Construction of a multi-soliton blow-up solution to the semilinear wave equation in one space dimension

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Abstract: We consider the semilinear wave equation with power nonlinearity in one space dimension. Given a blow-up solution with a characteristic point, we refine the blow-up behavior first derived by Merle and Zaag. We also refine the geometry of the blow-up set near a characteristic point, and show that except may be for one exceptional situation, it is never symmetric with the respect to the characteristic point. Then, we show that all blow-up modalities predicted by those authors do occur. More precisely, given any integer $k \geq 2$ and $\zeta_0 \in \mathbb{R}$, we construct a blow-up solution with a characteristic point a, such that the asymptotic behavior of the solution near $(a, T(a))$ shows a decoupled sum of k solitons with alternate signs, whose centers (in the hyperbolic geometry) have ζ_0 as a center of mass, for all times.

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1 Introduction

We consider the one-dimensional semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } \partial_t u(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
We solve equation (1) locally in time in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ (see Ginibre, Soffer and Velo [11], Lindblad and Sogge [15]). For the existence of blow-up solutions, we have the

following blow-up criterion from Levine [14]: If $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} \left(\frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 - \frac{1}{p+1}|u_0(x)|^{p+1} \right) dx < 0,$$

then the solution of (1) cannot be global in time. More blow-up results can be found in Caffarelli and Friedman [7, 6], Alinhac [1, 2] and Kichenassamy and Littman [12, 13].

If u is an arbitrary blow-up solution of (1), we define (see for example Alinhac [1]) a 1-Lipschitz curve $\Gamma = \{(x, T(x))\}$ such that the maximal influence domain D of u (or the domain of definition of u) is written as

$$D = \{(x,t) \mid t < T(x)\}. \tag{2}$$

$\bar{T} = \inf_{x\in\mathbb{R}} T(x)$ and Γ are called the blow-up time and the blow-up graph of u. A point x_0 is a non characteristic point if

$$\text{there are } \delta_0 \in (0,1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathscr{C}_{x_0, T(x_0), \delta_0} \cap \{t \geq t_0\} \tag{3}$$

where $\mathscr{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}$. We denote by $\mathscr{R}$ (resp. $\mathscr{S}$) the set of non characteristic (resp. characteristic) points.

In order to study the asymptotic behavior of u near a given $(x_0, T(x_0)) \in \Gamma$, it is convenient to introduce similarity variables defined for all $x_0 \in \mathbb{R}$ and $T_0 \in \mathbb{R}$ by

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{4}$$

If $T_0 = T(x_0)$, we will simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. The function $w = w_{x_0}$ satisfies the following equation for all $y \in (-1,1)$ and $s \geq -\log T(x_0)$:

$$\partial_s^2 w = \mathscr{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1}\partial_s w - 2y\partial_{y,s}^2 w, \tag{5}$$

$$\text{where} \quad \mathscr{L} w = \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) \quad \text{and} \quad \rho = (1-y^2)^{\frac{2}{p-1}}. \tag{6}$$

This equation can be put in the following first order form:

$$\partial_s \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ \mathscr{L} w_1 - \frac{2(p+1)}{(p-1)^2} w_1 + |w_1|^{p-1} w_1 - \frac{p-3}{p-1} w_2 - 2y\partial_y w_2 \end{pmatrix}. \tag{7}$$

The Lyapunov functional for equation (5)

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy \tag{8}$$

is defined for $(w, \partial_s w) \in \mathscr{H}$ where

$$\mathscr{H} = \left\{ (q_1, q_2) \;\mid\; \|(q_1,q_2)\|_{\mathscr{H}}^2 \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \tag{9}$$

We also introduce the projection of the space $\mathscr{H}$ (9) on the first coordinate:

$$\mathscr{H}_0 = \left\{ r \in H^1_{loc} \mid \|r\|^2_{\mathscr{H}_0} \equiv \int_{-1}^{1} \left(r^2 + (r')^2 (1-y^2) \right) \rho dy < +\infty \right\}.$$

Finally, we introduce for all $|d| < 1$ the following stationary solutions of (5) (or solitons) defined by

$$\kappa(d, y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \tag{10}$$

In [22, 23, 26, 27] (see also the note [24]), Merle and Zaag gave an exhaustive description of the geometry of the blow-up set on the one hand, and the asymptotic behavior of solutions near the blow-up set on the other hand (they also extended their results to the radial case with conformal or subconformal power nonlinearity outside the origin in [25]):

