Particle systems in the low density regime.

Part 2. Fluctuations

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1 Structure of correlations $\to$ a partial answer to P1

We can extract more information (especially on recollisions) by looking at exponential moments (possibly with different times)

$$\Lambda_t^\varepsilon(h) = \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon\left(\exp\left(\sum_{i=1}^N h(z_i(t))\right)\right)$$

This quantity is expanded in cumulants

$$\Lambda_t^\varepsilon(h) = \sum_{n=1}^\infty \frac{1}{n!} \int f_n^\varepsilon(t) \left(e^h - 1\right)^{\otimes n} dZ_n$$

$$f_n^\varepsilon(t) = \mu_\varepsilon^{n-1} \sum_{s=1}^n \sum_{\sigma \in \mathcal{P}_n^s} (-1)^s (s-1)! \prod_{i=1}^s F_{\sigma_i}^\varepsilon(t)$$

- Each cumulant encodes finer and finer correlations (which is the reason why we introduce the scaling factor μ_ε^{n-1})

- Contrary to correlation functions F_n^ε, they do not duplicate the information which is already encoded at lower orders

- They encode all the correlations, since we have the following inversion formula

$$F_n^\varepsilon(t) = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} \mu_\varepsilon^{-(n-s)} f_\sigma^\varepsilon(t)$$

<u>Theorem</u> : in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the cumulants associated to the hard-sphere dynamics satisfy the uniform estimates

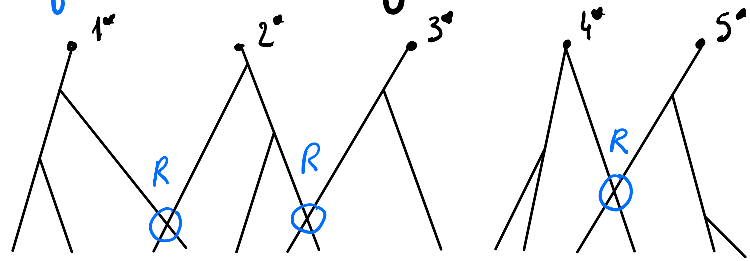
$$\| f_n^\varepsilon(t) \|_{L^1} \leq \left(C(t+\varepsilon) \right)^{n-1} n!$$

so that the cumulant generating function is absolutely convergent on a time interval $[0,T]$.

Furthermore one can characterize the limiting cumulants and prove that the limiting cumulant generating function is the solution of some Hamilton-Jacobi equation.

② Geometric representation of cumulants

We start from the geometric representation of F_n^ε with pseudotrajectories, and introduce a partition of $\{1, \dots n\}$ in forests connected by recollisions.



$\lambda_1 = \{1,2,3\}$

$\lambda_2 = \{4,5\}$

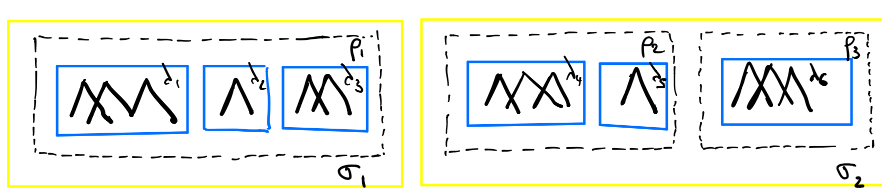
⚠ Forests are not independent as their parameters are constrained by the fact that no recollision should occur!

We then expand

$$\begin{cases} \displaystyle\prod_{i\neq j} 1_{\{\lambda_i \not\sim \lambda_j\}} = \sum_{s=1}^{n} \sum_{\sigma \in \mathcal{P}_n^s} \prod_{k=1}^{s} \varphi_{\sigma_k} \\ \displaystyle\varphi_n = \sum_{G \in \mathcal{C}_n} \prod_{\{i,j\} \in E(G)} \left(-1_{\{\lambda_i \sim \lambda_j\}}\right) \end{cases}$$

and group forests in jungles connected by overlaps.

The last source of correlation comes from the initial data. We end up with a cluster structure for the trees of the form



Short time estimate

By definition, the (scaled) cumulant f_n^ε corresponds to connected graphs of size n

$$f_n^\varepsilon(t) = \mu_\varepsilon^{n-1} \sum_{\lambda \in \mathcal{P}_n^\ell} \sum_{\rho \in \mathcal{P}_\ell^r} \int \Big(\prod_{i=1}^{\ell} \Delta_{\lambda_i} \Big) \varphi_\rho \, f^{\varepsilon 0}_{\{1,\dots,r\}}$$

We can then identify $(n-1)$ "independent" clustering constraints ($|\lambda_i| - 1$ in each forest, $|\rho_j| - 1$ in each jungle, $r-1$ at time 0) on the roots of the trees X_n^*.

