing plant.

(III) The base side is not a diagonal.

If there is something else than the would-be cycle (see the description of kind (III)), there is a peeling point by induction in one of the sub-non-crossing plants bounding this would-be cycle. This is also a peeling point in the full non-crossing plant.

If there is just a would-be cycle, one can pick as peeling point any vertex not in the base.

A *border leaf* is a peeling point in a based non-crossing tree that has only one incident side. In Figure 5.2, there are 3 border leaves between sides 2 and 3, sides 4 and 5 and sides 6 and 7 in the non-crossing tree on the left.

Lemme 5.5.2 *There is always at least one border leaf in a based non-crossing tree in the $n+1$ polygon, for $n \geq 2$.*

Proof. By induction on $n \geq 2$. This is clear if $n = 2$, for the based non-crossing trees  and . Assume that $n \geq 3$. In any based non-crossing tree, one can define a left and a right subtree, as the connected components of the tree minus its base side (every based non-crossing tree is of kind (II) as a non-crossing plant). At least one of them is not empty. It is enough to prove that there is a border leaf in one of them. One can therefore assume for instance that only the right subtree is not empty.

Then consider the leftmost diagonal (other than the base side) emanating from the right vertex of the base side. Either there is a non-empty non-crossing tree to its right, hence a border leaf inside it by induction, or it has nothing to its right (it is a side). In this case, one can build a new non-crossing tree by shrinking the base side to a point and taking the side to its right as new base side. By induction, this new non-crossing tree has a border leaf. This implies that the initial tree has one too.

5.5.2 The operad of non-crossing plants

Each diagonal is mapped to a linear function in variables $u_1, \dots, u_n$ as follows : the base side is mapped to $u_{1\dots n}$. The other sides are mapped to $u_1, \dots, u_n$ in the clockwise order. For diagonals which are not sides, one considers the half plane not containing the base side, with respect to this diagonal. This diagonal is mapped to the sum of the values of the sides that are in this half-plane.

To each non-crossing plant, one can then associate a rational function in Mould which is the product of the linear functions associated to its numerator diagonals divided by the product of the linear functions associated to its denominator diagonals (this justifies the chosen names). For instance, for the non-crossing tree in the example of Fig. 5.2, one gets

$$1/\left(u_{1\dots7}u_2u_{2\dots3}u_{4\dots7}u_5u_6u_{6\dots7}\right), \tag{5.50}$$

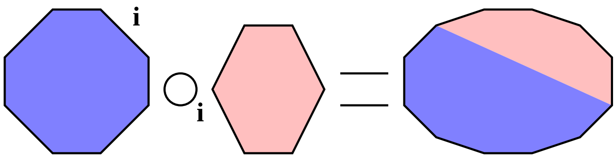

FIG. 5.4 – grafting of non-crossing plants : deformation of polygons

and for the non-crossing plant on the right of the same figure,

$$u_{1\ldots4}/\left(u_{1\ldots6}u_1u_{2\ldots4}u_{3\ldots4}u_4u_5u_6u_7\right). \tag{5.51}$$

The 3 non-crossing plants in a triangle ◣,◢ and ∧ are mapped to $1/(u_1u_{1\ldots2})$, $1/(u_{1\ldots2}u_2)$ and $1/(u_1u_2)$ in Mould.

This mapping from the set of non-crossing plants to Mould is obviously injective, as one can recover the non-crossing plant from the factorization of its image. Therefore, we will from now on identify non-crossing plants with their images in Mould.

One can check, using the the definition of the composition in Mould, that the set of non-crossing plants is closed under composition. Let us give a combinatorial description of the composition of non-crossing plants. Given two non-crossing plants f and g in some regular polygons and a side i of the polygon containing f, one has to define a new non-crossing plant in the grafted polygon as in Fig 5.4. This is simply the union of f and g, with some modification along the grafting diagonal. If this diagonal is present in both f and g, then it is kept in $f\circ_i g$. If it is present in exactly one of f and g, then it is not kept in $f\circ_i g$. If it is present in neither f or g, then it becomes a numerator diagonal in $f\circ_i g$.

Théorème 5.5.3 *The non-crossing plants form a suboperad* NCP *in the category of sets, which is contained in the image of* Dend. *This operad has the following presentation : three generators* ◣, ◢ *and* ∧, *subject only to the relations :*

$$◣\circ_2 ◢ = ◢\circ_1 ◣, \tag{5.52}$$
$$∧\circ_1 ∧ = ∧\circ_2 ∧, \tag{5.53}$$
$$◣\circ_1 ∧ = ◣\circ_2 ◣, \tag{5.54}$$
$$◢\circ_2 ∧ = ◢\circ_1 ◢. \tag{5.55}$$

Proof. The inclusion in the image of Dend will follow from the presentation by generators and relations, with generators ◣, ◢ and ∧ that are the images of the elements ⋎, ⋎ and ⋎ + ⋎ of Dend in Mould.

There remains to prove the presentation. Let us define an operad NCP′ by the presentation above, with ◣, ◢ and ∧ replaced by symbols L, R and M. As the relations are satisfied in NCP, there is a unique morphism of operads ∇ from NCP′ to NCP sending the generators L, R and M to ◣, ◢ and ∧. Let us prove by induction that there is an inverse Δ to ∇.

Note that, whenever this makes sense, Δ is of course a morphism of operads. The existence of Δ is clear for $n=2$. Assume that $n\geq 3$ and let T be in NCP(n). By Lemma 5.5.1, there is a peeling point in the non-crossing plant T.

Let i be the index of the side of the polygon which is left to this leaf. Then T can be written $S \circ_i \delta$ where δ is either ,  or  and $S \in \mathrm{NCP}(n-1)$.

Let us define

$$\Delta(T) = \Delta(S) \circ_i \Delta(\delta). \tag{5.56}$$

One has to prove that this definition does not depend on the choice of the peeling point. Let us assume that there is another peeling point. Without further restriction, one can assume that it is at the right of side j with $i < j$. Thus T can also be written $S' \circ_j \delta'$ where δ' is either ,  or  and $S' \in \mathrm{NCP}(n-1)$. One then has to distinguish two cases.

Far case

If $i+1 < j$, then there is still a peeling point in S' at the right of edge i, hence there exists $S'' \in \mathrm{NCP}(n-2)$ such that T can be written as

$$S' \circ_j \delta' = (S'' \circ_i \delta) \circ_j \delta' = (S'' \circ_{j-1} \delta') \circ_i \delta = S \circ_i \delta, \tag{5.57}$$

where the second equality is an axiom of operads. This implies that both choices of peeling point in T leads to the same value for $\Delta(T)$:

$$\begin{aligned} \Delta(S') \circ_j \Delta(\delta') &= (\Delta(S'') \circ_i \Delta(\delta)) \circ_j \Delta(\delta') \\ &= (\Delta(S'') \circ_{j-1} \Delta(\delta')) \circ_i \Delta(\delta) = \Delta(S) \circ_i \Delta(\delta). \end{aligned} \tag{5.58}$$

Near cases

If $i+1 = j$, then one can distinguish 4 cases. Either all three sides $i, i+1, i+2$ are diagonals, or just two of them are. The other possibilities are excluded by the second condition in the definition of non-crossing plants.

Let us consider the first case. Necessarily T can be written, for some $S'' \in \mathrm{NCP}(n-2)$, as

$(S'' \circ_i$ $) \circ_{i+1}$  $= S'' \circ_i ($ $\circ_2$ $) = S'' \circ_i ($ $\circ_1$ $) = (S'' \circ_i$ $) \circ_i$ . (5.59)

This implies that both choices of peeling point give the same value for $\Delta(T)$:

$\Delta(S'' \circ_i$$) \circ_{i+1} M = (\Delta(S'') \circ_i M) \circ_{i+1} M = (\Delta(S'') \circ_i M) \circ_i M = \Delta(S'' \circ_i$$) \circ_i M,$ (5.60)

where the middle equality follows from relation (5.53) for the generators L, R and M.

The three other cases are similar to this one, each one of them using one of the relations (5.52), (5.54) and (5.55) for the generators L, R and M.

Hence Δ is well-defined. Then, one has for all T in $\mathrm{NCP}(n)$,

$$\nabla(\Delta(T)) = \nabla(\Delta(S) \circ_i \Delta(\delta)) = S \circ_i \delta = T, \tag{5.61}$$

by induction hypothesis and because ∇ is a morphism of operad. Let x be in $\mathrm{NCP}'(n)$. Then x can written $y \circ_i d$ for some y in $\mathrm{NCP}'(n-1)$ and $d \in \{L, R, M\}$. Then

$$\Delta(\nabla(x)) = \Delta(\nabla(y) \circ_i \nabla(d)) = \Delta(\nabla(y)) \circ_i \Delta(\nabla(d)) = y \circ_i d = x. \tag{5.62}$$

Here for the computation of Δ we choose the peeling point corresponding to $\nabla(d)$. We have proved that Δ is the inverse of ∇ up to order n. This concludes the induction step.

⬟

Remarque 7 By adding the opposite of each non-crossing plant, one can get an anticyclic operad in the category of sets. The cyclic action is just given by the rotation of the regular polygon, up to sign.

5.5.3 The operad of based non-crossing trees

It is not hard to see, using the combinatorial description of the composition given above, that based non-crossing trees are closed under composition.

Proposition 5.5.4 *The suboperad of* NCP *generated by* $\angle$ *and* $\perp$ *is exactly the suboperad of based non-crossing-trees. These generators are only subject to the relation* (5.52).

Proof. As $\angle$ and $\perp$ are based non-crossing trees, the operad they generate is contained in the suboperad of non-crossing trees.

To prove the reverse inclusion, one proceeds by induction. By Lemma 5.5.2, there is a border leaf in any non-crossing tree T. Let i be the index of the side of the polygon which is left to this leaf of T. Then T can be written $S \circ_i \delta$ where δ is either $\perp$ or $\angle$ and S is a smaller based non-crossing tree. This implies the inclusion in the suboperad generated by $\perp$ and $\angle$.

The presentation is a consequence of that of the bigger operad NCP. ⬟

From results of Loday in [Lod02], one can deduce that

Proposition 5.5.5 *For any non-crossing plant* T *of degree* n*, the inverse image of* T *in* Dend *is a sum without multiplicities in* Dend(n) *and can therefore be considered as a subset of* $\mathcal{Y}(n)$.

We would like to draw the attention on the following conjecture, which has been checked in low degrees. Recall that the Tamari poset [FT67] is a partial order on the set $\mathcal{Y}(n)$ which indexes a basis of Dend(n).

Conjecture 5.5.6 *For any non-crossing tree* T *of degree* n *(not necessarily based) in* Mould*, the inverse image of* T *in* Dend *is*

$$\sum_{t \in I} t, \tag{5.63}$$

where I *is some interval in the Tamari poset* $\mathcal{Y}(n)$.

5.6 Other structures

Let us consider some other operations on the free Mould algebra on one generator.

5.6.1 Over and Under operations

Loday and Ronco have introduced in [LR02] two other associative products on the free dendriform algebra on one generator, called *Over* and *Under* and denoted by / and \. They are usually defined as simple combinatorial operations on planar binary trees, but can be restated using the Dendriform operad as follows :

$$f/g = (g \circ_1 \curlyvee) \circ_1 f \tag{5.64}$$

and

$$f \backslash g = (f \circ_m \curlyvee) \circ_{m+1} g, \tag{5.65}$$

where f is assumed to be of degree m. One can use this to extend these operations to the free Mould algebra on one generator. Explicitly, restated inside the Mould operad, these products are given by

$$(f/g)(u_1, \dots, u_{m+n}) = f(u_1, \dots, u_n) g(u_{1 \dots n+1}, u_{n+2}, \dots, u_{m+n}) \tag{5.66}$$

and

$$(f \backslash g)(u_1, \dots, u_{m+n}) = f(u_1, \dots, u_{n-1}, u_{n \dots m+n}) g(u_{n+1}, \dots, u_{m+n}). \tag{5.67}$$

5.6.2 Structures associated to the operad structure

The fact that Mould is an operad implies that one can define other operations on the free Mould algebra on one generator, namely another pre-Lie product $\circ$ (not to be confused with the one introduced before and denoted by $\curvearrowleft$, which is quite different) and the associated Lie bracket and group law.

The pre-Lie product $\circ$ is defined for $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$ by

$$f \circ g = \sum_{i=1}^{m} f \circ_i g. \tag{5.68}$$

More explicitly, $f \circ g$ is given by

$$\sum_{i=1}^{m} f(u_1, \dots, u_{i-1}, u_{i \dots i+n-1}, u_{i+n}, \dots, u_{m+n-1}) g(u_i, \dots, u_{i+n-1}) u_{i \dots i+n-1}. \tag{5.69}$$

This construction is clearly functorial from the category of operads to the category of pre-Lie algebras. Note that the product $f \circ g$ is in $\mathrm{Mould}(m+n-1)$.

Théorème 5.6.1 *The pre-Lie product $\circ$ preserves alternality, that is $f \circ g$ is alternal as soon as f and g are.*

Proof. Let f be in $\mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$. Let us fix $j \in \{1, \dots, m+n-2\}$. One has to check that

$$\sum_{\sigma \in \mathrm{Sh}(j, m+n-1-j)} (f \circ g)(u_{\sigma(1)}, \dots, u_{\sigma(m+n-1)}) = 0. \tag{5.70}$$

The sum to be computed is

$$\sum_{i=1}^{m} \sum_{\sigma \in \mathrm{Sh}(j, m+n-1-j)} f(u_{\sigma(1)}, \dots, u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}, \dots, u_{\sigma(m+n-1)}) \\ g(u_{\sigma(i)}, \dots, u_{\sigma(i+n-1)})(u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}). \tag{5.71}$$

Let us introduce the sets $E' = \{\sigma(1), \dots, \sigma(i-1)\}$ of cardinal $i-1$ and $E'' = \{\sigma(i+n), \dots, \sigma(m+n-1)\}$ of cardinal $m-i$. They have the following properties :

- $E' \cap \{1, \dots, j\}$ is an initial subset of $\{1, \dots, j\}$,
- $E' \cap \{j+1, \dots, m+n-1\}$ is an initial subset of $\{j+1, \dots, m+n-1\}$,
- $E'' \cap \{1, \dots, j\}$ is a final subset of $\{1, \dots, j\}$,
- $E'' \cap \{j+1, \dots, m+n-1\}$ is a final subset of $\{j+1, \dots, m+n-1\}$.

Let $E = \{\sigma(i), \ldots, \sigma(i+n-1)\}$ be the complement of $E' \cup E''$. It follows from the conditions above that $E \cap \{1, \ldots, j\}$ and $E \cap \{j+1, \ldots, m+n-1\}$ are sub-intervals.

Using standard properties of the set of shuffles, and the shorthand notation u_σ introduced at the end of §5.1, one can then rewrite the previous sum as

$$\sum_{i=1}^{m} \sum_{E',E''} \sum_{\sigma',\sigma''} f(u_{\sigma'}, \sum_{k\in E} u_k, u_{\sigma''}) \sum_{\nu} g(u_\mu) \left(\sum_{k\in E} u_k \right), \tag{5.72}$$

where E' of cardinal $i-1$ and E'' of cardinal $m-i$ are subsets with the above properties, σ' is a shuffle of $E' \cap \{1, \ldots, j\}$ and $E' \cap \{j+1, \ldots, m+n-1\}$, σ'' is a shuffle of $E'' \cap \{1, \ldots, j\}$ and $E'' \cap \{j+1, \ldots, m+n-1\}$ and ν is a shuffle of $E \cap \{1, \ldots, j\}$ and $E \cap \{j+1, \ldots, m+n-1\}$.

Using the alternality of g, one can see that the sum reduces to the cases when $E \subset \{1, \ldots, j\}$ or $E \subset \{j+1, \ldots, m+n-1\}$. Let us show that each of these two terms vanishes. As the proof is similar, we treat only the case when $E \subset \{1, \ldots, j\}$. In this case, there exists k such that $E = \{k, \ldots, k+n-1\}$. The corresponding term is

$$\sum_{i=1}^{m} \sum_{E',E''} \sum_{\sigma',\sigma''} f(u_{\sigma'}, u_{k\ldots k+n-1}, u_{\sigma''}) g(u_k, \ldots, u_{k+n-1}) u_{k\ldots k+n-1}, \tag{5.73}$$

where E' and E'' runs over the appropriate sets.

Once again by the usual properties of shuffles, this can be rewritten as

$$\sum_{k} g(u_k, \ldots, u_{k+n-1})(u_{k\ldots k+n-1}) \sum_{\mu} f(u_\mu), \tag{5.74}$$

where $1 \leq k \leq k+n-1 \leq j$ and μ is a shuffle of $\{1, \ldots, k-1, (k\ldots k+n-1), k+n, \ldots, j\}$ and $\{j+1, \ldots, m+n-1\}$, with the abuse of notation made in considering $(k \ldots k+n-1)$ as an index. This sum is zero because f is alternal. ⬟

Proposition 5.6.2 *If f and g are vegetal, then so is $f \circ g$.*

Proof. One has to compute

$$\sum_{\sigma \in \mathfrak{S}_{m+n-1}} \sum_{i=1}^{m} f(t\, u_{\sigma(1)}, \ldots, t(u_{\sigma(i)} + \cdots + u_{\sigma(i+n-1)}), \ldots, t\, u_{\sigma(m+n-1)})$$
$$u_1 \ldots u_{m+n-1} g(t\, u_{\sigma(i)}, \ldots, t\, u_{\sigma(i+n-1)}) t\, (u_{\sigma(i)} + \cdots + u_{\sigma(i+n-1)}). \tag{5.75}$$

Let us introduce the set $E = \{\sigma(i), \ldots, \sigma(i+n-1)\}$. One can then rewrite the previous sum as

$$u_1 \ldots u_{m+n-1}\, t \sum_{E} (\sum_{j\in E} u_j) \sum_{i=1}^{m} \sum_{\sigma'} f(t\, u_{\sigma'(1)}, \ldots, t(\sum_{j\in E} u_j), \ldots, t\, u_{\sigma'(m+n-1)})$$
$$\sum_{\sigma''} g(t\, u_{\sigma''(i)}, \ldots, t\, u_{\sigma''(i+n-1)}), \tag{5.76}$$

where E runs over the set of subsets of cardinal n of $\{1, \ldots, m+n-1\}$, σ' is a bijection from $\{1, \ldots, i-1\} \sqcup \{i+n, \ldots, m+n-1\}$ to the complement of E and σ'' is a bijection from $\{i, \ldots, i+n-1\}$ to E.

