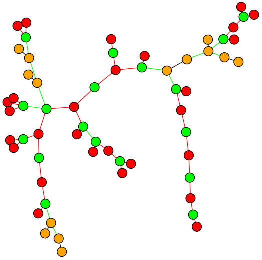
Cluster varieties for tree-shaped quivers and their cohomology



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Cluster algebras and the associated varieties

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$\implies$ cluster varieties (their spectrum) are algebraic varieties

Question: can we compute their cohomology rings ?

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Why is this interesting ?
$\rightarrow$ classical way to study algebraic varieties
$\rightarrow$ useful (necessary) to understand integration on them
(there are interesting periods involved)
$\rightarrow$ answer is not obvious, and sometimes nice
$\rightarrow$ there are interesting known differential forms

My choice goes to trees

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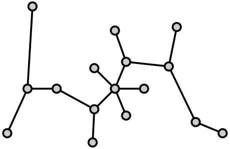
$\Longrightarrow$ restriction to quivers that are **trees**
(general quivers are more complicated)



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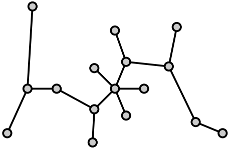

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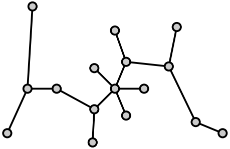

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$\to$ computing number of **points over finite fields** $\mathbb{F}_q$
can be seen as a first approximation towards determination of cohomology and is usually much more easy

First example (for babies)

Cluster algebra of type $\mathbb{A}_1$: $\boxed{\alpha} - x$ with one frozen vertex α.
Presentation by the unique relation

$$x\,x' = 1 + \alpha$$

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We will consider cluster algebras **with invertible coefficients**
So here α is assumed to be invertible.
One can then do **two different things**:
$\to$ (1) either let α vary in $\mathbb{C}^*$.
This gives an open sub-variety in $\mathbb{C}^2$ with coordinates x, x'.
$\to$ (2) or fix α to a generic invertible value (here $\alpha \neq -1, 0$)
This gives a variety isomorphic to $\mathbb{C}^*$ with coordinate x.

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The second case (2) (α fixed to a generic value) could be called a **cluster fiber variety**:
the inclusion of algebras $\mathbb{C}[\alpha, \alpha^{-1}] \to R$
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Note that the fiber at $\alpha = -1$ is singular.

General case of trees

Let us generalize this simple example.

For any tree, there is a well-defined cluster type
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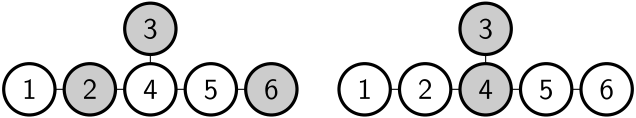
For any tree T, the aim is to define several varieties
that are a kind of **mixture** between cluster varieties and fibers

For that, need first to introduce some combinatorics on trees

Independent sets in graphs

By definition, an **independent set** in a graph G is
a subset S of the set of vertices of G
such that every edge contains at most one element of S



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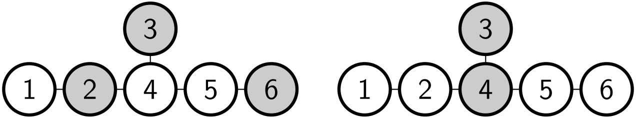

independent and not independent

A **maximum independent set** is an independent set of maximal cardinality among all independent sets.