- *The geometry of the blow-up set*: In Theorem 1 (and the following remark) in [23], and Theorems 1 and 2 (and the following remark) in [27]), the following is proved:

(i) $\mathscr{R}$ is a non empty open set, and $x \mapsto T(x)$ is of class C^1 on $\mathscr{R}$;
(ii) $\mathscr{S}$ is made of isolated points, and given $x_0 \in \mathscr{S}$, if $0 < |x - x_0| \le \delta_0$, then

$$\frac{|x-x_0|}{C_0 |\log(x-x_0)|^{\frac{(k(x_0)-1)(p-1)}{2}}} \le T(x) - T(x_0) + |x - x_0| \le \frac{C_0 |x-x_0|}{|\log(x-x_0)|^{\frac{(k(x_0)-1)(p-1)}{2}}} \tag{11}$$

for some $\delta_0 > 0$ and $C_0 > 0$, where $k(x_0) \ge 2$ is an integer. Moreover, estimate (11) *remains true after differentiation.*

- *The classification of the blow-up behavior near the singularity in* $(x_0, T(x_0))$: From Corollary 4 page 49, Theorem 3 page 48 in [22], Theorem 1 page 58 in [23] and Theorem 6 in [26], we recall the asymptotic behavior of $u(x,t)$ near the singular point $(x_0, T(x_0))$, according to the fact that x_0 is a non-characteristic point or not:

(iii) There exist $\mu_0 > 0$ and $C_0 > 0$ such that for all $x_0 \in \mathscr{R}$, there exist $\theta(x_0) = \pm 1$ and $s_0(x_0) \ge -\log T(x_0)$ such that for all $s \ge s_0$:

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(x_0)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \le C_0 e^{-\mu_0(s-s_0)}. \tag{12}$$

Moreover, $E(w_{x_0}(s)) \to E(\kappa_0)$ as $s \to \infty$.
(iv) If $x_0 \in \mathscr{S}$, then it holds that

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta_1 \begin{pmatrix} \sum_{i=1}^{k(x_0)} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to k(x_0) E(\kappa_0) \tag{13}$$

as $s \to \infty$, *where the integer* $k(x_0) \geq 2$ *has been defined in* (11), *for some* $\theta_1 = \pm 1$ *and continuous* $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ *for* $i = 1, ..., k(x_0)$. *Moreover, for some* $C_0 > 0$, *for all* $i = 1, ..., k(x_0)$ *and* s *large enough, we have*

$$\left| \zeta_i(s) - \left(i - \frac{(k(x_0)+1)}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0. \tag{14}$$

Our first result refines the expansion (14) up to the order $o(1)$. Let us introduce

$$\bar{\zeta}_i(s) = \left(i - \frac{(k+1)}{2} \right) \frac{(p-1)}{2} \log s + \bar{\alpha}_i(p,k) \tag{15}$$

where the sequence $(\bar{\alpha}_i)_{i=1,\dots,k}$ is uniquely determined by the fact that $(\bar{\zeta}_i(s))_{i=1,\dots,k}$ is an explicit solution with zero center of mass for the following ODE system:

$$\frac{1}{c_1}\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)}, \tag{16}$$

where $c_1 = c_1(p) > 0$ and $\zeta_0(s) \equiv \zeta_{k+1}(s) \equiv 0$ (see (31) below for a proof of this fact). Note that $c_1 = c_1(p) > 0$ is the constant appearing in system (28), itself inherited from Proposition 3.2 of [26]. With this definition, we can state our first result:

Theorem 1 (Refined asymptotics near a characteristic point)**.** *Consider* $u(x,t)$ *a blow-up solution of equation* (1) *and* x_0 *a characteristic point with* $k(x_0)$ *solitons. Then, there is* $\zeta_0(x_0) \in \mathbb{R}$ *such that estimate* (13) *holds with*

$$d_i(s) = -\tanh \zeta_i(s) \text{ and } \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0, \tag{17}$$

where $\bar{\zeta}_i(s)$ *is introduced above in* (15).

Remark. As one can see from (17) and (15), ζ_0 is the center of mass of the $\zeta_i(s)$ for any $s \geq -\log T(x_0)$.

Remark. Following the analysis of Merle and Zaag in [25], our result holds with the same proof for the higher-dimensional radial case

$$\partial_t^2 u = \partial_r^2 u + \frac{(N-1)}{r}\partial_r u + |u|^{p-1}u, \tag{18}$$

with

$$p \leq 1 + \frac{4}{N-1} \text{ if } N \geq 2, \tag{19}$$

provided that we consider a characteristic point different from the origin.