Using first the vegetal property of g, one gets

$$u_1 \dots u_{m+n-1} t \sum_E \frac{\sum_{j\in E} u_j}{\prod_{j\in E} u_j} \sum_{i=1}^{m} \sum_{\sigma'} f(t\, u_{\sigma'(1)}, \dots, t \left(\sum_{j\in E} u_j\right), \dots, t\, u_{\sigma'(m)})$$
$$n! g(t,\dots,t). \quad (5.77)$$

Then, this can be rewritten as

$$n! t g(t,\dots,t) \sum_E \left(\sum_{j\in E} u_j\right) \sum_\theta f(t\,\theta(1), \dots, t\,\theta(m)) \prod_{j\notin E} u_j. \quad (5.78)$$

where θ is a bijection from $\{1,\dots,m\}$ to $\{u_j\}_{j\notin E} \sqcup \{\sum_{j\in E} u_j\}$. By using the vegetal property of f, this becomes

$$n! \binom{m+n-1}{n} m! f(t,\dots,t) g(t,\dots,t) t. \quad (5.79)$$

Using once again the vegetal property of f in the special case $u_1 = n$ and $u_2 = \dots = u_m = 1$, one gets

$$(m+n-1)! \sum_{i=1}^{m} f(t,\dots,nt,\dots,t)\, g(t,\dots,t)\, nt, \quad (5.80)$$

which is $(m+n-1)!(f\circ g)(t,\dots,t)$. Hence $f\circ g$ is vegetal. ⬟

5.6.3 Forgetful morphism to formal vector fields

The aim of this section is to define a map $\mathscr{F}$ of pre-Lie algebras from moulds satisfying appropriate conditions to formal power series in one variable x, or rather to formal vector fields.

Let us consider here only moulds f such that f_n is homogeneous of weight $-n$ and has only poles at some $u_{i\dots j}$ with arbitrary multiplicity (nice poles). From Remarks 1 and 3, this subspace is a anticyclic suboperad, hence it is closed for $\circ$ by functoriality. Let us consider its intersection with the subspace of vegetal moulds, which is also closed for $\circ$ by Prop. 5.6.2. Let us note that this intersection contains the image of Dend by Prop. 5.4.1 and Prop. 5.3.2.

Let us recall that the usual pre-Lie product, also denoted by $\circ$, on vector fields is given by

$$F(x)\partial_x \circ G(x)\partial_x = (\partial_x F(x)) G(x) \partial_x. \quad (5.81)$$

Théorème 5.6.3 *The substitution $u_i \mapsto 1/x$ induces a morphism $\mathscr{F}$ of pre-Lie algebras $f \mapsto f(x^{-1},\dots,x^{-1})\partial_x$ from* Mould *(restricted as above to homogeneous vegetal moulds with nice poles) with the pre-Lie product $\circ$ to the pre-Lie algebra of vector fields in the variable x with formal power series in x as coefficients.*

Proof. Let $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$ satisfying the additional conditions stated before. One has to prove that

$$\partial_x f(x^{-1},\dots,x^{-1}) g(x^{-1},\dots,x^{-1})$$
$$= \sum_{i=1}^{m} n x^{-1} f(x^{-1},\dots,n x^{-1},\dots,x^{-1}) g(x^{-1},\dots,x^{-1}), \quad (5.82)$$

where we have used the assumed shape of the poles of f and g to ensure that the substitution makes sense. It is therefore enough to prove that

$$x\partial_x f(x^{-1},\dots,x^{-1}) = \sum_{i=1}^{m} n f(x^{-1},\dots,nx^{-1},\dots,x^{-1}). \tag{5.83}$$

From the homogeneity of f, one has to prove that

$$m f(x^{-1},\dots,x^{-1}) = \sum_{i=1}^{m} n f(x^{-1},\dots,nx^{-1},\dots,x^{-1}), \tag{5.84}$$

which is a special case of the vegetal property of f, with $t = x^{-1}$, $u_1 = n$ and $u_2 = \cdots = u_m = 1$. ⬟

As the group law associated to the usual pre-Lie product on formal power series is the classical composition of power series, one can see the group structure on moulds corresponding to $\circ$ as some kind of generalized composition.

Remarque 8 From (5.3), it is quite obvious that the associative product MU is mapped by $\mathscr{F}$ to the usual commutative product of formal power series.

5.6.4 Derivation

Let us introduce a map ∂ on moulds, which decreases the degree by 1.
For a mould $f \in \mathrm{Mould}(m)$, ∂f is the element of $\mathrm{Mould}(m-1)$ defined by

$$\partial f(u_1,\dots,u_{m-1}) = \sum_{j=1}^{m} \mathrm{Res}_{t=0}\, f(u_1,\dots,u_{j-1},t,u_j,\dots,u_{m-1}), \tag{5.85}$$

where Res is the residue.
The main motivation for this map is the following property.

Proposition 5.6.4 *The map ∂ is sent by the forgetful map $\mathscr{F}$ to the partial derivative with respect to x, i.e. for any $f \in \mathrm{Mould}(m)$ which is homogeneous, vegetal and has nice poles, one has $\mathscr{F}(\partial f) = \partial_x \mathscr{F}(f)$.*

Proof. Let $f \in \mathrm{Mould}(m)$. By homogeneity, $\mathscr{F}(f) = f(x^{-1},\dots,x^{-1}) = x^m f(1,\dots,1)$. Hence $\partial_x \mathscr{F}(f) = m x^{m-1} f(1,\dots,1)$.
On the other hand, $\mathscr{F}(\partial f)$ is

$$\mathrm{Res}_{t=0} \sum_{j=1}^{m} f(x^{-1},\dots,t,\dots,x^{-1}), \tag{5.86}$$

where t is in the jth position. By the vegetal property of f, this is

$$\mathrm{Res}_{t=0}\, m \frac{x^{m-1}}{t} f(1,\dots,1). \tag{5.87}$$

This proves the expected equality. ⬟

Remarque 9 If f is alternal and of degree at least 2, then $\partial f = 0$. This is obvious once the definition of ∂ is rewritten as the residue of a sum over shuffles of t with $\{u_1,\dots,u_{m-1}\}$.

Proposition 5.6.5 *The map ∂ is a derivation for the products $\prec, \succ, \curvearrowleft$,* MU *and* LIMU*. It is also a derivation for $\circ$, under the restriction that functions have nice poles.*

Proof. It is enough to prove this for $\succ$ and $\circ$. The case of $\prec$ is similar to the case of $\succ$ and the other cases can be deduced from these ones.

Let us consider the case of $\succ$. Let $f \in \mathrm{Mould}(m)$ and $g \in \mathrm{Mould}(n)$. One has to compute

$$\sum_{j=1}^{m} \mathrm{Res}_{t=0}\left(f(u_1,\ldots,t,u_j,\ldots,u_{m-1})g(u_m,\ldots,u_{m+n-1})\frac{u_{m\ldots m+n-1}}{u_{1\ldots m+n-1}+t}\right)+$$
$$\sum_{j=m+1}^{m+n} \mathrm{Res}_{t=0}\left(f(u_1,\ldots,u_m)g(u_{m+1},\ldots,t,u_j,\ldots,u_{m+n-1})\frac{u_{m+1\ldots m+n-1}+t}{u_{1\ldots m+n-1}+t}.\right). \quad (5.88)$$

By the properties of the residue, this becomes

$$\sum_{j=1}^{m} \mathrm{Res}_{t=0}\left(f(u_1,\ldots,t,u_j,\ldots,u_{m-1})\right)g(u_m,\ldots,u_{m+n-1})\frac{u_{m\ldots m+n-1}}{u_{1\ldots m+n-1}}+$$
$$\sum_{j=m+1}^{m+n} f(u_1,\ldots,u_m)\,\mathrm{Res}_{t=0}\left(g(u_{m+1},\ldots,t,u_j,\ldots,u_{m+n-1})\right)\frac{u_{m+1\ldots m+n-1}}{u_{1\ldots m+n-1}}. \quad (5.89)$$

This is $\partial f \succ g + f \succ \partial g$, which proves that ∂ is a derivation of $\succ$.

Let us now consider the case of $\circ$. One has to compute

$$\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathrm{Res}_{t=0}(f(u_1,\ldots,t,u_j,\ldots,u_{i-2},u_{i-1\ldots i+n-2},u_{i+n-1},\ldots,u_{m+n-2})$$
$$g(u_{i-1},\ldots,u_{i+n-2})u_{i-1\ldots i+n-2})+$$
$$\sum_{i=1}^{m}\sum_{j=i}^{i+n-1}\mathrm{Res}_{t=0}(f(u_1,\ldots,u_{i-1},u_{i\ldots i+n-2+t},u_{i+n-1},\ldots,u_{m+n-2})$$
$$g(u_i,\ldots,t,u_j,\ldots,u_{i+n-2})(u_{i\ldots i+n-2}+t))+$$
$$\sum_{i=1}^{m}\sum_{j=i+n}^{m+n-1}\mathrm{Res}_{t=0}(f(u_1,\ldots,u_{i-1},u_{i\ldots i+n-1},u_{i+n},\ldots,t,u_j,\ldots,u_{m+n-2})$$
$$g(u_i,\ldots,u_{i+n-1})u_{i\ldots i+n-1}). \quad (5.90)$$

By the properties of residues, and the assumption that f and g have nice poles, this becomes

$$\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathrm{Res}_{t=0}\left(f(u_1,\dots,t,u_j,\dots,u_{i-2},u_{i-1\dots i+n-2},u_{i+n-1},\dots,u_{m+n-2})\right)$$
$$g(u_{i-1},\dots,u_{i+n-2})u_{i-1\dots i+n-2}+$$
$$\sum_{i=1}^{m}\sum_{j=i+n}^{m+n-1}\mathrm{Res}_{t=0}\left(f(u_1,\dots,u_{i-1},u_{i\dots i+n-1},u_{i+n},\dots,t,u_j,\dots,u_{m+n-2})\right)$$
$$g(u_i,\dots,u_{i+n-1})u_{i\dots i+n-1}+$$
$$\sum_{i=1}^{m}\sum_{j=i}^{i+n-1} f(u_1,\dots,u_{i-1},u_{i\dots i+n-2},u_{i+n-1},\dots,u_{m+n-2})$$
$$\mathrm{Res}_{t=0}\left(g(u_i,\dots,t,u_j,\dots,u_{i+n-2})\right)u_{i\dots i+n-2}. \quad (5.91)$$

The first two terms give $\partial f\circ g$ and the third one gives $f\circ\partial g$. ⬟

As a corollary of Prop. 5.6.5, the map ∂ preserves the image by ψ of the free Dendriform algebra on one generator. One can be more precise : the action of ∂ is by vertex-removal, in the following sense. From the description of $\psi(T)$ for a planar binary tree T in Prop. 5.3.2, one can see that taking the residue with respect to one of the variables and then renumbering correspond to the removal of a top vertex in T. Hence $\partial\psi(T)$ is the sum over all top vertices of T of the image by ψ of some smaller binary tree.

5.6.5 The products ARIT and ARI

Let us define two other bilinear products on the free Mould algebra on one generator, denoted by ARIT and ARI :

$$\mathrm{ARIT}(f,g)=f\circ(g/Y)-f\circ(Y\backslash g) \quad (5.92)$$

and

$$\mathrm{ARI}(f,g)=\mathrm{ARIT}(f,g)-\mathrm{ARIT}(g,f)+\mathrm{LIMU}(f,g). \quad (5.93)$$

One can check, by writing the explicit expression for these products, that they do indeed reproduce the ARIT and ARI maps introduced by Ecalle in his study of moulds. In particular, it is known that ARI is a Lie bracket that preserves alternality. Note that we have defined as $\mathrm{ARIT}(f,g)$ what Ecalle denotes by $\mathrm{ARIT}(g)f$.

Lemme 5.6.6 *There holds $\partial(f/Y)=(\partial f)/Y$ and $\partial(Y\backslash f)=Y\backslash(\partial f)$.*

Proof. Let us consider only the first case, the other one being similar. Let $f\in\mathrm{Mould}(n)$. As

$$f/Y=\curlyvee\circ_1 f=\frac{1}{u_{1\dots n+1}}f(u_1,\dots,u_n), \quad (5.94)$$

one has to compute

$$\sum_{i=1}^{n}\mathrm{Res}_{t=0}\frac{1}{u_{1\dots n}+t}f(u_1,\dots,u_{i-1},t,u_i,\dots,u_{n-1})+\mathrm{Res}_{t=0}\frac{1}{u_{1\dots n}+t}f(u_1,\dots,u_n). \quad (5.95)$$

The second term vanishes, and what remains is

$$\frac{1}{u_{1\dots n}} \sum_{i=1}^{n} \mathrm{Res}_{t=0}\, f(u_1, \dots, u_{i-1}, t, u_i, \dots, u_{n-1}), \tag{5.96}$$

which is exactly $(\partial f)/Y$. ⬟

Corollaire 5.6.7 *The map ∂ is a derivation for* ARIT *and* ARI.

Proof. As ∂ is a derivation of $\circ$ by Prop. 5.6.5, one has

$$\partial(\mathrm{ARIT}(f,g)) = \partial(f)\circ(g/Y) + f\circ\partial(g/Y) - \partial(f)\circ(Y\backslash g) - f\circ\partial(Y\backslash g). \tag{5.97}$$

One can then conclude for ARIT using Lemma 5.6.6. The proof for ARI follows from this and from Prop. 5.6.5. ⬟

Let us now state some properties of the ARI and ARIT products.

Proposition 5.6.8 *The free dendriform algebra in* Mould *is closed under* ARIT *and* ARI. *The products* ARIT *and* ARI *preserves the vegetal property.*

Proof. For the first statement, it is enough to look at the definition of ARIT. One already knows that the Over and Under operations are defined on the dendriform subspace. The product $\circ$ has the same property, because it is a functorial construction on operads.

Let us prove the second statement. It is enough to prove this for ARIT, by Prop. 5.4.2. As this property is already known for $\circ$ by Prop. 5.6.2, it is enough to prove that f/Y and $Y\backslash f$ are vegetal if f is so. Let us consider only the first case, as the other one is just the same.

Let f be in Mould(n). One has to compute

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} f(u_{\sigma(1)}, \dots, u_{\sigma(n)}) \frac{1}{u_{1\dots n+1}\, t}. \tag{5.98}$$

One can separate the sum according to the value of $\sigma(n+1)$, obtaining

$$\frac{1}{u_{1\dots n+1}\, t} \sum_{i=1}^{n+1} \sum_{\sigma'} f(u_{\sigma'(1)}, \dots, u_{\sigma'(n)}), \tag{5.99}$$

where σ' runs over the bijections from $\{1, \dots, n\}$ to $\{1, \dots, n+1\} \setminus \{i\}$. Using then the vegetal property of f, this becomes

$$\frac{1}{u_{1\dots n+1}\, t} \sum_{i=1}^{n+1} \frac{u_i}{u_1 \dots u_{n+1}} n! f(t, \dots, t). \tag{5.100}$$

This gives

$$\frac{f(t, \dots, t)(n+1)!}{u_1 \dots u_{n+1}\,(n+1)\, t}, \tag{5.101}$$

which proves the expected vegetal property. ⬟

5.7 Examples of moulds

Let us describe the image in Mould of some special and nice elements of Dend.

Let AS be the mould defined by $AS_n = 1/(u_1 \dots u_n)$. The components of this mould provide the basis of the Associative suboperad. Hence it is in the image of the Dendriform operad in the Mould operad. On the other hand, it is known that the basis of the Associative suboperad of the Dendriform operad is given by the sum of all planar binary trees. Hence, one has

$$AS_n = \psi \left(\sum_{t \in \mathcal{Y}(n)} T \right) = \frac{1}{u_1 \dots u_n}. \tag{5.102}$$

On can note that the image by $\mathscr{F}$ of the mould AS is $x/(1-x)$.

Let us say that a planar binary tree is of type (p,q) if its left subtree has $p+1$ leaves and its right subtree has $q+1$ leaves. The sum over binary trees of type (p,q) is

$$\frac{u_p}{u_1 \dots u_n (u_{1\dots n})}. \tag{5.103}$$

This is an easy consequence of the previous result, using for instance the Over and Under products.