Independent sets in graphs

Independent sets are a very classical notion in graph theory.
$\rightarrow$ NP-complete problem for general graphs (Richard Karp, 1972)
$\rightarrow$ polynomial algorithm for bipartite graphs (Jack Edmonds, 1961).
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One has to distinguish three kinds of vertices:
- vertices belonging **to all** maximal independent sets: RED ●
- vertices belonging **to some** max. independent sets: ORANGE ●
- vertices belonging **to no** maximal independent set: GREEN ●

Colors chosen are "traffic light colors"

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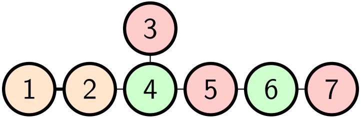
Nota Bene: this has nothing to do with *green sequences*

Canonical coloring

This gives a **canonical** coloring of every tree !
Here is one example of coloring



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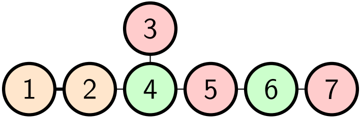

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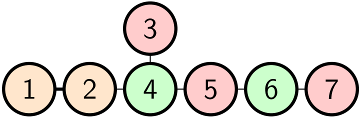

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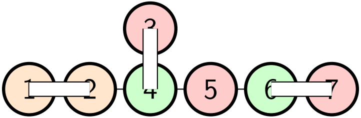
It turns out that this coloring is also related to **matchings**.

Coloring and matchings

A matching is a set of edges with no common vertices.
A **maximum matching** is a matching of maximum cardinality among all matchings.



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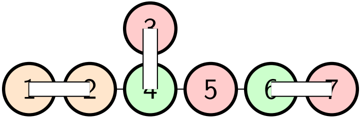

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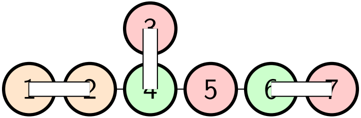

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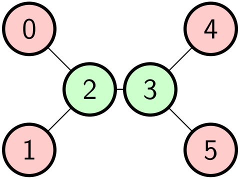
Theorem (Zito ; Bauer-Coulomb)

This coloring is the same as:

- *orange: vertices always in the same domino in all max. matchings*
- *green: vertices always covered by a domino in any max. matching*
- *red: vertices not covered by a domino in some max. matching*

Red-green components

One can then use this coloring to define **red-green components**: keep only the edges linking a red vertex to a green vertex; this defines a forest; take its connected components



An example with two red-green components $\{0, 1, 2\}$ and $\{3, 4, 5\}$

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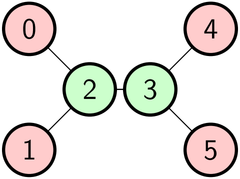
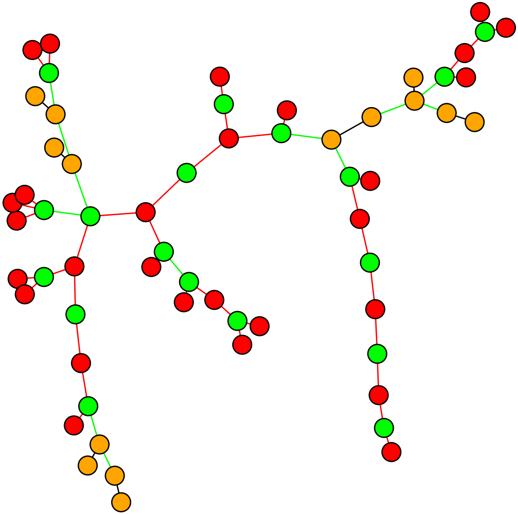


An example with two red-green components $\{0, 1, 2\}$ and $\{3, 4, 5\}$

For a tree T, let us call **dimension** $\dim T = \#$ red $- \#$ green.
This is always an integer $\dim(T) \geq 0$.
In the example above, the dimension is $4 - 2 = 2$.

Here is a big random example, with canonical coloring



So what are the varieties ?

Take a tree T and consider the alternating (bipartite) orientation on T. This gives a quiver, initial data for cluster theory.
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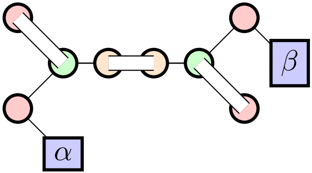
Pick a maximum matching of T and attach one frozen vertex to every vertex not covered by the matching.