The refined estimate of Theorem 1 enables us to refine estimate (11) proved in [27] and get a more refined estimated for $T(x)$ and $T'(x)$ when x is near a characteristic point. More precisely, we have the following:

Corollary 2 (Refined behavior for the blow-up set near a characteristic point)**.** *Consider $u(x,t)$ a blow-up solution of equation* (1) *and x_0 a characteristic point with $k(x_0)$ solitons and $\zeta_0(x_0) \in \mathbb{R}$ as center of mass of the solitons' center as shown in* (13) *and* (17)*. Then,*

$$T'(x) = -\theta(x)\left(1 - \frac{\gamma e^{-2\theta(x)\zeta_0(x_0)}(1+o(1))}{|\log|x-x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}}\right) \tag{20}$$

$$T(x) = T(x_0) - |x-x_0| + \frac{\gamma e^{-2\theta(x)\zeta_0(x_0)}|x-x_0|(1+o(1))}{|\log|x-x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}} \tag{21}$$

as $x \to x_0$, where $\theta(x) = \frac{x-x_0}{|x-x_0|}$ and $\gamma = \gamma(p) > 0$.

Remark. Unlike what one may think from the less accurate estimate (11), we surprisingly see from this corollary that the blow-up set is *never* symmetric with respect to a characteristic point x_0, except maybe when $\zeta_0(x_0) = 0$.

Remark. As usual in blow-up problems, the geometrical features of the blow-up set (here, $T(x)$ and $T'(x)$) are linked to the parameters of the asymptotic behavior of the solution (here, $k(x_0)$ the number of solitons in similarity variables and $\zeta_0(x_0)$, the location of their center of mass).

Following this classification given for arbitrary blow-up solutions, we asked the question whether all these blow-up modalities given above in (iii) and (iv) (refined by Theorem 1) do occur or not.

As far as non-characteristic points are concerned, the answer is easy.
Indeed, any blow-up solution (for example those constructed by Levine's criterion given on page 2) has non-characteristic points, as stated above in Result (i) of page 3.
Regarding the asymptotic behavior, any profile given in (12) does occur. Indeed, note first that for any $d \in (-1,1)$, the function

$$u(x,t) = (1-t)^{-\frac{2}{p-1}}\kappa\left(d, \frac{x}{1-t}\right) = \frac{\kappa_0(1-d^2)^{\frac{1}{p-1}}}{(1-t+dx)^{\frac{2}{p-1}}} \tag{22}$$

is a particular solution to equation (1) defined for all $(x,t) \in \mathbb{R}^2$ such that $1-t+dx > 0$, blowing up on the curve $T(x) = 1+dx$ and such that for any $x_0 \in \mathbb{R}$, $T'(x_0) = d$ and $w_{x_0}(y,s) = \kappa(d,y) = \kappa(T'(x_0), y)$, and (12) is trivially true. However, the problem with this solution is that it is not a solution of the Cauchy problem at $t=0$, in the sense that it is not even defined for all $x \in \mathbb{R}$ when $t=0$. This is in fact not a problem thanks to the finite speed of propagation. Indeed, performing a truncation of (22) at $t=0$, the new solution will coincide with (22) for all $|x_0| \le R$ and $t \in [0, T(x_0))$ for some $R > 0$, and (12) holds for the new solution as well, for all $|x_0| < R$.

Now, considering characteristic points, the answer is much more delicate. Unlike what was commonly believed after the work of Caffarelli and Friedman [7, 6], Merle and Zaag proved in Proposition 1 of [26] the *existence of solutions* of (1) such that

$$\mathscr{S} \neq \emptyset.$$

Since that solution was odd by construction, the number of solitons appearing in the decomposition (13) has to be even. No other information on the number of solitons was available. After this result, the following question remained open :

Given an integer $k \geq 2$, is there a blow-up solution of equation (1) *with a characteristic point x_0 such that the decomposition* (13) *holds with k solitons?*

In this paper, we show that the answer is *yes*, and we do better, by prescribing the location of the center of mass of the $\zeta_i(s)$ in (14). More precisely, this is our second result:

Theorem 3 (Existence of a solution with prescribed blow-up behavior at a characteristic point)**.** *For any integer $k \geq 2$ and $\zeta_0 \in \mathbb{R}$, there exists a blow-up solution $u(x,t)$ to equation* (1) *in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ with $0 \in \mathscr{S}$ such that*

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty, \tag{23}$$

with

$$d_i(s) = -\tanh \zeta_i(s), \quad \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 \tag{24}$$

and $\bar{\zeta}_i(s)$ defined in (15).

