# Operads and shuffles

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# The shuffle product on the tensor algebra

Let $k$ be a field and $V$ be a vector space over $k$. Let $T(V)$ be

$$\bigoplus_{n \geq 0} V^{\otimes n}.$$

This is an associative algebra for the concatenation product, called the tensor algebra.

There is also an associative and commutative product $ш$ on $T(V)$, called the **shuffle product**.

The element 1 in the base field is a unit for the shuffle product.

The $ш$ product has the following property:

$$(x_1 \otimes w_1) ш (x_2 \otimes w_2) = x_1 \otimes (w_1 ш (x_2 \otimes w_2)) + x_2 \otimes (w_2 ш (x_1 \otimes w_1)),$$

where $x_1$ and $x_2$ are letters in $V$ and $w_1$ and $w_2$ are words in $T(V)$.

## Half of a shuffle

Let us cut the shuffle product in two parts according to the previous formula. Let $T^+(V)$ be the augmented tensor algebra

$$\bigoplus_{n \geq 1} V^{\otimes n}.$$

Define a bilinear product (half-shuffle) $*$ on $T^+(V)$ by

$$(x_1 \otimes w_1) * (x_2 \otimes w_2) = x_1 \otimes (w_1 \amalg (x_2 \otimes w_2)).$$

Then one can recover the shuffle product from the $*$ product by symmetrization:

$$x \amalg y = x * y + y * x.$$

The $*$ product is not associative. It satisfies instead the relation

$$(x * y) * z = x * (y * z + z * y).$$

Let us call this algebraic structure a **Zinbiel algebra**.
Then $T^+(V)$ with $*$ is the free Zinbiel algebra on $V$. This provides a universal property for the shuffle algebra.

All this has been considered by Schützenberger a long time ago ("Sur une propriété combinatoire des algèbres de Lie libres pouvant être utilisée dans un problème de mathématiques appliquées", Séminaire Dubreil, novembre 1958, page 18).

The terminology "Zinbiel algebra" has been introduced much more recently by Loday, in relation with the notion of **Leibniz algebra**. There is a duality between Zinbiel and Leibniz.

## Operads

In order to deal with the many different kinds of algebras that are being used currently, it is useful to introduce **operads**.

An operad $P$ is a sequence of vector spaces $P(n)$ for $n \geq 1$ with an action of the symmetric group $\mathfrak{S}_n$ on $P(n)$, a distinguished element $\mathbf{1}$ in $P(1)$ and composition maps

$$\gamma : P(m) \otimes P(n_1) \otimes \cdots \otimes P(n_m) \longrightarrow P(n_1 + \cdots + n_m) \qquad (1)$$

satisfying some "associativity", "equivariance" and "unit" axioms. In a more fancy language, operads are associative monoids with unit in the monoidal category of $\mathfrak{S}$-modules.

The basic example is the following: fix a vector space $V$. Let $P(n) = \hom_k(V^{\otimes n}, V)$, let $\mathbf{1}$ be the identity map from $V$ to itself and let $\gamma$ be the usual composition map of multilinear applications.

Let us describe the operad Zin corresponding to Zinbiel algebras. The vector space $Zin(n)$ is $k\mathfrak{S}_n$, with the regular action of the symmetric group. Permutations are seen as words. The unit $\mathbf{1}$ is the unique permutation $1$ in $\mathfrak{S}_1$.

The composition $\gamma$ can be described using shuffles and renumbering. Let us give more details in a particular case. Let $\sigma_i \in \mathfrak{S}_{n_i}$ for $i = 1, \ldots, m$. First renumber the permutation $\sigma_1$ from $1$ to $n_1$, then renumber $\sigma_2$ from $n_1 + 1$ to $n_1 + n_2$ and so on.. Then

$$\gamma(12\ldots m, \sigma_1, \ldots, \sigma_m)$$

is the sum over all shuffles of the renumbered $\sigma$'s that preserves the order of the leftmost letters. For instance:

$$\gamma(12, 321, 21) = 32154 + 32514 + 32541 + 35214 + 35241 + 35421.$$

The operad Zin has more structure: it is an **anticyclic operad**. This implies in particular that there is an action of the bigger symmetric group $\mathfrak{S}_{n+1}$ on $Zin(n)$, extending the regular action of $\mathfrak{S}_n$.