Let TY be the mould defined by

$$TY_n = \frac{\sum_{i=1}^n t^{i-1} u_i}{u_1 \dots u_n u_{1\dots n}}, \tag{5.104}$$

with a parameter t. By the preceding discussion, the mould TY is also in the image of the Dendriform operad. The image of TY by $\mathscr{F}$ is

$$\frac{1}{1-t} \log \left(\frac{1-tx}{1-x} \right). \tag{5.105}$$

Another interesting mould has the following components :

$$\frac{\sum_{i=1}^n i\, u_i}{u_1 \dots u_n u_{1\dots n}}. \tag{5.106}$$

By the same argument as above, this mould belongs to the image of the Dendriform operad. This mould should be related to the series indexed by planar binary trees considered in [Cha06]. Its image by $\mathscr{F}$ is

$$\frac{x(2-x)}{2(1-x)^2}. \tag{5.107}$$

One can also compute the image of the Connes-Moscovici series. Let us first recall its definition. In the free pre-Lie algebra on one generator, where the product is denoted by $\curvearrowleft$, let CM_1 be the generator and let

$$CM_n = CM_{n-1} \curvearrowleft CM_1. \tag{5.108}$$

One can consider these objects as elements of the free dendriform algebra on one generator, endowed with the pre-Lie product $\curvearrowleft$. It follows from Prop. 5.4.3 that these elements are alternal.

Proposition 5.7.1 *One has*

$$\psi(CM_n) = \frac{1}{u_1 \dots u_n u_{1\dots n}} \sum_{k=1}^{n} (-1)^{n+k} \binom{n}{k} u_k. \tag{5.109}$$

The image of $\psi(CM)$ *by* $\mathscr{F}$ *is* x.

Proof. The proof is by induction. By definition of the pre-Lie operation $\curvearrowleft$ in the Dendriform operad, one has

$$CM_{n+1} = \curlyvee \circ_2 CM_n - \curlyvee \circ_1 CM_n. \tag{5.110}$$

Hence, in Mould, one gets

$$\psi(CM_{n+1}) = \frac{1}{u_1 u_{1\dots 2}} \circ_2 \psi(CM_n) - \frac{1}{u_{1\dots 2} u_2} \circ_1 \psi(CM_n). \tag{5.111}$$

Explicitly, $\psi(CM_{n+1})$ is given by

$$\frac{u_{2\dots n+1}}{u_1 u_{1\dots n+1}} \psi(CM_n)(u_2, \dots, u_{n+1}) - \frac{u_{1\dots n}}{u_{1\dots n+1} u_{n+1}} \psi(CM_n). \tag{5.112}$$

Then one can use the addition rule for binomial coefficients and the induction hypothesis. ⬟

Another interesting and natural mould PO has the following components :

$$PO_n = \frac{\prod_{i=2}^{n} u_{1\dots i-1} + t\, u_i}{u_1 \prod_{i=2}^{n} (u_i u_{1\dots i})}, \tag{5.113}$$

with a parameter t. This mould also belongs to the image of the Dendriform operad, as it satisfies the following equation :

$$PO_{n+1} = tPO_n \succ 1/u_1 + PO_n \prec 1/u_1. \tag{5.114}$$

Obviously, its image by $\mathscr{F}$ is given by the well-known exponential generating series for the Stirling numbers of the first kind :

$$\frac{(1-x)^{-t} - 1}{t}. \tag{5.115}$$

5.8 Relation with quivers and tilting modules

There is a nice relationship with the theory of tilting modules for the equi-oriented quivers of type $\mathbb{A}$ (in the classical list of simply-laced Dynkin diagrams). Some properties of this special case may be true in the general case of a Dynkin quiver.

Let Q be the equi-oriented quiver of type $\mathbb{A}_n$. It is known by a theorem of Gabriel that there is a bijection between indecomposable modules for Q and positive roots for the root system of type $\mathbb{A}_n$. These positive roots are the sums $\alpha_i + \dots + \alpha_j$ for $1 \leq i \leq j \leq n$, where $\alpha_1, \dots, \alpha_n$ are the simple roots. There is an obvious bijection $\underline{\dim}$ from the set of positive roots to the set of linear functions $u_{i\dots j}$ for $1 \leq i \leq j \leq n$, which is induced by the bijection $\alpha_i \mapsto u_i$.

A tilting module T for the quiver Q is a direct sum of n pairwise non-isomorphic indecomposable modules such that T has no self-extension. One can therefore describe a tilting module T as a set of positive roots, satisfying some condition. Taking the inverse of the product over the corresponding set of linear functions in the variables u, one gets a rational function $\psi(T)$ for each tilting module T. One can check that this set of rational functions is exactly the image in Mould of the set of planar binary trees in Dend by the operad morphism ψ. This gives a natural bijection between tilting modules and planar binary trees.

By this correspondence between tilting modules and trees, the action of the anticyclic rotation τ on the vector space $\mathrm{Dend}(n)$ is mapped to the action induced on the set of roots by the Auslander-Reiten functor on the derived category of the quiver Q.

On the other hand, the action of the anticyclic rotation τ on the vector space $\mathrm{Dend}(n)$ has been related in [Cha05b] to the square of the Auslander-Reiten translation for the derived category of the Tamari poset, which is a classical partial order on the set of planar binary trees. The Tamari poset also has a very natural interpretation in the setting of tilting modules, as a special case of the natural partial order defined by Riedtmann and Schofield [RS91] on the set of tilting modules of a finite-dimensional algebra.

One can try and generalize this to any quiver Q of finite Dynkin type, that is any quiver whose underlying graph is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$. The theorem of Gabriel still holds, hence there is a bijection between indecomposable modules and positive roots. One can similarly map positive roots to linear functions in variables u using the decomposition in the basis of simple roots. For an indecomposable module M, the corresponding linear function is

$$\underline{\dim}(M) = \sum_{i=1}^{n} \dim M_i \, u_i. \tag{5.116}$$

There is a finite set of tilting modules for Q. One can, just as above, define a rational function for each tilting module T, as the inverse of the product of the linear functions over the summands of T. For a tilting module $T = \oplus_j M_j$, one gets

$$\psi(T) = 1/\prod_j \underline{\dim}(M_j) \tag{5.117}$$

Then, one can ask the following questions :

Question 1 : are the functions $\psi(T)$ for all tilting modules T linearly independent ?

Let V_Q be the vector space spanned by the $\psi(T)$ for all tilting modules T.

Question 2 : is V_Q stable by the action induced by the action of the Auslander-Reiten translation τ for Q on the set of positive roots ?

Question 3 : if so, is this action of the Auslander-Reiten translation for Q related to the Auslander-Reiten translation for the poset of tilting modules for Q defined by Riedtmann and Schofield [RS91, HU05].

Let us note a result in the same spirit, that we have learned from L. Hille [Hil06] : for any Dynkin quiver Q, one has

$$\sum_T \psi(T) = 1/u_1 \dots u_n, \tag{5.118}$$

where the sum runs over the set of isomorphism classes of tilting modules. This identity comes from a fan related to tilting modules.

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Chapitre 6

Quivers with relations arising from clusters (A_n case)

This article, written in collaboration with Philippe Caldero and Ralf Schiffler, has been published in the Transactions of the A.M.S. 358 (2006), no. 3, 1347–1364.

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in connection with dual canonical bases. Let U be a cluster algebra of type A_n. We associate to each cluster C of U an abelian category $\mathcal{C}_C$ such that the indecomposable objects of $\mathcal{C}_C$ are in natural correspondence with the cluster variables of U which are not in C. We give an algebraic realization and a geometric realization of $\mathcal{C}_C$. Then, we generalize the "denominator Theorem" of Fomin and Zelevinsky to any cluster.

6.0 Introduction

Cluster algebras were introduced in the work of A. Berenstein, S. Fomin, and A. Zelevinsky, [BFZ05, FZ02, FZ03a, FZ03b]. This theory appeared in the context of dual canonical basis and more particularly in the study of the Berenstein-Zelevinsky conjecture. Cluster algebras are now connected with many topics : double Bruhat cells, Poisson varieties, total positivity, Teichmüller spaces. The main results on cluster algebras are on the one hand the classification of finite cluster algebras by root systems and on the other hand the realization of algebras of regular functions on double Bruhat cells in terms of cluster algebras.

Recall some facts about cluster algebras. Cluster algebras U of rank n form a class of algebras defined axiomatically in terms of a distinguished set of generators $\{u_1, \ldots, u_n\}$. A cluster is a set of "cluster variables" $\{w_1, \ldots, w_n\}$ obtained combinatorially from $\{u_1, \ldots, u_n\}$. The so-called Laurent phenomenon asserts that each cluster variable is a Laurent polynomial in the set of variables given by a cluster. For each cluster C, one can define combinatorially an oriented quiver Q_C.

Suppose from now on that U is a finite cluster algebra, *i.e.* there only exists a finite number of cluster variables. It is known that U can be described by the data of a root system X_n. Moreover, there exists a cluster Σ such that Q_Σ is the alternating quiver on X_n. S. Fomin and A. Zelevinsky give in [FZ03a] a more precise description of the Laurent phenomenon : there exists a one-to-one correspondence $\alpha \mapsto w_\alpha$ between the set of almost positive roots of X_n, *i.e.* positive roots and simple negative roots, and the set of cluster variables, such that the denominator of w_α as a Laurent polynomial in Σ is given by the decomposition of α in the basis of simple roots. Via Gabriel's Theorem, this property suggests a link between cluster algebras and representation theory of artinian rings. This relation was already investigated in [MRZ03] and is the core subject of the recent papers [BMR$^+$06, BMR07].

In this paper, we give conjectural relations R_C on the quiver Q_C, such that for any cluster C, the denominators of the cluster variables as Laurent polynomial in C are described by indecomposables of the category $\mathcal{C}_C$ of representations of Q_C with relations R_C.

The main result of this article is the proof of this conjecture in the A_n case.

Another important result of this paper is a geometric realization of the category $\mathcal{C}_C$ in the A_n case. Recall that the algebra of regular functions on the 2-grassmannian of $\mathbb{C}^{n+3}$ is a finite cluster algebra U of type A_n. Via this realization, the cluster variables are in natural bijection with the diagonals of a regular $(n+3)$ polygon. Moreover, a result of Fomin and Zelevinsky asserts that this bijection gives a one-to-one correspondence between the set of clusters of U and the set of diagonal triangulations of the polygon. Theorem 6.4.3 gives a simple realization of the category $\mathcal{C}_C$ in terms of the diagonals of the $(n+3)$ polygon. There also exists a more canonical category associated to a finite cluster algebra and also studied in [BMR$^+$06]. We give in the A_n case a geometric realization of this category, see Theorem 6.5.2.

Let us mention that so-called cluster-tilted rings, introduced in [BMR$^+$06, BMR07] as endomorphism rings of tilting objects in this canonical category, are conjectured there to be isomorphic to the path algebras of our quivers with relations. The view-point of these articles should provide a well-suited categorical background to generalize our constructions.

The authors would like to thank Robert Marsh and Bernhard Keller for useful conversations. Our thanks also go to the LACIM at UQAM in Montreal, where this work was started, for its great hospitality.

6.1 Quivers of cluster type

Let X_n be a simply-laced Dynkin diagram of rank n and finite type. First we need to recall some material on clusters.

Each cluster C of a cluster algebra of rank n is associated with a sign-skew-symmetric square matrix B_C whose rows and columns are indexed by the cluster variables of the cluster C. In the simply-laced case, coefficients of the matrices B_C belong to the set $\{-1, 0, 1\}$. Hence it is easy and convenient to depict these matrices using oriented graphs (once a convention is chosen for the orientation). This oriented graph is called the quiver associated to the cluster C and is denoted by Q_C. It is known that all triangles (and more generally cycles) in these quivers are oriented in a cyclic way [FZ03a, Proposition 9.7].

The mutation procedure of clusters contains in particular a mutation rule for the associated matrices, which can be translated as a mutation rule for the associated quivers. In the simply-laced case, the mutation rule can be further simplified. The result is as follows.

Let C be a cluster in a cluster algebra of simply-laced type. The mutation $\mu_i(Q_C)$ of the quiver Q_C at a vertex i is described as follows. First, all arrows incident to i in Q_C are reversed in the mutated quiver. Then, for each pair of one incoming arrow $j \to i$ and one outgoing arrow $i \to k$ in Q_C, the arrow $j \to k$ is in the mutated quiver if and only if the arrow $k \to j$ is not in Q_C. The other arrows of Q_C are kept unchanged in the mutated quiver.

By definition, a *shortest path* in the quiver Q_C is an oriented path (with no repeated arrow) contained in an induced subgraph of Q_C which is a cycle.

Définition 2 *Let Q_C be the quiver associated to the cluster C. For each arrow $i \to j$ in Q_C, a relation $\mathrm{Rel}_{i,j}$ is defined as follows. Consider the set of shortest paths from j to i :*

- *If there are exactly two distinct paths c and c' then $c = c'$.*
- *If there exists only one path c then $c = 0$.*
- *If there is no such path, there is no relation.*

To each cluster C, one defines the (abelian) category $\mathrm{Mod}\,\mathsf{A}_C$ as the category of modules over the algebra A_C which is the quotient of the path algebra of the quiver Q_C by the ideal generated by the relations $\mathrm{Rel}_{i,j}$ for all arrows $i \to j$ of Q_C.

Remark : it is clear in type A that there is at most one shortest path for each arrow. One could show using the geometric model of type D cluster algebras that there are at most two such paths in this case. In order for the conjecture to make sense also in the exceptional cases E, it remains to prove that this is also true in these cases. We will not consider this question here.

Obvious remark : there is a natural one-to-one correspondence between the vertices of the quiver Q_C and the simple modules in $\mathrm{Mod}\,\mathsf{A}_C$. Therefore, the isomorphism class of the simple module associated to the vertex i will be denoted α_i.

Let $\mathrm{Ind}(Q_C)$ be the set of isomorphism classes of indecomposables of $\mathrm{Mod}\,\mathsf{A}_C$.

Conjecture 6.1.1 *Let $C = \{u_1, \ldots, u_n\}$ be a cluster in a cluster algebra of simply-laced type and rank n. Let V be the set of all cluster variables for this cluster algebra. There exists a bijection $b : \mathrm{Ind}(Q_C) \to V \backslash C$, $\alpha \mapsto w_\alpha$ such that $w_\alpha = \frac{P(u_1,\ldots,u_n)}{\prod_i u_i^{n_i}}$, where P is a polynomial prime to u_i for all i and where $n_i = n_i(\alpha)$ is the multiplicity of the simple module α_i in the module α.*

Remarque 10 *Through Gabriel's celebrated Theorem relating indecomposables and positive roots, this conjecture generalizes the Theorem of Fomin and Zelevinsky ([FZ03a, Theorem 1.9]) which corresponds to the case of the alternating quiver.*

The main aim of the present article is to prove Conjecture 6.1.1 in the case of cluster algebras of type A_n. This will be done using the geometric realization in terms of triangulations given by Fomin and Zelevinsky [FZ03b, §3.5].

6.2 Categories of diagonals

In this section, we introduce some terminology on triangulations and define a category $\mathcal{C}_T$ for each triangulation T in a geometric way.

6.2.1 Triangulations and diagonals

Let us fix a nonnegative integer n and a triangulation T of a regular polygon with $n+3$ vertices. The diagonals of this polygon will be called roots and designed by Greek letters. Let us call *negative* the roots belonging to T and *positive* the other roots. Let Φ_+ be the set of positive roots with respect to T. Let I be the set of negative roots. By convention, the negative root corresponding to $i \in I$ will be called $-\alpha_i$. The *support* $\operatorname{Supp} \alpha \subseteq I$ of a positive root α is the set of negative roots which cross α. Note that a positive root α is determined by its support. Indeed it is possible to recover the vertices of a positive diagonal from the sequence of crossed negative diagonals. A positive root α is related to a positive root α' by a *pivoting elementary move* if the associated diagonals share a vertex on the border (the pivot), the other vertices of α and α' are the vertices of a border edge of the polygon and the rotation around the pivot is positive (for the trigonometric direction) from α to α'. Let P_v denote the pivoting elementary move with pivot v. A *pivoting path* from a positive root α to a positive root α' is a sequence of pivoting elementary moves starting at α and ending at α'.

6.2.2 Categories of diagonals

One can define a combinatorial $\mathbb{C}$-linear additive category $\mathcal{C}_T$ as follows. The objects are positive integral linear combinations of positive roots. By additivity, it is enough to define morphisms between positive roots. The space of morphisms from a positive root α to a positive root α' is a quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from α to α'.

The subspace which defines the quotient is spanned by the so-called *mesh relations* (see Figure 6.1). For any couple α, α' of positive roots such that α is related to α' by two consecutive pivoting elementary moves with distinct pivots, we define the mesh relation $P_{v_2'} P_{v_1} = P_{v_1'} P_{v_2}$, where v_1, v_2 (respectively v_1', v_2') are the vertices of α (respectively α') such that $P_{v_1'} P_{v_2}(\alpha) = \alpha'$. That is, any two consecutive pivoting elementary moves using different pivots can in some sense be "exchanged".

In these relations, negative roots or border edges are allowed, with the following conventions.

(i) If one of the intermediate edges is a border edge, the corresponding term in the mesh relation is replaced by zero.

(ii) If one of the intermediate edges is a negative root, the corresponding term in the mesh relation is replaced by zero.

More generally, a mesh relation is an equality between two pivoting paths which differ only in two consecutive pivoting elementary moves by such a change.

We can now define the set of morphisms from a positive root α to a positive root α' to be the quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from α to α' by the subspace generated by mesh relations.