Remark. Note from (24) and (15) that the barycenter of $\zeta_i(s)$ is fixed, in the sense that

$$\frac{\zeta_1(s) + \cdots + \zeta_k(s)}{k} = \frac{\bar{\zeta}_1(s) + \cdots + \bar{\zeta}_k(s)}{k} + \zeta_0 = \zeta_0, \quad \forall s \geq -\log T(0). \tag{25}$$

Remark. Note that this result uses our argument for Theorem 1, in particular our analysis of ODE (28) given in section 2 below. As we pointed out in a remark following Theorem 1, our result holds also in the higher-dimensional radial case (18) under the condition (19), in the sense that for any $r_0 > 0$, we can construct a solution of equation (18) such that its similarity variables version $w_{r_0}(y,s)$ behaves according to (23) with the parameters $d_i(s)$ given by (24).

Remark. We are unable to say whether this solution has other characteristic points or not. In particular, we have been unable to find a solution with $\mathscr{S}$ exactly equal to $\{0\}$. Nevertheless, let us remark that from the finite speed of propagation, we can prescribe more characteristic points as one can see from the following corollary:

Corollary 4 (Prescribing more characteristic points)**.** *Let $I = \{1, ..., n_0\}$ or $I = \mathbb{N}$ and for all $n \in I$, $x_n \in \mathbb{R}$, $T_n > 0$, $k_n \geq 2$ and $\zeta_{0,n} \in \mathbb{R}$ such that*

$$x_n + T_n < x_{n+1} - T_{n+1}.$$

Then, there exists a blow-up solution $u(x,t)$ of equation (1) *in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ with $\{x_n \mid n \in I\} \subset \mathscr{S}$, $T(x_n) = T_n$ and for all $n \in I$,*

$$\left\| \begin{pmatrix} w_{x_n}(s) \\ \partial_s w_{x_n}(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k_n} (-1)^{i+1} \kappa(d_{i,n}(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty,$$

with

$$\forall i = 1, \dots, k_n, \;\; d_{i,n}(s) = -\tanh \zeta_{i,n}(s), \;\; \zeta_{i,n}(s) = \bar{\zeta}_i(s) + \zeta_{0,n}$$

and $\bar{\zeta}_i(s)$ *defined in* (15)*.*

Remark. Once again, we are unable to construct a solution with $\mathscr{S}$ exactly equal to $\{x_n \mid n \in I\}$.

As one can see from (23) and (24), the solution we have just constructed in Theorem 3 behaves like the sum of k solitons as $s \to \infty$. In the literature, such a solution is called a *multi-soliton solution*. Constructing multi-soliton solutions is an important problem in nonlinear dispersive equations. It has already be done for the L^2 critical and subcritical nonlinear Schrödinger equation (NLS) (see Merle [19] and Martel and Merle [17]), the L^2 critical and subcritical generalized Korteweg de Vries equation (gKdV) (see Martel [16]), and for the L^2 supercritical case both for (gKdV) and (NLS) equations in Côte, Martel and Merle [8].

More generally, constructing a solution to some Partial Differential Equation with a prescribed behavior (not necessarily multi-solitons solutions) is an important question. We solved this question for (gKdV) in Côte [4, 5], and also for parabolic equations exhibiting blow-up, like the semilinear heat equation with Merle in [21, 20], the complex Ginzburg-Landau equation in [28] and with Masmoudi in [18], or a gradient perturbed heat equation with Ebde in [9]. In all these cases, the prescribed behavior shows a convergence to a limiting profile in some rescaled coordinates, as the time approaches the blow-up time.

Surprisingly enough, in both the parabolic equations above and the supercritical dispersive equations treated in [8], the same topological argument is crucial to control the directions of instability. This will be the case again for the semilinear wave equation (1) under consideration in our paper. More precisely, our strategy relies on two steps:

- Thanks to a dynamical system formulation, we show that controlling the similarity variables version $w(y, s)$ (5) around the expected behavior (23) reduces to the control of the unstable directions, whose number is finite. This dynamical system formulation is essentially the same as the one that allowed us to show that all characteristic points are isolated in [27]. Then, we solve the finite dimensional problem thanks to a topological argument based on index theory. This solves the problem without allowing us to prescribe the center of mass as required in (25).

- Performing a Lorentz transform on the solution we have just constructed, we are able to choose the center of mass as in (25).