(Claim: no loss in generality compared to arbitrary coefficients)
$\rightarrow$ every coefficient is attached to a red vertex

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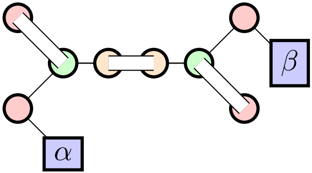

$\rightarrow$ the extended graph is still a tree, and has a perfect matching.
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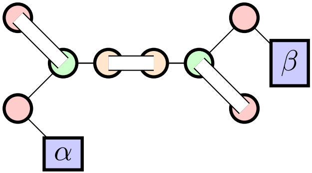

$\rightarrow$ the extended graph is still a tree, and has a perfect matching.
$\rightarrow$ the number of frozen vertices is $\dim(T)$.
Then **choose independently** for every red-green component:
- either to let all coefficients vary (but staying invertible)
- or to let all coefficients be fixed at generic (invertible) values

So what are the varieties ?

The equations are the cluster exchange relations for the alternating orientation (of the extended tree): $x_i x_i' = 1 + \prod_j x_j$.
One uses here a theorem of Berenstein-Fomin-Zelevinsky (in Cluster III) which gives a presentation by generators and relations of **acyclic** cluster algebras.

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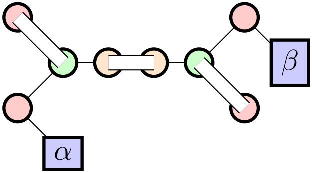

In this example, one can choose to fix β and let α vary.
This is really a mixture between the global cluster variety and the fibers of the coefficient morphism.

The matching does not matter

Theorem

This variety does not depend on the matching (up to isomorphism). All these varieties are smooth.

Proved using monomial isomorphisms ; smoothness by induction

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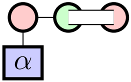
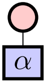
Note that the genericity condition can be made very explicit and is really necessary to ensure smoothness: Counter examples





$\mathbb{A}_3$ singular when $\alpha = 1$ and $\mathbb{A}_1$ when $\alpha = -1$
($\mathbb{A}_1$ was the baby example)

Points over finite fields

equations have coefficients in $\mathbb{Z} \to$ reduction to finite field $\mathbb{F}_q$.

Theorem

For X any of these varieties, there exists a polynomial P_X such that $\#X(\mathbb{F}_q)$ is given by $P_x(q)$.

This is a first hint that maybe cohomology may be "nice".

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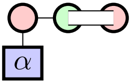

For $\mathbb{A}_3$ with α generic, one gets $q^3 - 1$ points.

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Consider the variety X_T associated with tree T and a choice for every red-green component of T between "varying" or "generic" coefficients. Let N be the sum of $\dim C$ over all "generic"-type red-green components C.

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Moreover the enumerating polynomial P_X can be written as $(q-1)^N$ times a reciprocal polynomial.

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Reciprocal means $P(1/q) = q^d P(q)$ (palindromic coefficients)

Free action: an example

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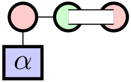

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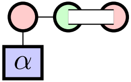

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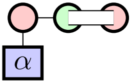

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The enumerating polynomial is $q^3 - 1 = (q-1)(q^2 - q + 1)$
This variety is not a product, but a non-trivial $\mathbb{C}^*$-principal bundle.

What about cohomology ?

Tools that can be used to study cohomology :
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$\implies$ coverings by two open sets, can apply Mayer-Vietoris long exact sequence
can also find covering by more open sets $\rightarrow$ use spectral sequences.
The Hodge structure sometimes help to prove that the spectral sequence degenerates at step 2.

Some simple classes in cohomology

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Not enough

The sub-algebra generated by those forms is **not** the full cohomology ring in general !