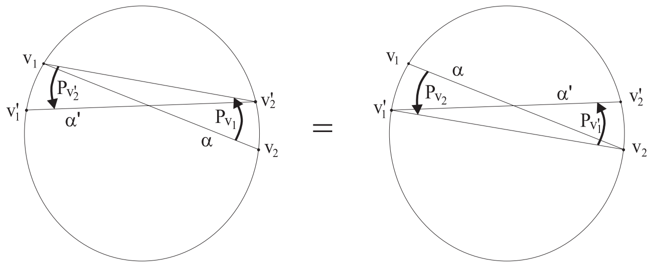

FIG. 6.1 – mesh relation $P_{v_2'}P_{v_1} = P_{v_1'}P_{v_2}$

Therefore, the image of a pivoting path in the space of morphisms is either the zero morphism or only depends on the class of the pivoting path modulo the equivalence relation on the set of pivoting paths generated by the mesh relations with no vanishing terms.

The following Lemma will be useful later.

Lemme 6.2.1 *The vector space* $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ *is not zero if and only if there exists* $i \in \mathrm{Supp}\,\alpha \cap \mathrm{Supp}\,\alpha'$ *such that the relative positions of* α, α' *and* $-\alpha_i$ *are as in Figure 6.2. That is, let* v_1, v_2 *be the endpoints of* $-\alpha_i$ *and* u_1, u_2 *(respectively* u_1', u_2'*) be the endpoints of* α *(respectively* α'*). Then ordering the vertices of the polygon in the positive trigonometric direction starting at* v_1*, we have* $v_1 < u_1 \leq u_1' < v_2 < u_2 \leq u_2'$*. In this case,* $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ *is of dimension one.*

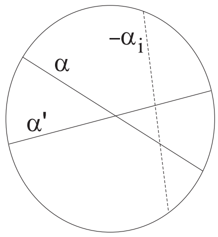

FIG. 6.2 – Relative position

Proof. Suppose that $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is not zero. Let $P \neq 0$ be a sequence $\alpha = \alpha^0 \xrightarrow{P_1} \alpha^1 \xrightarrow{P_2} \dots \xrightarrow{P_m} \alpha^m = \alpha'$ where the α^i are positive roots and the P_i are pivoting elementary moves. One can map this sequence to a path in a grid using the following rules. Let U and R be the two vertices of α. Then the grid path starts going up if the first pivot is U and right if the first pivot is R. Then two consecutive steps in the grid path are drawn in the same direction (up or right) if and only if the corresponding elementary pivoting moves share the same pivot. The sequence of elementary pivoting moves can be recovered uniquely from the grid path. Now using the mesh relations, one can replace any pair of consecutive steps (up,right) in the grid path by a pair of consecutive steps (right,up) or the other way round. This gives another non-vanishing sequence

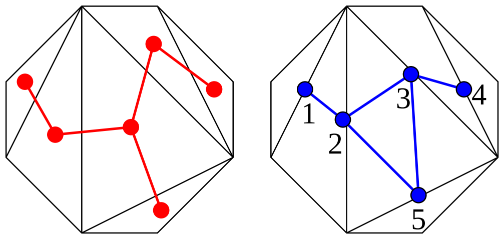

FIG. 6.3 – Tree t_T and Graph Q_T defined by a Triangulation T

of elementary pivoting moves from α to α'. In this way, one can reach every grid path with the same numbers of up and right steps *i.e.* ending at the same point of the grid. We may therefore suppose that the first k moves $P_1, \ldots, P_k$ have as pivot one of the vertices of α and the last $(m-k)$ moves $P_{k+1}, \ldots, P_m$ have as pivot the common vertex of α^k and α'. Denote by V_1 (respectively V_2) the set of vertices of $\alpha^1, \ldots, \alpha^k$ (respectively $\alpha^{k+1}, \ldots, \alpha^m$) other than the pivot of P_1 (respectively P_{k+1}). Since $P \neq 0$ this implies that α and α' intersect or share a vertex and that all diagonals with one vertex in V_1 and the other in V_2 are positive. Thus the vertices of α and α' form a quadrilateral in the polygon without any diagonals of the triangulation T crossing it from V_1 to V_2. Because T is a triangulation we must have a $-\alpha_i \in T$ crossing this quadrilateral in the other direction and we get the situation in the diagram. On the other hand, in the situation of the diagram it is clear that there is a non-zero morphism from α to α'. Finally, the dimension of $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is at most one, since any two non-zero pivoting paths from α to α' are in the same class. ⬟

6.3 Quivers and triangulations

In this section, we define a quiver directly from a triangulation and introduce the category of modules on this quiver with some relations.

6.3.1 Graphs and trees

Let T be a triangulation. Then one can define a planar tree t_T as follows. Its vertices are the triangles of T and its edges are between adjacent triangles (see left part of Figure 6.3). Vertices of t_T have valency 1, 2 or 3. It is clear that there is always at least one vertex of valency 1.

From T, one can also define a graph Q_T as follows. The vertices of Q_T are the inner edges of T and are related by an edge if they bound part of the same triangle (see right part of Figure 6.3).

In fact, it is possible to define the graph Q_T starting from the planar tree t_T. Vertices of Q_T are the edges of t_T. Two vertices of Q_T are related by an edge in Q_T if the corresponding edges of t_T share a vertex in t_T. The equivalence with the previous definition is obvious.

A *leaf* is an edge e of t_T such that at least one of its vertices has valency 1. As there is always a vertex of valency 1 in t_T, there always exists a leaf.

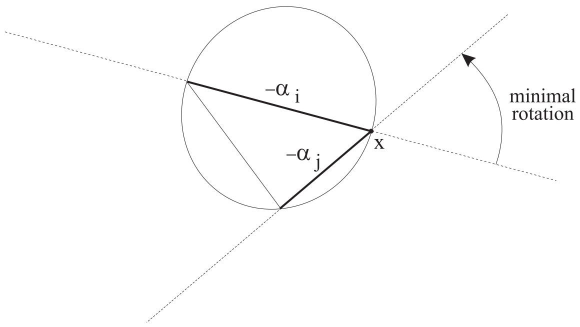

FIG. 6.4 – Definition of the arrows of Q_T

Recall that the mutation of a triangulation at one of its diagonal is the unique triangulation which can be obtained by replacing this diagonal with another one.

6.3.2 Quivers and triangulations

Let T be a triangulation. Let us define a quiver Q_T with underlying graph the graph Q_T defined in the previous section. Recall that its vertices are in bijection with I. Put a point at the middle of each negative root in T and draw an edge between points in two negative roots $-\alpha_i, -\alpha_j$ which bound part of the same triangle. The orientation of the edge is defined as follows : denote by x the common vertex of $-\alpha_i, -\alpha_j$, then $-\alpha_i \to -\alpha_j$ if the rotation with minimal angle around x that sends the line through $-\alpha_i$ to the line through $-\alpha_j$ is in positive trigonometric direction (see Figure 6.4). From this description, it follows that all triangles in Q_T are oriented.

Lemme 6.3.1 *The mutation of quivers, as defined in §6.1, corresponds to the mutation of triangulations described above.*

Proof. Left to the reader. ⬟

One can define a $\mathbb{C}$-linear abelian category $\operatorname{Mod} Q_T$ as follows. This is the category of modules over the quiver Q_T with the following relations, called *triangle relations.*

$$\text{In any triangle, the composition of two successive maps is zero.} \tag{6.1}$$

These relations are exactly the relations prescribed by Definition 2.

The next two Lemmas are steps for the proof of Lemma 6.3.4 which will be used later.

Lemme 6.3.2 *The support* $\operatorname{Supp}\alpha$ *of a positive root* α *is connected as a subset of the quiver* Q_T.

Proof. Let $-\alpha_i, -\alpha_j$ be two distinct diagonals in $\operatorname{Supp}\alpha$. We will show that there is an unoriented path from i to j in $Q_T \cap \operatorname{Supp}\alpha$. The diagonals $-\alpha_i$ and $-\alpha_j$ cut the polygon into three parts. Denote by R_{ij} the part that contains both $-\alpha_i$ and $-\alpha_j$. We proceed by induction on m the number of negative roots in

R_{ij}. If $m = 1$ then $-\alpha_i = -\alpha_j$ and there is nothing to prove. Let us assume that $m > 1$. Let Δ be the unique triangle in T that contains $-\alpha_i$ and lies in R_{ij}. Since α crosses both $-\alpha_i$ and $-\alpha_j$, it has to cross exactly one of the two sides different from $-\alpha_i$ in Δ. This side cannot be a border edge of the polygon, hence it has to be a negative root, call it $-\alpha_k$. Thus there is an edge between $-\alpha_i$ and $-\alpha_k$ in Q_T and $i, k \in \operatorname{Supp}\alpha$. We may suppose by induction that there is an unoriented path in $\operatorname{Supp}\alpha$ from k to j and we are done. ⬟

Lemme 6.3.3 *Let α, α' be positive roots, then $\operatorname{Supp}\alpha \cap \operatorname{Supp}\alpha'$ is connected.*

Proof. Suppose the contrary. Write $S = \operatorname{Supp}\alpha$ and $S' = \operatorname{Supp}\alpha'$ for short. Let $i, k \in S \cap S'$ be two vertices that belong to different connected components of $S \cap S'$. Since S and S' are connected (Lemma 6.3.2) we may choose two minimal paths $p : i = i_1, i_2, \ldots, i_p = k$ in S and $p' : i = j_1, j_2, \ldots, j_q = k$ in S'. Let m be the smallest integer such that $i_{m+1} \neq j_{m+1}$. In the triangulation T, each of the diagonals $-\alpha_{i_{m-1}}, -\alpha_{i_{m+1}}, -\alpha_{j_{m+1}}$ has a vertex in common with $-\alpha_{i_m}$. Since a positive root can only cross two sides of a triangle, we get that $-\alpha_{i_m}, -\alpha_{i_{m+1}}$ and $-\alpha_{j_{m+1}}$ form a triangle Δ in T. Moreover $i_{m+1} \in S \setminus S'$ and $j_{m+1} \in S' \setminus S$. Now cutting out the triangle Δ divides the polygon into three parts : $R_{i_m}, R_{i_{m+1}}$ and $R_{j_{m+1}}$ such that R_l contains $-\alpha_l$, $l = i_m, i_{m+1}, j_{m+1}$. Clearly all $-\alpha_{i_l}$, $l \geq m+1$ lie in $R_{i_{m+1}}$ and all $-\alpha_{j_l}$, $l \geq m+1$ lie in $R_{j_{m+1}}$. But this contradicts the fact $-\alpha_{j_q} = -\alpha_k = -\alpha_{i_p}$ and we have shown that $S \cap S'$ is connected. ⬟

Let us introduce objects of $\operatorname{Mod} Q_T$ indexed by the positive roots. The module $(M^\alpha, f^\alpha) = (M_i^\alpha, f_{ij}^\alpha)$ defined by

$$M_i^\alpha = \begin{cases} \mathbb{C} & \text{if } i \in \operatorname{Supp}\alpha, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{ij}^\alpha = \begin{cases} \mathrm{id}_{\mathbb{C}} & \text{if } M_i^\alpha = \mathbb{C} = M_j^\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

This is indeed an object in $\operatorname{Mod} Q_T$ because a positive root α can only cross two sides of a triangle in T, which implies that in each triangle in Q_T there is at most one arrow $i \to j$ such that $f_{ij}^\alpha \neq 0$ and hence the triangle relations (6.1) hold.

Then we have the following Lemma.

Lemme 6.3.4 *The vector space $\operatorname{Hom}_{\operatorname{Mod} Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is not zero if and only if the following conditions hold. Let $S = \operatorname{Supp}\alpha$ and $S' = \operatorname{Supp}\alpha'$ for short.*

(i) *$S \cap S'$ is not empty,*

(ii) *There is no arrow from $S \setminus S'$ to $S \cap S'$ in Q_T,*

(iii) *There is no arrow from $S \cap S'$ to $S' \setminus S$ in Q_T.*

In this case, $\operatorname{Hom}_{\operatorname{Mod} Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is of dimension one.

Proof. Let P be a non-zero element of $\operatorname{Hom}_{\operatorname{Mod} Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$. Then condition (i) is clearly true. Let us show conditions (ii) and (iii). Suppose that condition (ii) is not true, thus there is an arrow $i \to j$ in Q_T with $i \in S \setminus S'$, $j \in S \cap S'$ and such that the following diagram commutes.

$$\begin{array}{ccc} M_i^\alpha & \xrightarrow{\mathrm{id}_{\mathbb{C}}} & M_j^\alpha \\ {\scriptstyle P_i}\downarrow & & \downarrow{\scriptstyle P_j} \\ 0 & \longrightarrow & M_j^{\alpha'} \end{array} \tag{6.3}$$

Thus P_i and P_j are both zero. Now let k be any vertex in $S \cap S'$, We will show that P_k is zero. By Lemma 6.3.3, there is an unoriented path $k = k_0 — k_1 — \cdots — k_m = i$ in Q_T such that each $k_i \in S \cap S'$. We proceed by induction on m. The case $m = 0$ is done above, suppose $m > 0$. By induction P_{k_1} is zero and the commutativity of the diagram

$$\begin{array}{ccc} M_k^{\alpha} & \xrightarrow{\mathrm{id}_{\mathbb{C}}} & M_{k_1}^{\alpha} \\ {\scriptstyle P_k}\downarrow & & \downarrow{\scriptstyle 0} \\ M_k^{\alpha'} & \xrightarrow{\mathrm{id}_{\mathbb{C}}} & M_{k_1}^{\alpha'} \end{array} \tag{6.4}$$

implies that P_k is zero for both possibilities of orientation of $k — k_1$ in Q_T. By contradiction, this shows (ii). Condition (iii) is proved by a similar argument.

In order to show the converse statement, let α, α' be such that (i),(ii) and (iii) hold. Define $P \in \mathrm{Hom}_{\mathrm{Mod}\, Q_T}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$ by $P_i = \mathrm{id}_{\mathbb{C}}$ whenever $i \in S \cap S'$ and $P_i = 0$ otherwise. Then (i) implies that P is non-zero. We only have to check that P is a morphism of quiver modules, *i.e.* that the diagram

$$\begin{array}{ccc} M_i^{\alpha} & \xrightarrow{f_{ij}^{\alpha}} & M_j^{\alpha} \\ {\scriptstyle P_i}\downarrow & & \downarrow{\scriptstyle P_j} \\ M_i^{\alpha'} & \xrightarrow{f_{ij}^{\alpha'}} & M_j^{\alpha'} \end{array} \tag{6.5}$$

commutes for all $i \to j$ in Q_T. But this is true because of conditions (ii) and (iii). Finally, the dimension of $\mathrm{Hom}_{\mathrm{Mod}\, Q_T}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$ is at most one, since all vector spaces $M_i^{\alpha}, M_i^{\alpha'}$ are of dimension zero or one and the intersection of the supports is connected (Lemma 6.3.3). ⬟

6.4 Equivalence of categories

In this section we define a functor and prove that it is an equivalence of categories.

6.4.1 The Θ functor

Let us define a $\mathbb{C}$-linear additive functor Θ from $\mathcal{C}_T$ to $\mathrm{Mod}\, Q_T$. On objects, it is sufficient by additivity to define Θ on positive roots. The image of the positive root α is taken to be the module (M^{α}, f^{α}) defined by Formula (6.2). Now we define the functor Θ on morphisms. By additivity, it is sufficient to define the functor on morphisms from a positive root to a positive root. Our strategy is to define first the functor on pivoting elementary moves, then check that the mesh relations hold. For any pivoting elementary move $P : \alpha \to \alpha'$, define the morphism $\Theta(P)$ from (M^{α}, f^{α}) to $(M^{\alpha'}, f^{\alpha'})$ to be $\mathrm{id}_{\mathbb{C}}$ whenever possible and 0 else. Let us now check that this is indeed a morphism in $\mathrm{Mod}\, Q_T$. For a given arrow $j \to i$ in Q_T, we have to check the commutativity of the following diagram :

$$\begin{array}{ccc} M_j^{\alpha} & \xrightarrow{f_{ji}^{\alpha}} & M_i^{\alpha} \\ {\scriptstyle \Theta(P)_j}\downarrow & & \downarrow{\scriptstyle \Theta(P)_i} \\ M_j^{\alpha'} & \xrightarrow{f_{ji}^{\alpha'}} & M_i^{\alpha'} \end{array} \tag{6.6}$$

This is obvious if $M_j^{\alpha} = 0$ or $M_i^{\alpha'} = 0$ and also if both M_i^{α} and $M_j^{\alpha'}$ are 0. Suppose $M_j^{\alpha} \neq 0$ and $M_i^{\alpha'} \neq 0$. If $M_i^{\alpha} \neq 0$ and $M_j^{\alpha'} \neq 0$ then all four maps are $\mathrm{id}_{\mathbb{C}}$ and the diagram commutes. The only remaining case is if exactly one of M_i^{α}, $M_j^{\alpha'}$ is not zero. We will show that this cannot happen. Suppose that $M_j^{\alpha'} = 0$ and $M_i^{\alpha} \neq 0$, that is $j, i \in \operatorname{Supp}\alpha,\ i \in \operatorname{Supp}\alpha'$ and $j \notin \operatorname{Supp}\alpha'$. Since $\alpha \to \alpha'$ is a pivoting elementary move we get that $-\alpha_j$ crosses α, that $-\alpha_j$ and α' have a common point on the boundary of the polygon and that $-\alpha_i$ crosses α and α'. This contradicts the orientation $j \to i$ in the quiver Q_T. The other case can be excluded by a similar argument. To show that the functor is well defined, it only remains to check the mesh relations. Let $\alpha \xrightarrow{P^1} \beta$, $\beta \xrightarrow{P^2} \gamma$, $\alpha \xrightarrow{P^3} \beta'$, $\beta' \xrightarrow{P^4} \gamma$ be pivoting elementary moves with α, β, γ positive roots and $\beta \neq \beta'$. Note that we can exclude the case where β and β' are both negative roots or border edges because in this case either α or γ has to be negative too, since T is a triangulation. Suppose first that β' is positive. One has to check the commutativity of the diagram

$$\begin{array}{ccc} M_i^{\alpha} & \xrightarrow{\Theta(P^1)_i} & M_i^{\beta} \\ {\scriptstyle \Theta(P^3)_i}\downarrow & & \downarrow{\scriptstyle \Theta(P^2)_i} \\ M_i^{\beta'} & \xrightarrow{\Theta(P^4)_i} & M_i^{\gamma} \end{array} \tag{6.7}$$

for all i. The only non trivial case is when $i \in \operatorname{Supp}\alpha \cap \operatorname{Supp}\gamma$. In this case, we also have $i \in \operatorname{Supp}\beta \cap \operatorname{Supp}\beta'$ because any diagonal crossing both α and γ must also cross β and β'. Thus all maps are $\mathrm{id}_{\mathbb{C}}$ and the diagram commutes. Suppose now that β' is negative or a border edge. We have to show that the composition $M_i^{\alpha} \xrightarrow{\Theta(P^1)_i} M_i^{\beta} \xrightarrow{\Theta(P^2)_i} M_i^{\gamma}$ is zero for all i. But in this case no negative root can cross both α and γ. So $\operatorname{Supp}\alpha \cap \operatorname{Supp}\gamma$ is empty, therefore the composition is zero. Hence the mesh relations hold, with the conventions made in its definition.