This paper is organized in three sections: In Section 2, we refine the blow-up behavior at a characteristic point and the geometry of the blow-up set and prove Theorem 1 together with Corollary 2. Then, in Section 3, we construct a multi-soliton solution in similarity variables. Finally, in Section 4, we translate the similarity variables construction in the $u(x, t)$ formulation, and then use a Lorentz transform to prescribe the center of mass and finish the proof of Theorem 3 and Corollary 4.

2 Refined asymptotics near a characteristic point

In this section, we prove Theorem 1 and Corollary 2, refining the description given in [26] for the blow-up behavior at a characteristic point together with the geometry of the blow-up set. We proceed in two subsections, the first devoted to the proof of Theorem 1 and the second to the proof of Corollary 2.

2.1 Refined blow-up behavior near a characteristic point

We prove Theorem 1 here.

Proof of Theorem 1. . Consider $u(x,t)$ a blow-up solution of equation (1) and $x_0 \in \mathscr{S}$. From the result of [26] recalled in (iv) in page 3, we know that estimate (13) holds for some $k = k(x_0) \geq 2$ and $|\theta_1| = 1$, with continuous functions $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ satisfying (14). In order to conclude, we claim that it is enough to refine this estimate by showing that

$$\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 + o(1) \text{ as } s \to \infty, \tag{26}$$

where $(\bar{\zeta}_i(s))_i$ (15) is the explicit solution to system (16). Indeed, once this is proved, we can slightly modify the $\zeta_i(s)$ by setting $\zeta_i(s)$ exactly equal to $\bar{\zeta}_i(s) + \zeta_0$ (as required in (17)) and still have (13) hold, thanks to the following continuity result for the solitons $\kappa(d)$ (10):

$$\|\kappa(d_1) - \kappa(d_2)\|_{\mathscr{H}_0} \leq C|\arg\tanh d_1 - \arg\tanh d_2| \tag{27}$$

(see Lemma 3.7 below for a more general statement). Thus, our goal in this section is to show (26), where the $\zeta_i(s) = -\arg\tanh d_i(s)$ are the parameters shown in (13) proved in [26].

From Proposition 3.2 in [26], we recall that $(\zeta_i(s))_{i=1,\dots,k}$ is in fact a C^1 function satisfying the following ODE system for $i = 1, \dots, k$:

$$\frac{1}{c_1}\zeta_i' = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} + R_i \text{ where } R_i = O\left(\frac{1}{s^{1+\eta}}\right) \text{ as } s \to \infty, \tag{28}$$

for some explicit constant $c_1 = c_1(p) > 0$, and a fixed small constant $\eta = \eta(p) > 0$, with the convention that $\zeta_0(s) \equiv -\infty$ and $\zeta_{k+1}(s) \equiv +\infty$. (Systems similar to (28) also appear in other contexts, for example, in the boundary layer formation for the real Ginzburg-Landau equation, see [3, 10]).

We proceed in two parts: we first study system (28) without the rest term (i.e. when all $R_i \equiv 0$), then we take into account the full system and conclude the proof in the general case.

Part 1: The ODE system with no rest term

Our system (28) with no rest terms is stated in (16). We proceed in 4 steps: We first give explicit solutions for system (16). Then, we study its critical points and give a Lyapunov functional for it. In the third step, we find a compact in $\mathbb{R}^k$ stable by the flow of system (16). Finally, applying Lyapunov's theorem we show that any bounded solution is asymptotically close to one of the explicit solutions given in the first step.

Step 1: Explicit solutions for system (16)
Introducing

$$\gamma_i = (p-1)\left(-i + \frac{k+1}{2}\right) \tag{29}$$

and looking for a solution of system (16) obeying the following ansatz

$$\zeta_i(s) = -\frac{\gamma_i}{2}\log s + \alpha_i, \tag{30}$$

we get the following necessary and sufficient condition: for all $i = 2, \ldots, k$,

$$e^{-\frac{2}{p-1}(\alpha_i - \alpha_{i-1})} = \frac{1}{2c_1}\sum_{j=1}^{i-1}\gamma_j = -\frac{1}{2c_1}\sum_{j=i}^{k}\gamma_j = \frac{(p-1)}{4c_1}(i-1)(k+1-i),$$

which makes a one parameter family of solutions, for example characterized by its center of mass $\frac{1}{k}\sum_{i=1}^{k}\zeta_i(s)$ (which in fact remains independent of time). Fixing the center of mass to be zero, we obtain the following particular solution

$$\bar{\zeta}_i(s) = -\frac{\gamma_i}{2}\log s + \bar{\alpha}_i,$$

already defined in (15), where $\alpha_i = \bar{\alpha}_i(p,k)$ are uniquely defined by

$$\sum_{i=1}^{k}\bar{\alpha}_i = 0, \quad e^{-\frac{2}{p-1}(\bar{\alpha}_i - \bar{\alpha}_{i-1})} = \frac{(p-1)}{4c_1}(i-1)(k+1-i), \; i = 2, \ldots, k. \tag{31}$$