This means that $Zin(n)$ is a module of dimension $n!$ over the symmetric group $\mathfrak{S}_{n+1}$. It is known that this module is isomorphic to the module $Lie(n+1)$ which is the multi-homogeneous part of degree $(1, 1, \ldots, 1)$ of the free Lie algebra on $n+1$ generators.

One possible way to describe this module is by mapping permutations to fractions.

Let us introduce an operad $M$ of fractions.

Here $M$ stand for "moulds".

The space $M(n)$ is the field $k(u_1, \ldots, u_n)$ of rational functions in $n$ indeterminates $u_1, \ldots, u_n$.

The unit $\mathbf{1}$ is the fraction $1/u_1$ in $M(1)$.

The composition can be described using substitution and product of functions. For example:

$$\gamma\left(\frac{1}{u_1 u_2}, \frac{1}{u_1}, \frac{1}{u_1(u_1 + u_2)u_3}\right) = \frac{u_2 + u_3 + u_4}{u_1(u_2 + u_3 + u_4)} \frac{1}{u_2(u_2 + u_3)u_4}.$$

## Action of bigger symmetric group on fractions

There is an action of $\mathfrak{S}_n$ on $M(n)$ by renumbering of the indeterminates $u_1, \ldots, u_n$.

This can be extended to an action of $\mathfrak{S}_{n+1}$ as follows.

Introduce a new variable $u_0$ such that

$$u_0 + u_1 + \cdots + u_n = 0.$$

Any element $\sigma$ of $\mathfrak{S}_{n+1}$ sends a fraction $f$ in $M(n)$ to an element $\sigma(f)$ in $k(u_0, u_1, \ldots, u_n)$; then $\sigma(f)$ can be uniquely rewritten as an element of $M(n)$ using the linear relation above to eliminate all instances of $u_0$.

For instance, the orbit of $-1/u_1 u_2$ contains $1/u_1(u_1 + u_2)$ and $1/u_2(u_1 + u_2)$.

Let us now map permutations in $\mathfrak{S}_n$ to fractions in $M(n)$:
The permutation $[\sigma(1), \ldots, \sigma(n)]$ is sent to

$$\frac{1}{(u_{\sigma(1)} + u_{\sigma(2)} + \cdots + u_{\sigma(n)}) \ldots (u_{\sigma(n-1)} + u_{\sigma(n)})(u_{\sigma(n)})}. \quad (2)$$

Claim : this defines an injective morphism of operads from Zin to $M$.

Claim : this morphism is compatible with the actions of $\mathfrak{S}_{n+1}$.

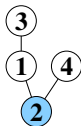Therefore there is an embedding of the Lie module inside the field of fractions.

A permutation can be seen as a linear tree, having its leftmost letter as root.

A **rooted tree** $T$ with vertices $1, \ldots, n$ can be seen as a **partial order**. Each tree can be mapped to the sum of all permutations compatible with this partial order.

For example, the rooted tree



is mapped to the following sum of permutations

$$2134 + 2143 + 2413.$$

Claim : the set of rooted trees in Zin is closed under composition.
The operation can be described as follows. The tree
$\gamma(T, S_1, \ldots, S_m)$ is obtained from the disjoint union of the trees
$S_1, \ldots, S_m$ by adding an edge between the root of $S_i$ and the root
of $S_j$ if there is an edge between $i$ and $j$ in $T$. The root is taken to
be the root of $S_k$ where $k$ is the root of $T$.
In this way, one recovers the operad NAP introduced by Livernet.
The acronym NAP means non-associative permutative.