6.4.2 Theorem of equivalence

First we need a Lemma.

Lemme 6.4.1 *Let α and α' be two positive roots. Then $\operatorname{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is not zero if and only if $\operatorname{Hom}_{\operatorname{Mod} Q_T}((M^{\alpha}, f^{\alpha}), (M^{\alpha'}, f^{\alpha'}))$ is not zero.*

Proof. We have to show that the conditions in Lemma 6.2.1 and in Lemma 6.3.4 are equivalent.

Suppose α, α' are as in Lemma 6.2.1. Then $i \in S \cap S'$ which implies (i). Suppose there is an arrow $j \to k$ in Q_T such that $j \in S \setminus S'$ and $k \in S \cap S'$. Then $-\alpha_k$ crosses both α and α' while $-\alpha_j$ crosses only α. Since $j \to k$, we know that $-\alpha_j$ and $-\alpha_k$ are related as in figure 6.4. This is impossible because of the way that $-\alpha_i$ intersects α and α'. We have shown that condition (ii) holds; condition (iii) can be shown similarly. This proves one direction of the Lemma.

Suppose now that α, α' satisfy the conditions (i),(ii) and (iii) of Lemma 6.3.4. By (i), there exists $-\alpha_i$ in $S \cap S'$. Let v_1, v_2 be the endpoints of $-\alpha_i$. Consider the two parts of the polygons R_l and R_r delimited by $-\alpha_i$. Each of them contains exactly one vertex of α and exactly one vertex of α'. Consider the positive roots α, α' as a paths running from R_r to R_l. The sequence of negative roots given by the successive intersections of the path α (respectively α') with elements of T yields an ordering of $\operatorname{Supp}\alpha$ (respectively $\operatorname{Supp}\alpha'$). Let $S_l = \{-\alpha_i = -\alpha_{i_1}, -\alpha_{i_2}, \ldots, -\alpha_{i_p}\}$ (respectively $S'_l = \{-\alpha_i = -\alpha_{j_1}, -\alpha_{j_2}, \ldots, -\alpha_{j_q}\}$) be the set of negative roots in R_l crossing α (respectively α') in that order. Let m be the greatest integer such that $-\alpha_{i_m} = -\alpha_{j_m}$. We will distinguish four cases.

1. $m = p = q$, then on the boundary of R_l, going from an endpoint of $-\alpha_i$ in positive direction, we meet α and α' at the same time.
2. $m = p < q$, then $-\alpha_{j_{m+1}}$ and $-\alpha_{j_m}$ bound part of the same triangle in T. The corresponding edge in Q_T is oriented $j_{m+1} \to j_m$ by (iii). This implies that going from an endpoint of $-\alpha_i$ in positive direction on the boundary of R_l, we meet α first and then α'.
3. $m = q < p$, then $-\alpha_{i_{m+1}}$ and $-\alpha_{i_m}$ bound part of the same triangle in T. The corresponding edge in Q_T is oriented $i_{m+1} \leftarrow i_m$ by (ii). This implies again that going from an endpoint of $-\alpha_i$ in positive direction on the boundary of R_l, we meet α first and then α'.
4. $m < p$ and $m < q$, then $-\alpha_{i_{m+1}}$, $-\alpha_{i_m}$ and $-\alpha_{j_{m+1}}$ are three different diagonals that bound part of the same triangle in T. The corresponding edges in Q_T are oriented $i_{m+1} \leftarrow i_m$ by (ii) and $j_{m+1} \to j_m$ by (iii). This implies once more that going from an endpoint of $-\alpha_i$ in positive direction on the boundary of R_l, we meet α first and then α'.

By symmetry, we obtain the same results in the other part R_r. This implies that the relative positions of α, α' and α_i are exactly as described in Lemma 6.2.1. ⬟

Proposition 6.4.2 *The functor Θ is fully faithful.*

Proof. Using Lemma 6.4.1, it only remains to show that the image of a non-zero morphism is a non-zero morphism. It is sufficient to show this for all non-zero morphisms between positive roots. Let $P \in \operatorname{Hom}(\alpha, \alpha')$ be such a morphism. Then P is given by a sequence of pivoting elementary moves $\alpha = \alpha^1 \to \ldots \to \alpha^m = \alpha'$. This sequence being a non-zero morphism implies that there exists a negative root $-\alpha_i$ crossing all the α^k, $k = 1, \ldots, m$, by Lemma 6.2.1. By definition, $\Theta(P)_i$ is $\mathrm{id}_{\mathbb{C}}$, hence non-zero. ⬟

Remarque 11 *If i is a leaf (see §6.3.1) and α a positive root, then $i \in \operatorname{Supp}\alpha$ if and only if one endpoint of α is the vertex x of the polygon that is cut off by $-\alpha_i$. Therefore there exists one positive root α^{Pr_i} such that the set of all positive roots with i in their supports is equal to the set*

$$\{\alpha^{Pr_i}, P_x(\alpha^{Pr_i}), P_x(P_x(\alpha^{Pr_i})), \ldots, P_x^{n-1}(\alpha^{Pr_i})\},$$

where P_x is the pivoting elementary move with pivot x.

Théorème 6.4.3 *The functor Θ gives an equivalence of categories from $\mathcal{C}_T$ to* $\operatorname{Mod} Q_T$.

Proof. It only remains to show that the functor Θ is essentially surjective, *i.e.* that each indecomposable module in $\operatorname{Mod} Q_T$ is the image of a positive root under Θ. In fact, we will characterize the indecomposable modules of $\operatorname{Mod} Q_T$ with the help of the Auslander-Reiten theory, and this will enable us to conclude by proving that there are $\frac{n(n+1)}{2}$ indecomposable Q_T-modules.

In the following, we refer to [Gab80], see also [ARS95] for definitions, notation and results in representation theory of finite dimensional algebras. In this proof, we use the following notations in $\operatorname{Mod} Q_T$: P_i (respectively I_i) is the $i-$th projective (respectively injective) indecomposable module, with the convention that $P_\emptyset = I_\emptyset = 0$. Hence $(P_i)_l = \mathbb{C}$ if there is an oriented path in Q_T modulo the triangle relations from i to l and $(P_i)_l = 0$ otherwise. Similarly, $(I_i)_l = \mathbb{C}$ if there is an oriented path in Q_T modulo the triangle relations from l to i and $(I_i)_l = 0$ otherwise. The maps of P_i and of I_i are $\mathrm{id}_{\mathbb{C}}$ whenever possible and zero otherwise. In particular, these modules are multiplicity free. Fix a triangulation T. In the sequel, we set $Q = Q_T$ when no confusion occurs and we denote by Q_0 its set of vertices. Given a subset of vertices S of Q_0, a *full subquiver* of Q with vertices S will be the set S together with the set of all arrows (with relations) of Q joining vertices of S. We say that a Q_T-module M is *of type A* if the full subquiver of Q_T on the support of M is of type A_k for some $k \in \mathbb{N}$.

Lemme 6.4.4 *Let M be an indecomposable Q-module of type A and let N be any indecomposable Q-module. If* $\operatorname{Hom}_Q(N, M)$ *or* $\operatorname{Hom}_Q(M, N)$ *contains an irreducible morphism, then N is of type A.*

Proof. The proof is based on the construction of irreducible morphisms via the Nakayama functor [Gab80, §4.4]. The dual functor gives an (anti)-equivalence between $\operatorname{Mod} Q$ and $\operatorname{Mod} Q^{opp} = \operatorname{Mod} Q_{T^*}$, where T^* is the triangulation obtained from T by a reflection w.r.t a line containing the center of the regular polygon. Using this (contravariant) functor, we can easily reduce the proof to the case where $\operatorname{Hom}_Q(N, M)$ contains an irreducible.

Suppose that the support of M is given by the set $Q'_0 := \{1, \ldots, m\}$. Let Q' be the full subquiver of Q given by $\operatorname{Supp} M$. By assumption, Q' is of type A_m with extremal vertices 1 and m and we can suppose that the edges link i with $i \pm 1$. Remark that, as M is indecomposable of type A, M is multiplicity free.

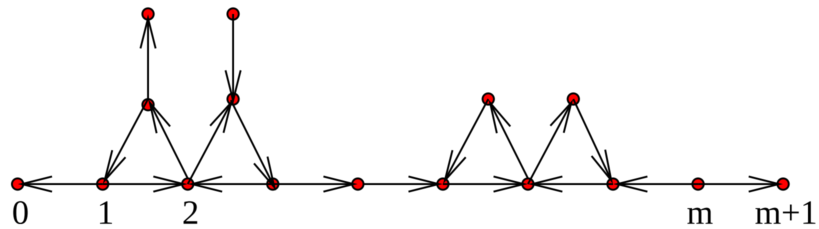

FIG. 6.5 – Quiver Q and subquiver Q' of type A

In order to go further, we need some more precise description of the quiver Q' inside Q. In the sequel, we freely use the section 6.3.1. The reader may like to follow the argument on the example provided by Figure 6.5. Suppose that the module M is not simple, *i.e.* $m \neq 1$. Now we define the vertices 0 and $m+1$ of Figure 6.5, which may or may not exist. There exists at most one vertex 0 in $Q_0 \backslash Q'_0$ such that $1 \to 0$ and such that there exist no other edges between Q'_0 and 0. In the same way, there exists at most one vertex $m+1$ in $Q_0 \backslash Q'_0$ such that $m \to m+1$ and such that there exist no other edges between Q'_0 and $m+1$. Note also that for all k in Q'_0, there exist at most two vertices $i(k)^{\pm}$ such that $i(k)^{\pm} \notin Q'_0$ and $i(k)^{\pm} \to k$ is an arrow of Q. We can define $i(k)^+$, resp. $i(k)^-$, to be the vertex such that $k+1 \to i(k)^+$, resp. $k-1 \to i(k)^-$. By convention, if the vertices $i(k)^+$, $i(k)^-$, 0, $m+1$ do not exist, we define the corresponding symbol to be the empty set.

Let S_0, resp. S_{m+1}, be the support of the injective module associated to 0, resp. $m+1$, in the full subquiver of Q with set of vertices $Q \backslash 1$, resp. $Q \backslash m$. It is clear from the tree structure of Q, see §6.3.1, that the set $Q''_0 := S_0 \cup Q'_0 \cup S_{m+1}$ is the set of vertices of a full subquiver Q'' of Q of type A.

For each source k or for $k = 1, m$, let $S_k^{\pm}$ be the support of the injective module associated to $i(k)^{\pm}$ in the full subquiver of Q with set of vertices $Q_0 \backslash Q'_0$.

Note that if M is projective, then the module N is a direct summand of the radical of M and the lemma is true in this case. Suppose now that M is not projective. Then, we can calculate the Auslander-Reiten translate τM via the Nakayama functor. We have a minimal projective presentation of M : $P^1 \to P^0 \to M \to 0$ by setting $P^0 := \oplus_{i \in R} P_i$, where i runs over the set R of sources of Q', and $P^1 := \oplus_{j \in R'} P_j$, where j runs over the set R' given by the union of $\{0, m+1\}$ with the sinks of $Q'_0 \backslash \{1, m\}$.

By [Gab80, Remark 3.1], with the help of the Nakayama functor, we obtain a minimal injective representation of the Auslander-Reiten translate τM of M : $0 \to \tau M \to I^1 \to I^0$, with $I^0 := \oplus_{i \in R} I_i$, and $I^1 := \oplus_{j \in R'} I_j$. For k in R', let k^+, resp. k^-, be the source of Q' succeeding, resp. preceding, k, with the convention that $k^+ = m$, resp. $k^- = 1$, if there is no such source.

Then, for each sink k of Q'_0, the support of I_k is given by $\mathrm{Supp}(I_k) = S^-_{k^-} \cup S^+_{k^+} \cup (\mathrm{Supp}(I_k) \cap Q'_0)$. The support of I_0, resp. I_{m+1}, is $S_l^+ \cup (\mathrm{Supp}(I_0) \cap Q''_0)$, resp. $S_g^- \cup (\mathrm{Supp}(I_{m+1}) \cap Q''_0)$, where l, resp. g, is the lowest, resp. greatest, source of Q'_0. The support of I_k, $k \in R$ contains $S_k^{\pm}$.

This implies that the support of τM is a subset of Q''_0.

Now, let X be the middle term in the Auslander-Reiten sequence $0 \to \tau M \to X \to M \to 0$. We have $\mathrm{Supp}\, X \subset \mathrm{Supp}\, \tau M \cup \mathrm{Supp}\, M \subset Q''_0$. Hence, X is of type A. The Auslander-Reiten Theorem asserts that the module N of the Lemma is a direct summand of X. So, we obtain the Lemma in this case. The case where M is simple is very similar and left to the reader. ⬟

We can now prove the Theorem. Let M be an indecomposable Q-module of type A. By the Lemma, the component of M in the Auslander-Reiten quiver contains only modules of type A, therefore can only be finite. Hence, by [Gab80, Proposition 6.3], every indecomposable module is of type A and in particular is multiplicity free. So, there exists a one-to-one correspondence between indecomposable Q-modules and full subquivers of Q of type A. Let i be a leaf of Q and j be any vertex of Q. By §6.3.1, there exists a unique full subquiver of type A of Q whose extreme vertices are i and j. This implies by induction that the number of such subquivers is $\frac{n(n+1)}{2}$. Hence, there are $\frac{n(n+1)}{2}$ indecomposable Q-modules as required.

♠

Corollaire 6.4.5 *The category $\mathcal{C}_T$ is abelian.*

Corollaire 6.4.6 *There exists a bijection φ between $\mathrm{Ind}(Q_T)$ and the diagonals of the polygon not in T. Moreover, for M in $\mathrm{Ind}(Q_T)$ and any vertex i of Q_T, the multiplicity of the simple module S_i in the module M is 1 if $\varphi(M)$ crosses the i^{th} diagonal of T and 0 if not. In particular, for two isoclasses M, M' in $\mathrm{Ind}(Q_T)$, we have $M = M'$ if and only if $n_i(M) = n_i(M')$ for all i.*

6.5 The orbit category

This section is not used in the sequel. We give here a description of the category $\mathcal{C}_T$, using the equivalence of categories proved above. Then, we prove that the orbit category introduced by [BMR+06] has a nice geometric realization in the A_n case. Let r^+, resp. r^-, be the elementary rotation of the polygon in the positive, resp. negative, direction.

Théorème 6.5.1 *Let T be a triangulation of the $n+3$ polygon, and let $\mathcal{C}_T$ be the corresponding category, then :*
- (i) *The irreducible morphisms of $\mathcal{C}_T$ are direct sums of the generating morphisms given by pivoting elementary moves.*
- (ii) *The mesh relations of $\mathcal{C}_T$ are the mesh relations [ARS95] of the Auslander-Reiten quiver of $\mathcal{C}_T$.*
- (iii) *The Auslander-Reiten translate is given on diagonals by r^-.*
- (iv) *The projective indecomposable objects of $\mathcal{C}_T$ are diagonals in $r^+(T)$.*
- (v) *The injective indecomposable objects of $\mathcal{C}_T$ are diagonals in $r^-(T)$.*

Proof. (i) and then (ii) are clear by construction of the category $\mathcal{C}_T$. By (i) and (ii), extremal terms of an almost split sequence are given by the diagonals α and α' of Figure 2. This proves (iii). (iv) and (v) follow from (iii). ♠

The assertions (iv) and (v) of Theorem 6.5.1 suggest an interpretation of the diagonals of the triangulation T in terms of a category. Indeed, we will consider those diagonals, at least in the hereditary case, as shifts of the projectives in the derived category $\mathcal{D}\mathcal{C}_T$.