In particular, all the other solutions obeying the ansatz (30) are obtained as

$$\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 \tag{32}$$

where ζ_0 is the constant value of the center of mass $\frac{1}{k}\sum_{i=1}^{k}\zeta_i(s)$. Let us remark that

$$\forall s > 0, \;\; \bar{\zeta}_i(s) = -\bar{\zeta}_{k-i}(s). \tag{33}$$

Indeed, from the definition (29) of γ_i and system (16), we see that $(-\bar{\zeta}_{k-i}(s))_i$ is also a solution of system (16) obeying the ansatz (30). Therefore, as in (32), we have for all $i = 1, \ldots, k$ and $s > 0$, $-\bar{\zeta}_{k-i}(s) = \bar{\zeta}_i(s) + \bar{\zeta}_0$, where $\bar{\zeta}_0 = \frac{1}{k}\sum_{i=1}^{k}(-\bar{\zeta}_{k-i}(s)) = -\frac{1}{k}\sum_{j=1}^{k}\bar{\zeta}_j(s) = 0$, and (33) follows.

Step 2: Critical points and a Lyapunov functional for system (16)
We now look at a perturbation $\zeta(s) = (\zeta_i(s))_{i=1,\ldots,k}$ of this solution. Denote

$$\xi_i(\tau) = \frac{2}{p-1}(\zeta_i(s) - \bar{\zeta}_i(s)) \text{ where } \tau = \log s \tag{34}$$

and assume that the maximal solution exists on some interval $[0, \tau_\infty)$ where either τ_∞ is finite or $\tau_\infty = \infty$. We assume that

$$\sum_{i=1}^{k} \xi_i(0) = 0 \tag{35}$$

Then, $\sum_{i=1}^{k} \xi_i(\tau) = 0$ for all $\tau \in [0, \tau_\infty)$ and the ξ_i satisfy the system

$$\begin{cases} \dot{\xi}_1 = -\sigma_1(e^{-(\xi_2-\xi_1)} - 1), \\ \dot{\xi}_i = \sigma_{i-1}(e^{-(\xi_i-\xi_{i-1})} - 1) - \sigma_i(e^{-(\xi_{i+1}-\xi_i)} - 1), \quad i = 2, \dots, k-1 \\ \dot{\xi}_k = \sigma_{k-1}(e^{-(\xi_k-\xi_{k-1})} - 1). \end{cases} \tag{36}$$

where

$$\sigma_i = \frac{i(k-i)}{2}. \tag{37}$$

Denote

$$b_i(\tau) = \sigma_{i-1}(e^{-(\xi_i(\tau)-\xi_{i-1}(\tau))} - 1) \text{ for } i = 2, \dots, k-1, \;\; b_1 = b_{k+1} = 0, \tag{38}$$

so that

$$\forall i = 1, \dots, k, \;\; \dot{\xi}_i = b_i - b_{i+1} \tag{39}$$

and consider

$$b(\tau) = \min\{b_i(\tau) | i = 1, \dots, k+1\}, \quad B(\tau) = \max\{b_i(\tau) | i = 1, \dots, k+1\}. \tag{40}$$

Note from (38) that

$$b(\tau) \leq 0 \leq B(\tau). \tag{41}$$

Proposition 2.1 (The critical point and Lyapunov functionals of system (36) under the condition (35))**.** *The only critical point of system* (36) *under the condition* (35) *is* $\xi_i = 0$*. Moreover, the functions* B *and* $-b$ *are Lyapunov functionals for system* (36)*. In addition,* $B - b$ *is (strictly) decreasing, except if* $\xi_1(\tau) \equiv \cdots \equiv \xi_k(\tau) \equiv 0$*.*

Proof. Regarding the critical points: note that $\dot{\xi}_1 = 0$ if and only if $\xi_2 = \xi_1$. By a straightforward induction one sees that all ξ_i are equal. As their sum is 0, the only critical point is $\xi_1 = \cdots = \xi_k = 0$.

