Stable finiteness of group algebras of surjunctive groups and model theory

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This is joint work with Tullio Ceccherini-Silberstein and Xuan Kien Phung [CCPcs].

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(A module M is called *Hopfian* if every surjective endomorphism of M is an automorphism.)

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There exist directly finite rings that are not stably finite [Coh66], [Lam07, Exercise 1.18].

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R stably finite $\implies$ R directly finite (since $\mathrm{Mat}_1(R) = R$).
There exist directly finite rings that are not stably finite [Coh66], [Lam07, Exercise 1.18].
For any $d \geq 1$, there exist rings R such that $\mathrm{Mat}_d(R)$ is directly finite but $\mathrm{Mat}_{d+1}(R)$ is not [Coh66].

Group algebras

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The group algebra of G with coefficients in K is the K-algebra $K[G]$ constructed as follows:

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- the multiplication on $K[G]$ is obtained by extending K-linearly the group operation on G.

Thus, every $\alpha \in K[G]$ can be uniquely written in the form

$$\alpha = \sum_{g \in G} \alpha_g g$$

with $\alpha_g \in K$ for all $g \in G$ and $\alpha_g = 0$ for all but finitely many $g \in G$.

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$$\alpha + \beta = \sum_{g \in G} (\alpha_g + \beta_g) g,$$

$$\lambda\alpha = \sum_{g \in G} (\lambda\alpha_g) g,$$

$$\alpha\beta = \sum_{g \in G} \left(\sum_{h_1, h_2 \in G : h_1 h_2 = g} \alpha_{h_1} \beta_{h_2} \right) g$$

for all $\alpha, \beta \in K[G]$ and $\lambda \in K$.

Kaplansky's stable finiteness conjecture

Theorem (Kaplansky [Kap69])

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Let G be a group and let K be a field of characteristic 0. Then the group algebra $K[G]$ is stably finite.

The proof is analytical after reducing to the case $K = \mathbb{C}$.
Kaplansky's stable finiteness conjecture: The group algebra $K[G]$ is stably finite for every group G and every field K.

Symbolic dynamics

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$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx := x \circ L_{g^{-1}} \end{aligned}$$

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The prodiscrete topology on A^G is the product topology obtained by taking the discrete topology on every factor A of $A^G = \prod_{g \in G} A$. The G-shift on A^G is continuous. The space A^G is homeomorphic to the Cantor space for $|A| \geq 2$ and G countably infinite.

Surjunctive groups

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The notion of surjunctivity goes back to Gottschalk [Got73].

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Gottschalk's surjunctivity conjecture: Every group is surjunctive.

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Definition

A group G is called sofic if for every $\varepsilon > 0$ and for every finite subset $F \subset G$, there exist a non-empty finite set X and a map $\phi \colon F \to \mathrm{Sym}(X)$ such that

- $\forall g, h \in F, \quad gh \in F \implies d_X(\phi(gh), \phi(g)\phi(h)) \leq \varepsilon;$
- $\forall g, h \in F, \quad g \neq h \implies d_X(\phi(g), \phi(h)) \geq 1 - \varepsilon.$

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Corollary (Elek et Szabó [ES04])

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- If two fields are elementary equivalent then they have the same characteristic.

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Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. The fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are elementary equivalent.
Observe that the fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are not isomorphic since $\overline{\mathbb{Q}}$ is countable while $\mathbb{C}$ is uncountable.

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Theorem (Second Lefschetz principle)

Let ψ be a first-order sentence in the language of rings which is satisfied by some (and hence any) algebraically closed field of characteristic 0. Then there exists an integer N such that ψ is satisfied by every algebraically closed field of characteristic $p \geq N$.

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Lemma 1

Let G be a group, let $d \geq 1$ be an integer, and let S be a finite subset of G. Then there exists a first-order sentence ψ in the language of rings such that a field K satisfies ψ if and only if there exist matrices $A, B \in \mathrm{Mat}_d(K[G])$ such that

1. *the support of each entry of A and of each entry of B is contained in S;*
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2. *$AB = I_d$ and $BA \neq I_d$.*

Lemma 2

Let G be a group and suppose that K and L are elementary equivalent fields. Then $K[G]$ is stably finite if and only if $L[G]$ is stably finite.

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Case 1: the field K is finite Let $d \geq 1$. Set $A := K^d$. A result in [CC07] (see also [CC10, Corollary 8.15.6]) says that $\mathrm{Mat}_d(K[G])$ is directly finite if and only if every injective, K-linear, G-equivariant and continuous map $\tau \colon A^G \to A^G$ is surjective.

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For $n \geq 1$, define $K_n \subset K$ by

$$K_n \coloneqq \mathrm{Fix}(\underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}).$$

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Proof of Theorem A (continued)

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Proof of Theorem A (continued)

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