Integrating with respect to $(a_i, t_i, v_i, \omega_i)_{i \leq m}$, we get

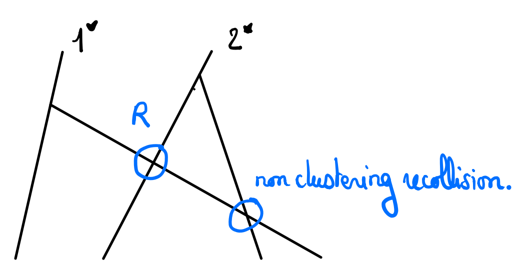
$$\| f_n^\varepsilon(t) \|_{L^1} \leq n! \, (C(t+\varepsilon))^{n-1}$$

Removing non clustering recollisions

When the size of particles $\varepsilon \to 0$, we expect pseudotrajectories $\Psi^\varepsilon_{n,m}$ will converge to some $\Psi_{n,m}$

- addition of particle i at t_i (without shift) + scattering
- overlap/recollision at $(\Theta_k)_{1 \leq k \leq n-1}$ (pointwise)
- backward free transport in the meantime

This convergence can be proved provided that $\Psi^\varepsilon_{m,n}$ has no additional recollision (which holds for almost all integration parameters)



③ Fluctuations $\to$ a partial answer to Q2

<u>Theorem 1</u>: in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the fluctuation field ζ_t^ε defined by

$$\zeta_t^\varepsilon(h) = \sqrt{\mu_\varepsilon}\left(\pi_t^\varepsilon(h) - \int F_1^\varepsilon(t) h \, dz\right)$$

converges in law (for short times) to the solution of the fluctuating Boltzmann equation (cf Spohn)

$$\begin{cases} d\zeta_t = \mathcal{L}_t \zeta_t \, dt + d\eta_t \\ \mathcal{L}_t h = -v \cdot \nabla_x h + Q(f, h) + Q(h, f) \\ \eta_t \text{ Gaussian noise } \delta\text{-correlated in } (t, x) \end{cases}$$

- $\mathcal{L}_t$ is the linearized collision operator around the solution f of the Boltzmann equation.
  In general, it is non autonomous, non self-adjoint and the corresponding semi group is not a contraction.

- η_t is the noise resulting from the initial randomness at very small scales, combined with the instability of the microscopic dynamics

- At equilibrium, the dissipation from the linearized operator $\mathcal{L}_M$ is exactly compensated by the noise.

- The proof relies on the study of the characteristic function

$$\log \mathbb{E}_\varepsilon\left(\exp\left(\zeta_t^\varepsilon(h)\right)\right) = \mu_\varepsilon \sum_{n=1}^{\infty} \int f_n^\varepsilon(t)\left(e^{h/\sqrt{\mu_\varepsilon}}-1\right)^{\otimes n} - \sqrt{\mu_\varepsilon}\int F_1^\varepsilon(t)\, h$$
$$= \frac{1}{2}\int f_1^\varepsilon(t)\, h^2 + \frac{1}{2}\int f_2^\varepsilon(t)\, h^{\otimes 2} + O(\mu_\varepsilon^{-1/2})$$

Extending this formula when $H(z([0,t])) = \sum_1^p h_p(z(\theta_p))$ shows that the limiting process is a Gaussian process.

- Close to equilibrium, taking advantage of the invariant measure, we have designed an alternative method (based on duality) extending Theorem 1 to long times (and even diffusive times!).

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(4) Large deviations → a partial answer to Q3

Theorem 2: in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the empirical measure π^ε satisfies the following large deviation estimates in the Skorohod space $\mathcal{D}([0,T],\mathcal{M})$ (for short times)

$$\limsup \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in F \text{ compact}\right) \leq \inf_F \mathcal{F}(\varphi)$$
$$\liminf \frac{1}{\mu_\varepsilon} \log \mathbb{P}_\varepsilon\left(\pi^\varepsilon \in \mathcal{O} \text{ open}\right) \geq -\inf_{\text{on } \mathcal{R}} \mathcal{F}(\varphi)$$

The large deviation functional coincides on a restricted set with the one obtained for stochastic dynamics (cf Rezakhanlou, Bouchet)

$$\mathcal{F}(\varphi) = H(\varphi_0 | f_0) + \sup_p \left[\int_0^T \langle p, D_s\varphi \rangle - \mathcal{H}(\varphi, p)\right]$$

- The Hamiltonian is a very singular functional similar to the Boltzmann collision operator

$$\begin{cases} \mathcal{H}(\varphi,p) = \frac{1}{\mathcal{L}} \int dz_1 dz_2\, d\omega \left((v_1-v_2)\cdot\omega\right)_+ \delta_{x_2-x_1}\, \varphi(z_1)\varphi(z_2)\left[e^{\Delta p(z_1,z_2,\omega)} - 1\right] \\ \Delta p(z_1,z_2,\omega) = p(z_1') + p(z_2') - p(z_1) - p(z_2) \end{cases}$$

- We have therefore to consider very smooth and decaying test functions, which explains why we do not get a complete large deviation principle, nor a clean identification of $\mathcal{F}$.

- The proof relies on the study of the limiting cumulant generating function Λ_t extended to $H(z([0,t]))$ with

$$H(z([0,t])) = h_t(z(t)) - \int_0^t D_s h(s,z(s))\, ds.$$

- It satisfies the Hamilton Jacobi equation.

$$\partial_t \mathcal{J}(t,h) = \frac{1}{2} \int \frac{\partial \mathcal{I}}{\partial h_t}(z_1) \frac{\partial \mathcal{I}}{\partial h_t}(z_2) \left(e^{h_t(z_1')+h_t(z_2')-h_t(z_1)-h_t(z_2)} - 1\right) d\mu(z_1,z_2,\omega)$$

which is a reversible equation (no information is lost up to exponentially small remainders), characterizing $\mathcal{F}$

$$\mathcal{F}(\varphi) = \sup_h \left[\langle \varphi_T, h_T\rangle - \int_0^T \langle \varphi, D_s h\rangle - \mathcal{J}(T,h)\right]$$