Let us now prove that B is nonincreasing along the flow; the argument for $-b$ will be similar. Hence, let $\boldsymbol{\xi}(\tau) = (\xi_1(\tau), \dots, \xi_k(\tau))$ be a solution of (36) such that

$$\boldsymbol{\xi}(\tau) \neq 0 \text{ for any } \tau \text{ in the domain of definition.} \tag{42}$$

Define

$$J(\tau) = \{i \in [\![1, k+1]\!] \mid b_i(\tau) = B(\tau)\},$$

the set of indices i for which b_i is maximum at time τ. The following lemma allows us to conclude:

Lemma 2.2. *For all $\tau_0 \in [0, \tau_\infty)$, there exist $\varepsilon = \varepsilon(\tau_0) > 0$ such that for all $i \in J(\tau_0) \cap [\![2, k]\!]$, $b_i(t) < B(\tau_0)$ for all $t \in (\tau_0, \tau_0 + \varepsilon)$.*

Indeed, assuming this lemma, let us show that for all $i = 1, \dots, k+1$, there exists $\epsilon_i > 0$ such that

$$\forall t \in (\tau_0, \tau_0 + \epsilon_i), \;\; b_i(\tau) \leq B(\tau_0). \tag{43}$$

If $i = 1$, or $i = k+1$, then (43) is obvious from (38) and (41).
If $2 \leq i \leq k$ and $i \in J(\tau_0)$, then (43) is clear from Lemma 2.2.
If $2 \leq i \leq k$ and $i \notin J(\tau_0)$, then $b_i(\tau_0) < B(\tau_0)$ by definition of $J(\tau_0)$ and (43) follows by continuity of $b_i(\tau)$.
By connectedness, it follows that B is nonincreasing on the whole interval of definition of the solution. The argument for $-b$ is quite similar.
In particular $B(\tau) - b(\tau)$ is nonincreasing too. Let us show that it is in fact decreasing. Since (42) holds, it follows that either $B(\tau_0) > 0$ or $-b(\tau_0) > 0$ (otherwise $B(\tau_0) = b(\tau_0) = 0$ by (41), hence $b_i(\tau_0) = 0$ and $\xi_i(\tau_0) = \xi_1(\tau_0) = 0$ by (40), (38) and (35), which is a contradiction by (42)). Hence, using a similar argument to the proof of (43), we see that either B or $-b$ is decreasing. Thus, $B - b$ is decreasing, which is the desired conclusion for Proposition 2.1. It remains to prove Lemma 2.2 in order to finish the proof of Proposition 2.1.

Proof of Lemma 2.2. Let $[\![m, \overline{m}]\!] \subset J(\tau_0)$ be a maximal interval of integers included in $J(\tau_0)$. As $J(\tau_0)$ is a union of such intervals, it is enough to prove Lemma 2.2 for all $i \in [\![m, \overline{m}]\!] \cap [\![2, k]\!]$.
Notice that for $i = 2, \dots, k$, b_i has the sign of $(\xi_{i-1} - \xi_i)$, and that

$$\dot{b}_i = \sigma_{i-1} e^{-(\xi_i - \xi_{i-1})} (\dot{\xi}_{i-1} - \dot{\xi}_i) \tag{44}$$

has the sign of $(\dot{\xi}_{i-1} - \dot{\xi}_i)$. Now, using (39), we write

$$\dot{\xi}_{i-1} - \dot{\xi}_i = b_{i-1} - 2b_i + b_{i+1}. \tag{45}$$

Case 1: $[\![m, \overline{m}]\!] \subset [\![2, k]\!]$.
In particular, as $m - 1 \notin J(\tau_0)$ and $\overline{m} + 1 \notin J(\tau_0)$, we get $b_{m-1}(\tau_0) < B(\tau_0)$ and $b_{\overline{m}+1}(\tau_0) < B(\tau_0)$, and this shows that

$$\begin{cases} \dot{b}_m(\tau_0) < 0, \quad \dot{b}_{\overline{m}}(\tau_0) < 0, \\ \dot{b}_i(\tau_0) = 0, \quad \text{for all } i \text{ such that } m < i < \overline{m}. \end{cases} \tag{46}$$

If $i = m$ or $i = \overline{m}$, as $b_i(\tau_0)$ is maximum, we see that $\dot{b}_i(\tau_0) < 0$, so that Lemma 2.2 holds for this i.