In order to simplify the construction, suppose that T is a triangulation corresponding to the unioriented quiver A_n with simple projective P_1. The category $\mathcal{C}_T \simeq \mathrm{Mod}\, Q_T$ is hereditary, so the indecomposable objects of the derived category $\mathcal{D}\,\mathrm{Mod}\, Q_T$ are the shifts $M[m]$, $m \in \mathbb{Z}$, of the indecomposables M of $\mathrm{Mod}\, Q_T$. Let F be the functor of $\mathcal{D}\,\mathrm{Mod}\, Q_T$ given by $M \mapsto \tau^{-1}M[1]$, where τ is the Auslander-Reiten translate in the derived category. We define the orbit category $\overline{\mathcal{D}\,\mathrm{Mod}\, Q_T}$ whose objects are objects of $\mathcal{D}\,\mathrm{Mod}\, Q_T$ and morphisms are given by $\mathrm{Hom}_{\overline{\mathcal{D}\,\mathrm{Mod}\, Q_T}}(M, N) := \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}\,\mathrm{Mod}\, Q_T}(M, F^i(N))$, M, $N \in \mathcal{D}\,\mathrm{Mod}\, Q_T$. The set $\mathrm{Ind}(Q_T) \cup \{P_i[1],\ 1 \le i \le n\}$ is the set of indecomposable objects of $\overline{\mathcal{D}\,\mathrm{Mod}\, Q_T}$ up to isomorphism. Note that the category $\overline{\mathcal{D}\,\mathrm{Mod}\, Q_T}$ is not abelian in general. By a result of Bernhard Keller [Kel05], it is a triangulated category.

We can also construct the total category $\mathcal{C}$ generated by all the diagonals of the $(n+3)$ polygon. The construction is analogue to the construction of $\mathcal{C}_T$: indecomposable objects are positive roots and simple negative ones. The homomorphisms and the mesh relations are defined as in §6.2.2 without the point (ii) in the convention made there.

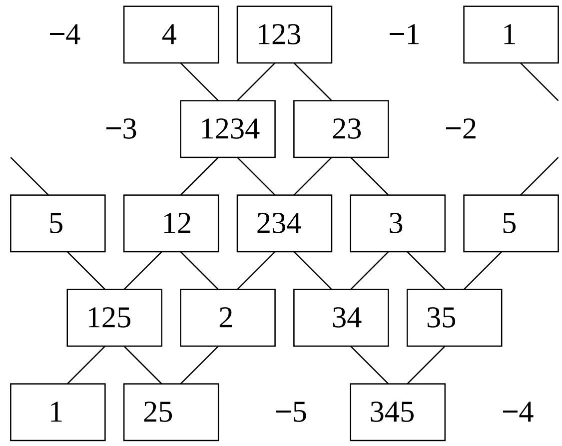

FIG. 6.6 – Example of Auslander-Reiten Quiver

Théorème 6.5.2 *Let T be a triangulation corresponding to an orientation of the quiver A_n. Then, the categories $\mathcal{C}$ and $\overline{\mathcal{D}\,\mathrm{Mod}\,Q_T}$ are equivalent.*

We give here a sketch of the proof. The derived category of representations of the an oriented quiver of type A is well-known. In particular, it does not depend on the orientation of the quiver. Hence, we can suppose that T correponds to the unioriented quiver as above. The indecomposable objects of $\mathcal{D}\,\mathrm{Mod}\,Q_T$ can be indexed by $\kappa : \mathbb{Z} \times \{1, \ldots, n\} \to \mathrm{Ind}\,\mathcal{D}\,\mathrm{Mod}\,Q_T$, by the rule $\kappa(1, i) = P_i$, $\kappa(i+1, j) = \tau^{-1}(\kappa(i, j))$, $\kappa(i+j, n+1-j) = \kappa(i, j)[1]$. This implies that the indecomposable objects of $\overline{\mathcal{D}\,\mathrm{Mod}\,Q_T}$ can be indexed by $\mathbb{Z} \times \{1, \ldots, n\}/ (i, j) \equiv (i+j+1, n-j+1)$.

For all (i, j) in $\mathbb{Z}\times\{1, \ldots, n\}$, we define the quadrilateral $R_{(i,j)}$ by its vertices (i, j), (i, n), $(i+j-1, 1)$, $(i+j-1, n-j+1)$. Let M be the indecomposable object in $\overline{\mathcal{D}\,\mathrm{Mod}\,Q_T}$ indexed by (i, j), and let N be any indecomposable object. Then, $\mathrm{Hom}_{\overline{\mathcal{D}\,\mathrm{Mod}\,Q_T}}(M, N) \neq 0$ if and only if N is indexed by a point inside $R_{(i,j)}$. In this case, it is $\mathbb{C}$ as a space and the composition of morphisms is given by the multiplication. Now, let us index the vertices of the $(n+3)$ polygon by the group $\mathbb{Z}/(n+3)\mathbb{Z}$ and let $[i, j]$ be the diagonal from i to j, $j - i \neq 1, 0, -1$. By the description above, the additive functor defined by $[i, j] \mapsto \kappa(i, j-i-1)$ gives an equivalence of categories.

Remarque 12 *Using the equivalence above, it is easy to see that given two diagonals α and α', the group $\mathrm{Ext}^1(\alpha, \alpha')$ is non zero if and only if α and α' cross. Hence, a triangulation of the polygon correspond to a maximal set of pairwise extension free diagonals.*

Remarque 13 *The orbit category $\overline{\mathcal{D}\,\mathrm{Mod}\,Q_T}$ was introduced in [BMR$^+$06] for all simply-laced root systems. Its construction was given to us by Bernhard Keller.*

An example of the Auslander-Reiten quiver is provided in Figure 6.6, for the Quiver with relation shown in the right part of Figure 6.3.

6.6 Denominators of Laurent polynomials

In this section, we will prove Conjecture 6.1.1 for cluster algebras of type A. The existence of the bijection will be an easy consequence of section 6.4 and older results of Fomin and Zelevinsky. Most of this section is concerned with the calculation of the exponents in the denominators of the Laurent polynomials, in order to prove the equation of the Conjecture.

Throughout this section we will use the following setup. Let $\Phi_{\geq -1}$ be the set of almost positive roots and τ_+ and τ_- the involutions on $\Phi_{\geq -1}$ defined by Fomin and Zelevinsky in [FZ03b]. For any $\alpha \in \Phi_{\geq -1}$, let w_α be the cluster variable corresponding to α by the bijection of Fomin and Zelevinsky [FZ03a, Theorem 1.9]. Let $C = \{u_1, \ldots, u_n\}$ be a cluster and let $\beta_1, \ldots, \beta_n$ be the almost positive roots such that $w_{\beta_i} = u_i$. Recall the following properties of $\tau_\pm$:

Proposition 6.6.1 1. *Every $\langle \tau_-, \tau_+ \rangle$–orbit in $\Phi_{\geq -1}$ contains a negative simple root.*

2. *There is a unique function $(\,||\,) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ such that*
   (i) $(-\alpha_i \,||\, \beta) = \max(n_i(\beta), 0)$,
   (ii) $(\tau_\pm \alpha \,||\, \tau_\pm \beta) = (\alpha \,||\, \beta)$.
   Furthermore $(\,||\,)$ is symmetric for simply-laced root systems.
3. *The set $\tau_\pm(C) = \{w_{\tau_\pm(\beta_i)} \mid i = 1, \ldots, n\}$ is a cluster.*

Proof. 1. is shown in [FZ03b, Theorem 2.6], 2. in [FZ03b, section 3.1] and 3. in [FZ03b, Proposition 3.5]. ⬟

By the Laurent phenomenon [FZ02], we can write, for any almost positive root α,

$$w_\alpha = \frac{R_{\alpha,C}}{\prod_{i=1}^n u_i^{[\alpha,\beta_i,C]}}, \tag{6.8}$$

where $R_{\alpha,C}$ is a polynomial in the variables $u_1, \ldots, u_n$ such that none of the u_i divides $R_{\alpha,C}$, and $[\alpha, \beta_i, C] \in \mathbb{Z}$. The following Lemma is crucial.

Lemme 6.6.2 *For any pair of almost positive roots α, β_i and any pair of clusters C, C' such that $u_i = w_{\beta_i} \in C \cap C'$, we have*

$$[\alpha, \beta_i, C] = [\alpha, \beta_i, C'].$$

Proof. It is sufficient to prove assertions (a) and (b) below.

(a) All clusters containing the given cluster variable u_i are connected in the mutation graph.

(b) In mutations which do not exchange u_i, the exponent of u_i in the denominator of w_α is unchanged.

Assertion (a) can either be seen as a classical statement on the link of a simplex in a simplicial sphere, or can be checked directly using the recursive properties of clusters. First the adjacency graph of clusters containing a fixed cluster variable is mapped to an isomorphic graph by the action of τ_+ and τ_- because this action respects the compatibility function. Thus one can suppose that the fixed cluster variable corresponds to a negative simple root. But then the graph of clusters containing this negative simple root is isomorphic to a product of whole cluster adjacency graphs for smaller sub-root systems. Therefore it is connected as a product of connected graphs.

Let us show (b) now. Consider the mutation $C \to C'$ that exchanges the cluster variables $u_j \in C$ and $u'_j \in C'$. The exchange relation gives $u_j = \frac{M_1+M_2}{u'_j}$, where M_1 and M_2 are monomials without common divisors in the variables $C \setminus \{u_j\}$. From equation (6.8), we obtain by substitution

$$w_\alpha = \frac{R_{\alpha,C}(u_1, \dots, u_{j-1}, \frac{M_1+M_2}{u'_j}, u_{j+1}, \dots, u_n)}{\prod_{l \neq j} u_l^{[\alpha,\beta_l,C]} (\frac{M_1+M_2}{u'_j})^{[\alpha,\beta_j,C]}}.$$

By the Laurent phenomenon, we know that w_α is a Laurent polynomial in the cluster variables $u_1, \dots, u'_j, \dots, u_n$. We want to prove that the exponent of u_i in the denominator is still $[\alpha, \beta_i, C]$. Clearly, by properties of M_1 and M_2, this is true if and only if the Laurent polynomial $R_{\alpha,C}(u_1, \dots, \frac{M_1+M_2}{u'_j}, \dots, u_n)$ is not zero after evaluation at $u_i = 0$. To conclude, remark that, by properties of the monomials M_1 and M_2 stated above, the value of this Laurent polynomial at $u_i = 0$ is obtained by an invertible substitution from the value of $R_{\alpha,C}$ at $u_i = 0$, which is known not to be zero. ⬟

Thus $[\alpha, \beta, C]$ will be denoted simply $[\alpha, \beta]$ from now on.

The following Lemma is proved for all simply-laced root-systems.

Lemme 6.6.3 *Let α,β be two almost positive roots. Then*

$$[\alpha, \beta] = [\tau_\pm \alpha, \tau_\pm \beta]. \tag{6.9}$$

Proof.

Let us consider a sequence of adjacent clusters

$$C_0 \leftrightarrow C_1 \leftrightarrow \cdots \leftrightarrow C_N, \tag{6.10}$$

where $\alpha \in C_0$ and $\beta \in C_N$. The exchange relations depend only on the matrices $B(C_0), \dots, B(C_N)$ associated to these clusters. As the action of τ respects the compatibility relation, one gets another chain of adjacent clusters

$$\tau(C_0) \leftrightarrow \tau(C_1) \leftrightarrow \cdots \leftrightarrow \tau(C_N), \tag{6.11}$$

where $\tau(\alpha) \in \tau(C_0)$ and $\tau(\beta) \in \tau(C_N)$. By Lemma 4.8 in [FZ03a], one has

$$B_{\tau\gamma',\tau\gamma''}(\tau(C)) = -B_{\gamma',\gamma''}(C), \tag{6.12}$$

for any cluster C and roots γ', γ'' in it. This minus sign does not change the exchange relations. From this one deduces that the expression of the cluster variable u_α in the variables of the cluster C_N is the same as the expression of the cluster variable $u_{\tau(\alpha)}$ in the variables of the cluster $\tau(C_N)$. This proves the Lemma. ⬟

Lemme 6.6.4 *Let $-\alpha_i$ be a simple negative root and α an almost positive root. Then*

$$[\alpha, -\alpha_i] = n_i(\alpha).$$

Proof. The quantity $[\alpha, -\alpha_i]$ is computed using the expression of the cluster variables α in the cluster made of negative roots. Then the bijection of Fomin and Zelevinsky between cluster variables and roots ([FZ03a, Theorem 1.9]) implies that this is $(\alpha \mid\mid -\alpha_i)$. By Proposition 6.6.1.2.(i) and symmetry of $(\mid\mid)$ in the simply-laced cases, the conclusion follows. ⬟

Proposition 6.6.5 *Let α,β be two distinct almost positive roots. Then*

$$[\alpha, \beta] = (\alpha \mid\mid \beta).$$

Proof. Define a function $b : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ by

$$b(\alpha, \beta) = \begin{cases} [\alpha, \beta] & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

This function is well defined by Lemma 6.6.2. Moreover

(1) $b(-\alpha_i, \beta) = \max(n_i(\beta), 0)$ (by Lemma 6.6.4)

(2) $b(\tau_{\pm}\alpha, \tau_{\pm}\beta) = b(\alpha, \beta)$ (by Lemma 6.6.3).

By Proposition 6.6.1(2), the function $(\mid\mid) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ is the unique function having the properties (1) and (2), thus $b(\alpha, \beta) = (\alpha \mid\mid \beta)$. Therefore $[\alpha, \beta] = (\alpha \mid\mid \beta)$ if $\alpha \neq \beta$. ⬟

The following Theorem establishes the Conjecture 6.1.1 for the type A_n.

Théorème 6.6.6 *Let $C = \{u_1, \ldots, u_n\}$ be a cluster of a cluster algebra of type A_n and let V be the set of all cluster variables of the algebra. Let Q_C be the quiver with relations associated to C and $\mathrm{Ind}(Q_C)$ the set of isoclasses of indecomposable modules. Then there is a bijection*

$$\mathrm{Ind}(Q_C) \to V \setminus C, \ \alpha \mapsto w_\alpha,$$

such that

$$w_\alpha = \frac{P(u_1, \ldots, u_n)}{\prod_{i=1}^n u_i^{n_i(\alpha)}},$$

where P is a polynomial such that none of the u_i divides P $(i = 1, \ldots, n)$ and $n_i(\alpha)$ is the multiplicity of the simple module α_i in the module α.

Proof. Let $T_C = \{\beta_1, \ldots, \beta_n\}$ be the triangulation of the $(n + 3)$ polygon corresponding to the cluster C and let D be the set of diagonals of the polygon ; thus $T_C \subset D$. Let $T_0 = \{-\alpha_1, \ldots, -\alpha_n\}$ be the "snake triangulation" [FZ03a, 12.2]. Q_{T_0} is the alternating quiver of type A_n and the diagonals $-\alpha_i \in T_0$ are the negative simple roots. Fomin and Zelevinsky have shown that there is a bijection $\alpha \mapsto w_\alpha$ between the set of almost positive roots $\Phi_{\geq -1}$ and the set of cluster variables V. In type A, they identified $\Phi_{\geq -1}$ with D and proved that for any cluster C there is a bijection $\alpha \mapsto w_\alpha$ between $D \setminus T_C$ and $V \setminus C$. In section 6.4, we have shown the bijection $M_\alpha \mapsto \alpha$ between $\mathrm{Ind}(Q_C)$ and $D \setminus T_C$. This establishes a bijection $\mathrm{Ind}(Q_C) \to V \setminus C$. This bijection sends the simple module in $\mathrm{Ind}(Q_C)$ at the vertex $i \in Q_C$ to the variable w_β where β is the unique diagonal in $D \setminus T_C$ that crosses β_i and does not cross any diagonal in $T_C \setminus \{\beta_i\}$

Let M_α be an element of $\mathrm{Ind}(Q_C)$ with α the corresponding diagonal in $D \setminus T_C$. By Lemma 6.6.2, we have $w_\alpha = \frac{P(u_1,\ldots,u_n)}{\prod_{i=1}^n u_i^{[\alpha,\beta_i]}}$. We have to show that $[\alpha, \beta_i] = n_i(\alpha)$ for all $i = 1, \ldots, n$. Note that $\alpha \neq \beta_i$ since $\alpha \notin T_C$, hence using Proposition 6.6.5, we get $[\alpha, \beta_i] = (\alpha \mid\mid \beta_i)$ and by [FZ03a, 12.2] this is equal to 1 if the diagonals α and β_i are crossing, and zero otherwise. Thus

$$[\alpha, \beta_i] = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathrm{Supp}\, \alpha \text{ in the sense of section 6.4} \\ 0 & \text{otherwise.} \end{array} \right\} = n_i(\alpha).$$

⬟

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Chapitre 7

Cohomology rings of toric varieties assigned to cluster quivers : the case of equioriented quivers of type A

The theory of cluster algebras of S. Fomin and A. Zelevinsky has assigned a fan to each Dynkin diagram. Then A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov have generalized this construction using arbitrary quivers on Dynkin diagrams. In the special case of the equioriented quiver of type A, we describe the cohomology ring of the toric variety associated to this fan. A natural basis is obtained and an explicit rule is given for the product of any two generators.

7.0 Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [6, 7, 8] for studying dual canonical bases in quantum groups and total positivity in Lie groups. One important result of this theory is the classification theorem of cluster algebras of finite type by the Killing-Cartan list of root systems. Part of the proof consists of building a cluster algebra of finite type starting from a given root system. In this construction, a simplicial fan is associated with each finite root system, hence also a smooth toric variety. It was proved in [4] that these toric varieties are projective.

Later, in the case of simply-laced Dynkin diagrams, this fan was seen to be a special case of a construction starting from any quiver on the given Dynkin diagram [2]. The fan studied before corresponds in this setting to the alternating quiver. It would be worth proving that all the associated toric varieties of these quiver fans, known to be smooth, are also projective.

In fact, this may be only the tip of something. There should be a systematic way to define a fan starting from any cluster in a cluster algebra of finite type, in such a way that the associated toric variety is smooth and projective. This construction should of course recover the preceding one, in case the chosen cluster is associated with a quiver.