Let us now forget about the actions of symmetric groups on $Zin(n)$.
We get a non-symmetric operad $Z$.
One can consider the direct sum MR of all spaces $Zin(n)$.

$$MR = \bigoplus_{n \geq 1} k\mathfrak{S}_n.$$

This is the free $Z$-algebra on one generator.

The element $12 + 21$ in Zin(2) defines an morphism from the symmetric operad Comm in the symmetric operad Zin

The element $12 + 21$ in $Z(2)$ defines an morphism from the non-symmetric operad Assoc in the non-symmetric operad $Z$

Hence one has associative product on MR.

This is nothing but the algebra structure of the Malvenuto-Reutenauer Hopf algebra, also known as the algebra **FQSym** of Free Quasi-symmetric functions.

The product is sometimes called the shifted shuffle.

But the operad structure gives more.

In the same way as the shuffle product was decomposed into two half-shuffles, the associative product on MR can be decomposed. Using 12 and 21, one gets on MR the structure of a **Dendriform algebra**.

Definition (Loday): A dendriform algebra is a vector space with two binary operations $\succ$ and $\prec$ such that

$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z, \quad (3)$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \quad (4)$$

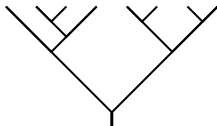$$x \prec (y \succ z) + x \prec (y \prec z) = (x \prec y) \prec z. \quad (5)$$

Then $x * y = x \prec y + x \succ y$ is associative.

And $x \lhd y = y \succ x - x \prec y$ is a pre-Lie product.

The Dendriform operad has been described by Loday using **planar binary trees**
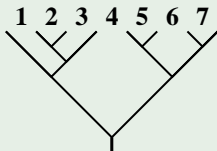


The free Dendriform algebra on one generator is the Hopf algebra of binary trees of Loday and Ronco.

There is an inclusion of the Dendriform operad in the Zinbiel operad, hence in the operad $M$ of fractions.

# The fraction of a planar binary tree

The fraction associated to a planar binary tree has a nice description.

## Example



This planar binary tree is mapped to

$$\frac{1}{(u_1 + u_2 + u_3)(u_2)(u_2 + u_3)(u_1 + \cdots + u_7)(u_5)(u_5 + u_6 + u_7)(u_7)}.$$

There is a link with tilting modules for the quivers of type $A$.

## A group structure on MR

As the algebra MR is the free algebra on one generator on the non-symmetric version of Zinbiel,
By a general construction for non-symmetric operads, the completion of MR becomes a group.
One can define a pre-Lie product $\circ$ using the operad composition as follows

$$x \circ y = \sum_{i=1}^{n} x \circ_i y,$$

when $x$ has degree $n$ and

$$x \circ_i y = \gamma(x, \mathbf{1}, \ldots, \mathbf{1}, y, \mathbf{1}, \ldots, \mathbf{1}) \tag{6}$$

with $y$ in $i$th position.
Then there is an associated Lie algebra and an associated group.

## Dendriform inside Zinbiel

The dendriform operad is contained in Zinbiel
In fact, it is the suboperad generated by 12 and 21
Therefore, there is a subalgebra of MR (The Loday-Ronco Hopf algebra of binary trees)
Let us call "dendriform" an element in this subalgebra.
This subspace is also a sub-group of the group structure on MR
As a subspace of the symmetric group ring, it contains the Descent algebra.

The operad $M$ was inspired by Ecalle's work and the notion of mould.

One can define a notion of alternal element in $M$, by a vanishing condition on summations over non-trivial shuffles. This is related to the usual notion of primitive element or Lie element,.

## Conjecture

An element of $M$ is dendriform and alternal if and only if it belongs to the image of the free Pre-Lie algebra inside the free Dendriform algebra.

(There is an injective map from the free Pre-Lie algebra to the free Dendriform algebra)

This would imply that Lie idempotents in the descent algebra come from elements in the free pre-Lie algebra.