There are other classes in cohomology

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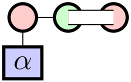

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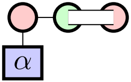

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• For α generic fixed, $\mathrm{WP} = \frac{dxdy}{xy} + \frac{dzdy}{zy}$, but cohomology has dimensions

$$H^* = \mathbb{Q}, \quad 0, \quad \mathbb{Q}, \quad \color{red}{\mathbb{Q}^2}$$

Mixed Tate-Hodge structures

They form an Abelian category, with a forgetful functor to $\mathbb{Q}$-vector spaces, and with one simple object $\mathbb{Q}(i)$ for every $i \in \mathbb{Z}$
no morphisms $\mathbb{Q}(i) \to \mathbb{Q}(j)$ if $i \neq j$.
Some extensions $\mathbb{Q}(i) \to E \to \mathbb{Q}(j)$ if $j > i$.
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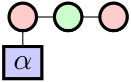
One can prove by induction that it is Hodge-Tate in the varieties under consideration. This means that there are no "more complicated factors".

Mixed Tate-Hodge structures: one example

Consider the type $\mathbb{A}_3$ for generic α



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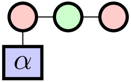

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Knowing this decomposition allows to recover the number of points over finite fields. Essentially every direct summand $\mathbb{Q}(i)$ in the cohomology group H^j gives a summand $(-1)^j q^i$. (But beware that one must use cohomology with compact support).

The cohomological information above gives back $q^3 - 1$.

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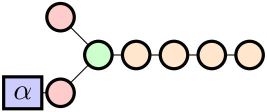
- Something can also be said about some trees of general shape H, in particular for $\mathbb{E}_6$ and $\mathbb{E}_8$
- case $\mathbb{E}_7$ with generic coefficient not fully understood.

Some details on type $\mathbb{D}$



One concrete example : $\mathbb{D}_n$ with n odd and generic coefficient α

Theorem

The cohomology is given by

$$\begin{cases} \mathbb{Q}(k) & \text{if } k = 0 \mod (2) \\ \mathbb{Q}(k-1) & \text{if } k = 1 \mod (2) \text{ and } k \neq 1, n \\ \mathbb{Q}(n-1) \oplus \mathbb{Q}(n) & \text{if } k = n \end{cases}$$

For $n = 3$, this coincide with the answer for $\mathbb{A}_3$, as it should.

Some things being skipped

$\rightarrow$ Results on counting points over $\mathbb{F}_q$ (nice formulas)
Just one tiny example in type $\mathbb{E}_n$ for n even:

$$(q^2 - q + 1)\frac{(q^{n-1} - 1)}{(q-1)}$$

$\rightarrow$ cellular decomposition (when coefficients are variables)
$\implies$ sum formula for the number of points over $\mathbb{F}_q$.

$\rightarrow$ Simple algorithm to compute the coloring.

Some perspectives (many things to do)

- at least complete the case of type $\mathbb{A}$ and Dynkin diagrams

- go beyond trees to all acyclic quivers and general matrices (announced article by David E Speyer and Thomas Lam.)

- say something about the periods ($\zeta(2)$ and $\zeta(3)$ are involved)

- try to organize all the cohomology rings of type $\mathbb{A}$ into some kind of algebraic structure (Hopf algebra, operad ?)

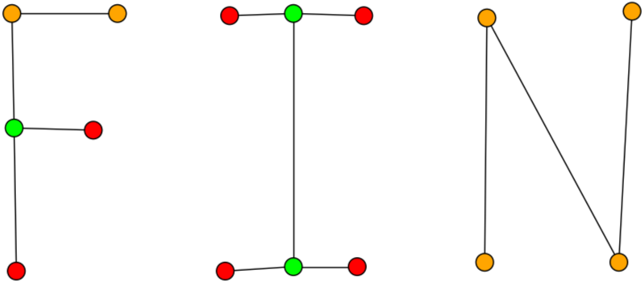
- study the topology of the real points (in relation with $q = -1$)

- topology of the set of non-generic parameters

- what about K-theory instead of cohomology ?

- understand the mysterious palindromic property

- some amusing relations with Pisot numbers



(THE END)

감사합니다