Moreover, based on experimental evidence, these fans should have the following properties. First, they should not only be simplicial but even smooth, meaning that each cone is spanned by an integral basis. Then the cone of ample divisors in the second integer cohomology group of the toric variety should be smooth in the same sense. If true, this would provide a natural basis of this cohomology group and therefore a natural set of generators of the cohomology ring. Then there should be a quadratic presentation of the cohomology ring and a basis of the cohomology ring consisting of monomials in the distinguished generators.

All these properties have been checked for low-dimensional alternating quivers of type A. The aim of the present article is to prove part of these statements in the case of equioriented quivers of type A.

More precisely, we obtain a basis and a quadratic presentation of the cohomology ring. The distinguished generators should be the extremal vectors of the cone of ample divisors, but we do not prove that here. To say the truth, this was however the way we guessed them by looking at low-dimensional cases.

Let us remark that the rings studied here have some obvious similarity with some rings related to the hyperplane arrangement of a root system and to nonnesting partitions, which were considered in [3]. We do not known what should be the meaning of this resemblance.

7.1 A toric variety associated to a Dynkin quiver

Let us fix an integer n once and for all and denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

For each quiver of Dynkin type, a fan has been defined on the set of almost positive roots in [2]. A similar construction is expected to hold starting from any seed in any cluster algebra of finite type.

Let us recall the construction of [2] in the case of the equioriented quiver of type A_n. So let Q_n be the quiver with n vertices and arrows from i to $i+1$ for $1 \leq i < n$.

By a simple instance of a famous Theorem of Gabriel, indecomposable modules over Q_n are in bijection with positive roots in the A_n root system. Recall that these positive roots are indexed by the intervals $[i, j]$ in the set $[n]$. Here, the indecomposable module associated to $[i, j]$ is given by the space $\mathbb{C}$ on each vertex k between i and j, the null space elsewhere and identity maps when possible.

Let us introduce some notation. Let $\Phi_{>0}$ be the set of positive roots, *i.e.* the set of intervals in $[n]$. The intervals $[i, i]$ are called simple roots and the set of simple roots is denoted by Π. Let $\Phi_{>1}$ be the set of non-simple roots. Let $\Phi_{\geq -1}$ be the disjoint union of $\Phi_{>0}$ with a copy of Π denoted by $-\Pi$. The elements of $\Phi_{\geq -1}$ are called almost positive roots and the elements of $-\Pi$ are called negative simple roots. In the sequel, we will denote by Greek letters the roots *i.e.* elements of $\Phi_{>0}$ and by Latin letters (corresponding to elements of $[n]$) the simple roots or their opposite.

One says that $i \in \alpha$ if $\alpha = [j,k]$ and $j \leq i \leq k$.

Let us say that two roots $\alpha = [i,j]$ and $\beta = [k,\ell]$ in $\Phi_{>0}$ *overlap* if one has $i \leq k \leq j \leq \ell$ or $k \leq i \leq \ell \leq j$. Let us say that they *overlap strictly* if they overlap and neither $\alpha \subseteq \beta$ nor $\beta \subseteq \alpha$.

To define the fan on the set of vectors $\Phi_{\geq -1}$, one needs a symmetric binary relation on the set $\Phi_{\geq -1}$, called the compatibility relation. The general definition is given in term of Ext-groups in a triangulated category called the cluster category, which is defined as a quotient of the derived category of modules on the chosen quiver. We will just state the result for the quiver Q_n.

An element $-i$ of $-\Pi$ is compatible with any other element $-j$ of $-\Pi$.

An element $-i$ of $-\Pi$ is compatible $\alpha \in \Phi_{>0}$ if and only if $i \notin \alpha$.

Two elements α and β in $\Phi_{>0}$ are not compatible if and only if

(i) either $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta \in \Phi_{>0}$ (adjacent roots),

(ii) or $\alpha \not\subseteq \beta$, $\beta \not\subseteq \alpha$ and $\alpha \cap \beta \neq \emptyset$ (strictly overlapping roots).

Here comes the description of the fan $\Sigma(Q_n)$. First we map the set $\Phi_{\geq -1}$ into the free abelian group generated by variables $\{\alpha_1, \ldots, \alpha_n\}$ by

$$\begin{cases} -i \mapsto -\alpha_i, \\ \alpha \mapsto \sum_{i \in \alpha} \alpha_i. \end{cases} \tag{7.1}$$

Then a subset of (the image of) $\Phi_{\geq -1}$ spans a cone of $\Sigma(Q_n)$ if and only if it is made of pairwise compatible elements.

It is known that the number of maximal cones of $\Sigma(Q_n)$ is the number of clusters of type A_n, which is the Catalan number

$$c_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}. \tag{7.2}$$

Let us define $\max \alpha$ and $\min \alpha$ for $\alpha = [i,j] \in \Phi_{>0}$ to be i and j respectively. Let us define $\mathscr{R}\alpha$ and $\mathscr{L}\alpha$ for $\alpha = [i,j] \in \Phi_{>1}$ to be $\alpha \setminus \min \alpha$ and $\alpha \setminus \max \alpha$ respectively.

For $\ell \in [n]$ and $\alpha = [i,j] \in \Phi_{>1}$ with $\ell \in \mathscr{L}\alpha$, we define ℓ/α to be the root $[\ell, j]$ in $\Phi_{>1}$ obtained by cutting the left-hand side of α. Similarly, if $\ell \in \mathscr{R}\alpha$, let α/ℓ be the root $[i,\ell]$ in $\Phi_{>1}$ defined by cutting the right-hand side of α.

7.2 Standard presentation of the cohomology

Let Σ be a smooth complete fan. Then there exists a standard description of the integer cohomology ring of the smooth toric variety X_Σ, see [5] and [9, §5.2]. Let us recall briefly this construction.

The cohomology ring $\mathbf{H}^*(X_\Sigma)$ is generated by variables T_u indexed by the set of 1-dimensional cones in the fan Σ. Then there are linear and quadratic relations between these generators. The linear relations are

$$\sum_u \langle v, w_u \rangle \mathsf{T}_u = 0, \tag{7.3}$$

where v runs through a basis of the dual lattice and w_u is the unique integral generating vector for the cone u. The quadratic relations are the following : the product $\mathsf{T}_u\mathsf{T}_v$ vanishes as soon as there is no cone σ in Σ containing both u and v.

It is also known that $\mathbf{H}^*(X_\Sigma)$ is a free abelian group of rank the number of maximal cones of Σ [5, Th. 10.8].

For the fans $\Sigma(Q_n)$ we are interested in, this amounts to the following description.

Proposition 7.2.1 *The cohomology ring $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is presented by the generators T_{-i} for $i \in [n]$ and T_α for $\alpha \in \Phi_{>0}$, the linear relations*

$$\mathsf{T}_{-i} = \sum_{i \in \alpha \in \Phi_{>0}} \mathsf{T}_\alpha \quad \textit{for } i \in [n], \tag{7.4}$$

and the quadratic relations

$$\mathsf{T}_{-i}\mathsf{T}_\alpha = 0 \quad \textit{when} \quad i \in \alpha \in \Phi_{>0}, \tag{7.5}$$

and

$$\mathsf{T}_\alpha\mathsf{T}_\beta = 0 \tag{7.6}$$

when

(i) either $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta \in \Phi_{>0}$ (adjacent roots),

(ii) or $\alpha \not\subseteq \beta$, $\beta \not\subseteq \alpha$ and $\alpha \cap \beta \neq \emptyset$ (strictly overlapping roots).

The rank of the free abelian group $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is the Catalan number c_{n+1}.

When $\alpha = [i,j] \in \Phi_{>1}$, we will sometimes denote T_α by $\mathsf{T}_{i,j}$.

One can rewrite the quadratic relations involving the variables T_i by using the linear relations (7.4) to eliminate these variables.

The relation $\mathsf{T}_i\mathsf{T}_{i+1} = 0$ becomes

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} - \sum_{i+1<k'} \mathsf{T}_{-i}\mathsf{T}_{i+1,k'} - \sum_{j<i} \mathsf{T}_{j,i}\mathsf{T}_{-i-1} + \sum_{\substack{j \leq i \leq k;\, j<k \\ j' \leq i+1 \leq k';\, j'<k' \\ \text{inclusion}}} \mathsf{T}_{j,k}\mathsf{T}_{j',k'} = 0, \tag{7.7}$$

where "inclusion" means that either $[j,k]$ is contained in $[j',k']$ or vice-versa.

The relation $\mathsf{T}_i\mathsf{T}_\alpha = 0$, with $\alpha \in \Phi_{>0}$, $i \notin \alpha$ and α containing either $i+1$ or $i-1$, becomes

$$\mathsf{T}_{-i}\mathsf{T}_\alpha = \sum_{i \in \beta\,;\, \alpha \subseteq \beta} \mathsf{T}_\alpha\mathsf{T}_\beta. \tag{7.8}$$

Lemme 7.2.2 *The square of T_{-i} is zero for all $i \in [n]$.*

Proof. One has

$$\mathsf{T}_{-i}^2 = \mathsf{T}_{-i}\left(\sum_{i \in \alpha \in \Phi_{>0}} \mathsf{T}_\alpha\right), \tag{7.9}$$

which vanishes by relations (7.5). ⬟

7.3 A ring with a quadratic presentation

7.3.1 Presentation

Let us introduce a ring $\mathbf{M}^*(n)$. Our aim will be to show that this ring is isomorphic to $\mathbf{H}^*(X_{\Sigma(Q_n)})$.

The ring $\mathbf{M}^*(n)$ is the commutative ring generated by variables S_i for $i \in [n]$ and S_α for $\alpha \in \Phi_{>1}$, modulo the following relations :

$$\mathsf{S}_i^2 = 0, \qquad \mathsf{S}_i \mathsf{S}_\alpha = \sum_{j \in \alpha, j \neq i} \mathsf{S}_i \mathsf{S}_j \quad \text{when } i \in \alpha, \tag{7.10}$$

and

$$\mathsf{S}_\alpha \mathsf{S}_\beta = \sum_{i<j \in \alpha\cap\beta} \mathsf{S}_i \mathsf{S}_j + \sum_{\ell \in \mathscr{R}\alpha \cap \mathscr{L}\beta} \mathsf{S}_{\alpha/\ell} \mathsf{S}_{\ell/\beta} - \sum_{\ell,\ell+1 \in \mathscr{R}\alpha \cap \mathscr{L}\beta} \mathsf{S}_{\alpha/\ell} \mathsf{S}_{\ell+1/\beta}, \tag{7.11}$$

whenever α and β overlap with $\alpha \cap \beta$ of cardinal at least 2 (one assumes that α is on the left of β).

Remark : the ring $\mathbf{M}^*(n)$ is obviously graded with generators of degree one.

When $\alpha = [i, j] \in \Phi_{>1}$, we will sometimes denote S_α by $\mathsf{S}_{i,j}$.

7.3.2 Combinatorial preliminaries : codes and U-sets

A *code* is a word C of length n in the alphabet $\{L, R, LR, V\}$ such that
– It contains as many letters L as letters R.
– Any left prefix contains at least as many letters L as letters R.

Note that L is for "links", R for "rechts" and V for "vakuum".

The *degree* $\deg(C)$ of a code C is the number of symbols L seen in the word, *i.e.* the number of letters L plus the number of letters LR. There is a natural duality operation $C \mapsto C^*$ on codes given by the replacement of all occurrences of LR by V and vice-versa. This involution maps a code of degree k to a code of degree $n-k$. Hence there is a unique code of length n and degree n, made of n letters LR.

It should be a simple combinatorial exercise for the reader to check that the number of codes of length n is the Catalan number c_{n+1}.

A U-set is a subset u of $[n] \sqcup \Phi_{>1}$ such that
(i) If $i \in u$ and $\alpha \in u$, then $i \notin \alpha$.
(ii) If α and β in u are overlapping, then $\alpha \cap \beta$ is a singleton.

Then U-sets are in bijection with codes as follows. A U-set u is mapped to the code C obtained by writing a L at position i for each non-simple root α starting at i in u, a R at position i for each non-simple root α ending at i in u, a LR at position i for each i in u and then filling the word with V. Note that the letter LR can either be obtained directly as such or as the successive writing of L and R at the same place.

For example, the U-set $\{[1], [3, 4], [4, 6], [6, 7]\}$ is mapped to the code

$$(LR)(V)(L)(LR)(V)(LR)(R). \tag{7.12}$$

The reverse bijection from codes to U-sets is easy and left to the reader.

By this correspondence between codes and U-sets, the degree of a code is mapped to the cardinality of the associated U-set. There is an induced duality on U-sets which will be used later.

7.3.3 Spanning set

We want to show that there is a spanning set of $\mathbf{M}^*(n)$ indexed by U-sets. First for each U-set u, one can define a monomial S^u in $\mathbf{M}^*(n)$ as the product of variables S_i and S_α over the elements of u.

Let the height of any monomial in $\mathbf{M}^*(n)$ be the sum of the height of its variables, where the generator S_i has height 1 and S_α has height $\#\alpha$.

Lemme 7.3.1 *The ring $\mathbf{M}^*(n)$ is spanned by the monomials S^u, where u runs over the set of U-sets.*

Proof. Using the defining relations (7.10) and (7.11) of $\mathbf{M}^*(n)$, one can replace any monomial not of the form S^u for some U-set u by a linear combination of monomials of strictly smaller height. The Lemma follows by induction on height. ⬟

7.4 Isomorphism and consequences

7.4.1 Isomorphism

Let us now describe a map ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$ and prove that it is an isomorphism.

Define ψ on the generators of $\mathbf{M}^*(n)$ by

$$\psi(\mathsf{S}_i) = \mathsf{T}_{-i} \quad \text{for} \quad i \in [n], \tag{7.13}$$

and

$$\psi(\mathsf{S}_\alpha) = \sum_{i\in\alpha} \mathsf{T}_{-i} - \sum_{\alpha\subseteq\beta} \mathsf{T}_\beta \quad \text{for} \quad \alpha \in \Phi_{>1}. \tag{7.14}$$

Proposition 7.4.1 *Formulas (7.13) and (7.14) define a morphism of rings ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$.*

Proof. Let us first check that relations (7.10) hold. By (7.13), one has

$$\psi(S_i^2) = \mathsf{T}_{-i}^2, \tag{7.15}$$

which vanishes by Lemma 7.2.2. One also has

$$\psi(\mathsf{S}_i\mathsf{S}_\alpha) = \mathsf{T}_{-i}\left(\sum_{j\in\alpha} \mathsf{T}_{-j} - \sum_{\alpha\subseteq\alpha'} \mathsf{T}_{\alpha'}\right). \tag{7.16}$$

By relations (7.5), this becomes, as expected,

$$\mathsf{T}_{-i} \sum_{j\in\alpha, j\neq i} \mathsf{T}_{-j} = \psi\left(\sum_{j\in\alpha, j\neq i} \mathsf{S}_i\mathsf{S}_j\right). \tag{7.17}$$

Let us now check that relations (7.11) hold. It is necessary to distinguish two cases.

First consider the case when $\mathscr{R}\alpha \cap \mathscr{L}\beta$ is empty. One can show that this implies that α and β are the same $[i, i+1]$ for some i. One has to check the vanishing of the image by ψ of

$$\mathsf{S}_{i,i+1}^2 - \mathsf{S}_i \mathsf{S}_{i+1}. \tag{7.18}$$

This is given by

$$\left(\mathsf{T}_{-i} + \mathsf{T}_{-i-1} - \sum_{j \le i < i+1 \le k} \mathsf{T}_{j,k} \right)^2 - \mathsf{T}_{-i}\mathsf{T}_{-i-1}. \tag{7.19}$$

By relations (7.5), this is

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} + \left(\sum_{j \le i < i+1 \le k} \mathsf{T}_{j,k} \right)^2. \tag{7.20}$$

Using relations (7.6), this becomes

$$\mathsf{T}_{-i}\mathsf{T}_{-i-1} + \sum_{\substack{j \le i < i+1 \le k \\ j' \le i < i+1 \le k' \\ \text{inclusion}}} \mathsf{T}_{j,k}\mathsf{T}_{j',k'}. \tag{7.21}$$

where “inclusion” means that either $[j,k]$ is contained in $[j',k']$ or vice-versa. Then using relation (7.7) to eliminate $\mathsf{T}_{-i}\mathsf{T}_{-i-1}$, one gets

$$\sum_{i+1<k'} \mathsf{T}_{-i}\mathsf{T}_{i+1,k'} + \sum_{j<i} \mathsf{T}_{j,i}\mathsf{T}_{-i-1} - \sum_{\substack{j' \le i \le i+1 \le k' \\ j' \le j < i \le k'}} \mathsf{T}_{j,i}\mathsf{T}_{j',k'} - \sum_{\substack{j \le i \le k \\ j \le i+1 < k' \le k}} \mathsf{T}_{i+1,k'}\mathsf{T}_{j,k}. \tag{7.22}$$

Then using relations (7.8), the first and fourth term annihilate as do the second and third term.

Let us now consider the case when $\mathscr{R}\alpha \cap \mathscr{L}\beta$ is not empty. We have to prove the vanishing of the image by ψ of

$$\mathsf{S}_\alpha \mathsf{S}_\beta - \sum_{i<j \in \alpha\cap\beta} \mathsf{S}_i\mathsf{S}_j - \sum_{\ell \in \mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell/\beta} + \sum_{\ell,\ell+1 \in \mathscr{R}\alpha\cap\mathscr{L}\beta} \mathsf{S}_{\alpha/\ell}\mathsf{S}_{\ell+1/\beta}. \tag{7.23}$$

This is

$$\begin{aligned} &\left(\sum_{i\in\alpha} \mathsf{T}_{-i} - \sum_{\alpha\subseteq\alpha'} \mathsf{T}_{\alpha'} \right) \left(\sum_{j\in\beta} \mathsf{T}_{-j} - \sum_{\beta\subseteq\beta'} \mathsf{T}_{\beta'} \right) - \sum_{i<j\in\alpha\cap\beta} \mathsf{T}_{-i}\mathsf{T}_{-j} \\ &\quad - \sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \left(\sum_{\substack{i\in\alpha \\ i\le\ell}} \mathsf{T}_{-i} - \sum_{\alpha/\ell\subseteq\alpha'} \mathsf{T}_{\alpha'} \right) \left(\sum_{\substack{j\in\beta \\ j\ge\ell}} \mathsf{T}_{-j} - \sum_{\ell/\beta\subseteq\beta'} \mathsf{T}_{\beta'} \right) \\ &+ \sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta} \left(\sum_{\substack{i\in\alpha \\ i\le\ell}} \mathsf{T}_{-i} - \sum_{\alpha/\ell\subseteq\alpha'} \mathsf{T}_{\alpha'} \right) \left(\sum_{\substack{j\in\beta \\ j\ge\ell+1}} \mathsf{T}_{-j} - \sum_{\ell+1/\beta\subseteq\beta'} \mathsf{T}_{\beta'} \right). \end{aligned} \tag{7.24}$$

In this sum, consider first the terms of the shape $\mathsf{T}_{-\star}\mathsf{T}_{-*}$. Let us prove that their sum vanishes. First, using the fact that α and β overlap with α on the left, and reversing summations, one gets

$$\sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}} \mathsf{T}_{-i}\mathsf{T}_{-j} - \sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}} \sum_{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]} \mathsf{T}_{-i}\mathsf{T}_{-j} + \sum_{\substack{i\in\alpha\\ j\in\beta\\ i<j}} \sum_{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]} \mathsf{T}_{-i}\mathsf{T}_{-j}. \tag{7.25}$$

Then it is enough to show that $\mathscr{R}\alpha\cap\mathscr{L}\beta\cap[i,j]$ is not empty. This is clear if $i+1<j$, as any $i<k<j$ will do the job. Then if $j=i+1$, the intersection can be empty only if $\mathscr{R}\alpha\cap\mathscr{L}\beta$ is already empty, which is excluded by hypothesis.

Then consider the terms of the shape $\mathsf{T}_{-\star}\mathsf{T}_{\alpha'}$ in (7.24). Using the left-right symmetry of the situation, let us compute only the terms of the shape $\mathsf{T}_{-j}\mathsf{T}_{\alpha'}$ where $\alpha\subseteq\alpha'$ and $j\in\beta$. After reversal of summations, this sum is

$$-\sum_{\substack{j\in\beta\\ \alpha\subseteq\alpha'\\ j\notin\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'} + \sum_{\substack{j\in\beta\\ \min\alpha\in\alpha'\\ j\notin\alpha'}} \sum_{\substack{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'} - \sum_{\substack{j\in\beta\\ \min\alpha\in\alpha'\\ j\notin\alpha'}} \sum_{\substack{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'}} \mathsf{T}_{-j}\mathsf{T}_{\alpha'}. \tag{7.26}$$

The sum of the last two terms under the additional assumption that $\alpha\subseteq\alpha'$ annihilates with the first term. Here we used that $\mathscr{R}\alpha\cap\mathscr{L}\beta$ is not empty. Let us therefore assume that $\alpha\nsubseteq\alpha'$ in the two right terms. This means that $\max\alpha\notin\alpha'$. Then both terms vanish unless α' meets $\mathscr{R}\alpha\cap\mathscr{L}\beta$. In this case, the sum vanishes unless $\max\alpha'=\max\mathscr{R}\alpha\cap\mathscr{L}\beta$.

This situation is possible if and only if $\max\alpha=\max\beta$, in which case one gets

$$\sum_{\alpha\cap\alpha'=\mathscr{L}\alpha} \mathsf{T}_{-\max\alpha}\mathsf{T}_{\alpha'}. \tag{7.27}$$

A similar proof for the left-right symmetric summation, gives that the corresponding sum vanishes unless $\min\alpha=\min\beta$, in which case it is given by

$$\sum_{\beta\cap\beta'=\mathscr{R}\beta} \mathsf{T}_{-\min\beta}\mathsf{T}_{\beta'}. \tag{7.28}$$

Then, at last, consider the terms of the shape $\mathsf{T}_{\alpha'}\mathsf{T}_{\beta'}$ in (7.24). This is given by

$$\sum_{\substack{\alpha\subseteq\alpha'\\ \beta\subseteq\beta'\\ \text{inclusion}}} \mathsf{T}_{\alpha'}\mathsf{T}_{\beta'} - \sum_{\substack{\min\alpha\in\alpha'\\ \max\beta\in\beta'\\ \text{inclusion}}} \sum_{\substack{\ell\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'\cap\beta'}} \mathsf{T}_{\alpha'}\mathsf{T}_{\beta'} + \sum_{\substack{\min\alpha\in\alpha'\\ \max\beta\in\beta'\\ \text{inclusion}}} \sum_{\substack{\ell,\ell+1\in\mathscr{R}\alpha\cap\mathscr{L}\beta\\ \ell\in\alpha'\\ \ell+1\cap\beta'}} \mathsf{T}_{\alpha'}\mathsf{T}_{\beta'}. \tag{7.29}$$

In each term, as $\alpha'\cap\beta'$ is necessarily not empty, the summation on α' and β' can be restricted using relations (7.6) to the cases where $\alpha'\subseteq\beta'$ or vice-versa. This is the meaning of the "inclusion" subscripts. The sum of the last two terms under the additional assumption that $\alpha\subseteq\alpha'$ and $\beta\subseteq\beta'$ annihilates with the first term. Here we used once again that $\mathscr{R}\alpha\cap\mathscr{L}\beta$ is not empty.

Then one can assume in the right two terms that either $\alpha\nsubseteq\alpha'$ or $\beta\nsubseteq\beta'$. It turns out that these possibilities exclude each other because of the inclusion $\alpha'\subseteq\beta'$ or vice-versa.

Let us compute the sum when $\alpha \not\subseteq \alpha'$, $\alpha \cup \beta \subseteq \beta'$ and $\alpha' \subseteq \beta'$. This is given by

$$- \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \alpha' \subseteq \beta'}} \sum_{\substack{\ell \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'} + \sum_{\substack{\min \alpha \in \alpha' \\ \max \beta \in \beta' \\ \alpha' \subseteq \beta'}} \sum_{\substack{\ell, \ell+1 \in \mathscr{R}\alpha \cap \mathscr{L}\beta \\ \ell \in \alpha'}} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}. \tag{7.30}$$

Then both terms vanish unless α' meets $\mathscr{R}\alpha \cap \mathscr{L}\beta$. In this case, the sum vanishes unless $\max \alpha' = \max \mathscr{R}\alpha \cap \mathscr{L}\beta$.

This situation is possible if and only if $\max \alpha = \max \beta$, in which case one gets

$$- \sum_{\alpha \cap \alpha' = \mathscr{L}\alpha} \sum_{\alpha' \cup \alpha \subseteq \beta'} \mathsf{T}_{\beta'} \mathsf{T}_{\alpha'}. \tag{7.31}$$

Similarly, the sum when $\alpha \cup \beta \subseteq \alpha'$, $\beta \not\subseteq \beta'$ and $\beta' \subseteq \alpha'$ vanish unless $\min \alpha = \min \beta$, in which case it is given by

$$- \sum_{\beta \cap \beta' = \mathscr{R}\beta} \sum_{\beta' \cup \beta \subseteq \alpha'} \mathsf{T}_{\alpha'} \mathsf{T}_{\beta'}. \tag{7.32}$$

Then gathering the terms (7.27),(7.31) and the terms (7.28),(7.32) and using relations (7.8), one gets the expected vanishing of (7.24) in all cases. ⬟

Théorème 7.4.2 *The morphism ψ from $\mathbf{M}^*(n)$ to $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is an isomorphism.*

Proof. Let us first prove that ψ is surjective. First it is clear from (7.13) that each T_{-i} is in the image of ψ. Then one can see by Möbius inversion on (7.14) that each T_α for $\alpha \in \Phi_{>1}$ is also in the image of ψ. But these variables together generates $\mathbf{H}^*(X_{\Sigma(Q_n)})$ because of the linear relations (7.4).

Now the ring $\mathbf{H}^*(X_{\Sigma(Q_n)})$ is a free abelian group of rank the Catalan number c_{n+1}. By Lemma 7.3.1, the surjectivity of ψ then implies that the monomials S^u, for u in the set of U-sets, are linearly independent in $\mathbf{M}^*(n)$. Hence they form a basis of $\mathbf{M}^*(n)$ and their images must be a basis of $\mathbf{H}^*(X_{\Sigma(Q_n)})$. So ψ is an isomorphism.

⬟

7.4.2 Consequences

The first consequence of this isomorphism is of course that the monomials S^u for U-sets u form a basis of $\mathbf{M}^*(n)$. Let us call it the natural basis. From now on, we will identify $\mathbf{M}^*(n)$ with $\mathbf{H}^*(X_{\Sigma(Q_n)})$ by the mean of ψ.

There is a unique element of degree n in the natural basis, which is the product of all S_i.

From Poincaré duality in the cohomology ring, one gets

Corollaire 7.4.3 *The ring $\mathbf{M}^*(n)$ is a graded Frobenius ring.*

Théorème 7.4.4 *The set of relations (7.10) and (7.11) is a (quadratic) Gröbner basis for the term order where variables of greater height are dominant.*

Proof. If this is not true, then there would exist another element in the Gröbner basis with a leading monomial of the form S^u for some U-set u. This would contradict the fact that the monomials associated to U-sets are linearly independent. ⬟

Théorème 7.4.5 *The ring $\mathbf{M}^*(n)$ is Koszul as an associative algebra.*

Proof. This follows from the fact that it admits a quadratic Gröbner basis, see for example [1]. ⬟

Proposition 7.4.6 *The ring $\mathbf{M}^*(n)$ is filtered by the subspaces spanned by monomials S^u of height less than a fixed bound.*

Proof. Indeed, the procedure of rewriting the product of two monomials in the natural basis as a sum of elements of this basis uses the Gröbner basis reduction, which can only decrease the height. ⬟

7.4.3 Duality between bottom T and top S

As said before, the natural basis of $\mathbf{M}^*(n)$ contains a unique element of degree n, which is simply

$$\prod_{i\in[n]} \mathsf{S}_i. \tag{7.33}$$

The symmetric bilinear form $\langle\, , \rangle$ defining the Frobenius structure of the graded ring $\mathbf{M}^*(n)$ is given by the coefficient of this unique element of degree n in the expression in the natural basis of the product of two elements of $\mathbf{M}^*(n)$.

By the graded Frobenius property, this bilinear map restricts to a non-degenerate pairing between the subspace of degree 1 (spanned by generators) and the subspace of degree $n-1$.

Let us consider the natural basis in degree $n-1$. It is indexed by U-sets of cardinality $n-1$. Using the duality on U-sets coming from the duality on codes, one can instead index this basis by $\Phi_{>0}$. Let S'_α be the element of the natural basis in degree $n-1$ assigned in this way to $\alpha \in \Phi_{>0}$.

By the Frobenius pairing, the natural basis in degree $n-1$ has a simple dual basis in degree 1 :

Proposition 7.4.7 *The basis $(\mathsf{T}_\alpha)_{\alpha\in\Phi_{>0}}$ in degree 1 is dual to the basis $(\mathsf{S}'_\alpha)_{\alpha\in\Phi_{>0}}$ in degree $n-1$ for the Frobenius pairing : for all α, β in $\Phi_{>0}$, one has*

$$\mathsf{S}_\alpha = \sum_\beta \langle \mathsf{S}_\alpha, \mathsf{S}'_\beta\rangle \mathsf{T}_\beta. \tag{7.34}$$

Proof. The proof is based on the comparison between the explicit computation of the coefficients $\langle \mathsf{S}_\alpha, \mathsf{S}'_\beta\rangle$ and the change of basis between the natural basis S_α in degree 1 and the basis T_α.

Let us start with the change of basis between S and T. Using (7.13), (7.14) and (7.4), one finds that

$$\mathsf{S}_i = \sum_{i\in\alpha\in\Phi_{>0}} \mathsf{T}_\alpha, \tag{7.35}$$

and for $\alpha \in \Phi_{>1}$,

$$\mathsf{S}_\alpha = \sum_{\alpha\subseteq\beta} (\#\alpha - 1)\mathsf{T}_\beta + \sum_{\alpha\not\subseteq\beta\in\Phi_{>0}} (\#\alpha\cap\beta)\mathsf{T}_\beta. \tag{7.36}$$

Then it only remains to show that these formulas coincide with the value of the pairing. This is done below. ⬟

Let us first state two useful Lemmas.

Lemme 7.4.8 *For $1 \leq i < j \leq n$, one has*

$$\mathsf{S}_{i,j}\,(\mathsf{S}_{i,i+1} \dots \mathsf{S}_{j-1,j}) = (j-i)\mathsf{S}_i \dots \mathsf{S}_j. \tag{7.37}$$

Proof. This is a simple inductive computation in $\mathbf{M}^*(n)$. This is easy if $j = i+1$. The inductive step first computes the product $\mathsf{S}_{i,j}\mathsf{S}_{i,i+1}$. ⬟

Lemme 7.4.9 *For $1 \leq i < j \leq n$, one has*

$$\mathsf{S}_{i,j}\,(\mathsf{S}_i \dots \mathsf{S}_j) = 0. \tag{7.38}$$

Proof. Quite obvious from the defining relations, by induction. ⬟

Let us now compute the pairing between elements of degree 1 and elements of degree $n-1$ in the natural basis of $\mathbf{M}^*(n)$.

Proposition 7.4.10 *The following equations hold for α, β in $\Phi_{>1}$:*

$$\langle \mathsf{S}_j, \mathsf{S}'_i \rangle = \delta_{i=j}, \tag{7.39}$$

$$\langle \mathsf{S}_\beta, \mathsf{S}'_i \rangle = \delta_{i\in\beta}, \tag{7.40}$$

$$\langle \mathsf{S}_j, \mathsf{S}'_\alpha \rangle = \delta_{j\in\alpha}, \tag{7.41}$$

$$\langle \mathsf{S}_\beta, \mathsf{S}'_\alpha \rangle = \begin{cases} \#\beta - 1 \text{ if } \beta \subseteq \alpha, \\ \#\alpha \cap \beta \text{ else.} \end{cases} \tag{7.42}$$

Proof. First, note that

$$\mathsf{S}'_i = \prod_{j\neq i} \mathsf{S}_j. \tag{7.43}$$

This implies the first relation using that $\mathsf{S}_i^2 = 0$ and the second relation using Lemma 7.4.9. Then note that for $\alpha = [i,j]$ with $i < j$, one has

$$\mathsf{S}'_\alpha = (\mathsf{S}_{i,i+1} \dots \mathsf{S}_{j-1,j}) \prod_{k\notin\alpha} \mathsf{S}_k. \tag{7.44}$$

This easily implies the third relation. The fourth relation can be checked by distinguishing whether $\beta \subseteq \alpha$ or not and using Lemma 7.4.8. ⬟

7.5 Parabolic inclusions

It follows from the presentation of the rings $\mathbf{M}^*(n)$ that, for any n_1 and n_2, there are morphisms of rings

$$\mathbf{M}^*(n_1) \otimes \mathbf{M}^*(n_2) \to \mathbf{M}^*(n_1+n_2), \tag{7.45}$$

mapping the generators $\mathsf{S} \otimes 1$ and $1 \otimes \mathsf{S}$ to some generators S according to the decomposition of the interval $[n_1+n_2]$ into two consecutive intervals $[n_1]$ and $[n_2]$.

These morphisms map the tensor product of the natural bases into the natural basis, hence they are injective. As the sum of the ranks is smaller than the rank in general, they are not surjective.

One can even see that, for a fixed n, the span of all the images of these maps for varying n_1, n_2 of sum n can not be the full ring $\mathbf{M}^*(n)$, for it can not contain the element $\mathsf{S}_{1,n}$. One can compute that the number of elements of the natural basis which cannot be reached in this way is the Catalan number c_{n-1}.

Through the isomorphisms with cohomology rings, these morphisms should come from refinements of fans, inducing maps of toric varieties, hence maps at the level of cohomology.

7.6 Conjectural deformation

It seems that one can replace the relations $\mathsf{S}_i^2 = 0$ in the presentation of $\mathbf{M}^*(n)$ by the relation $\mathsf{S}_i^2 = \mathsf{S}_i$ without much harm.

Let $\mathbf{M}^{\mathrm{def}}(n)$ be the commutative ring generated by variables S_i for $i \in [n]$ and S_α for $\alpha \in \Phi_{>1}$, modulo the right half of relations (7.10), all relations (7.11) and relations $\mathsf{S}_i^2 = \mathsf{S}_i$.

Of course, this ring is not graded as one relation is no longer homogeneous.

Conjecture 7.6.1 *The ring* $\mathbf{M}^{def}(n)$ *has dimension* c_{n+1}.

This has been checked by computer for $n \leq 6$. A strategy of proof would be to show that this set of relations is still a Gröbner basis. As part of this check, it is easy to see that the reduction of the monomials $\mathsf{S}_i^2 \mathsf{S}_\alpha$ for $i \in \alpha \in \Phi_{>1}$ works well.

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