Now assume $m < i < \overline{m}$, then $b_{i-1}(\tau_0) = b_i(\tau_0) = b_{i+1}(\tau_0) = B(\tau_0)$, so that $\dot{\xi}_{i-1}(\tau_0) - \dot{\xi}_i(\tau_0) = 0$ and then $\dot{b}_i(\tau_0) = 0$ from (44) and (45). We will show in fact that a higher derivative of b_i is negative at $\tau = \tau_0$ which will conclude the proof of Lemma 2.2 for this i. More precisely, we prove the following:

Claim. Let $d(i) = \min\{i - m, \overline{m} - i\}$. If $d(i) \geq 1$, then

$$\dot{b}_i(\tau_0) = \cdots = b_i^{(d(i))}(\tau_0) = 0 \text{ and } b_i^{(d(i)+1)}(\tau_0) < 0. \tag{47}$$

To prove the Claim, for $n \in [\![1, \lfloor(\overline{m} - m)/2\rfloor]\!]$ where $\lfloor z \rfloor$ stands for the integer part of $z \in \mathbb{R}$, consider the proposition

$$P_n: \quad b_{m-1+n}^{(n)}(\tau_0) < 0, \quad b_{\overline{m}+1-n}^{(n)}(\tau_0) < 0, \quad \text{and } \forall i \in [\![m+n, \overline{m}-n]\!],\ b_i^{(n)}(\tau_0) = 0.$$

In some sense, this proposition relies on an inductive mechanism, where a negative higher derivative propagates from $i = \overline{m}$ in the left direction, affecting after each step the next derivative of the left neighbor. A similar phenomenon starts from $i = m$ and goes to the right. We will prove proposition P_n by induction on n.
Note that (46) proves P_1. Let $n \geq 2$ and assume that $P_1, \ldots P_{n-1}$ hold. In particular, P_{n-1} gives

$$b_{m-2+n}^{(n-1)}(\tau_0) < 0, \quad b_{\overline{m}+2-n}^{(n-1)}(\tau_0) < 0, \quad \text{and } \forall i \in [\![m-1+n; \overline{m}+1-n]\!],\ b_i^{(n-1)}(\tau_0) = 0.$$

Differentiating (45) $(n-1)$ times gives

$$\xi_{i-1}^{(n)} - \xi_i^{(n)} = b_{i-1}^{(n-1)} - 2b_i^{(n-1)} + b_{i+1}^{(n-1)}.$$

From the previous two statements, we see that

$$\begin{gathered}\xi_{m-2+n}^{(n)}(\tau_0) - \xi_{m-1+n}^{(n)}(\tau_0) < 0, \quad \xi_{\overline{m}-n}^{(n)}(\tau_0) - \xi_{\overline{m}+1-n}^{(n)}(\tau_0) < 0, \\ \text{and for } i \in [\![m+n, \overline{m}-n]\!], \quad \xi_{i-1}^{(n)}(\tau_0) - \xi_i^{(n)}(\tau_0) = 0.\end{gathered}$$

Propositions $P_1, \ldots P_{n-1}$ show (in the same way) that for $i \in [\![m-1+n; \overline{m}+1-n]\!]$

$$\dot{\xi}_{i-1}(\tau_0) - \dot{\xi}_i(\tau_0) = \cdots = \xi_{i-1}^{(n-1)}(\tau_0) - \xi_i^{(n-1)}(\tau_0) = 0. \tag{48}$$

Now differentiate (44) $(n-1)$ times using the Leibniz and Faà di Bruno formulas, we see that at τ_0, the only term remaining is the one with n derivative on $\xi_{i-1} - \xi_i$, i.e.

$$b_i^{(n)}(\tau_0) = \frac{(i-1)(k+1-i)}{2\tau_0} e^{-\frac{2}{p-1}(\xi_i(\tau_0) - \xi_{i-1}(\tau_0))} (\xi_{i-1}^{(n)}(\tau_0) - \xi_i^{(n)}(\tau_0)).$$

Hence, we then deduce that

$$b_{m-1+n}^{(n)}(\tau_0) < 0, \quad b_{\overline{m}+1-n}^{(n)}(\tau_0) < 0, \quad \text{and for } i \in [\![m+n, \overline{m}-n]\!],\ b_i^{(n)}(\tau_0) = 0.$$

This is P_n, which concludes the induction. Fixing i and $d \in [\![1, \ldots, d(i)]\!]$, we see that P_d gives $b_i^{(d)}(\tau_0) = 0$. $P_{d(i)+1}$ gives $b_i^{(d(i)+1)}(\tau_0) < 0$. Hence the Claim is proved.

From the Claim (and (46) in the case $d(i) = 0$) and Taylor's expansion, we see that $b_i(t) - b_i(\tau_0) \sim b_i^{(d(i)+1)}(\tau_0)(t - \tau_0)^{d(i)+1}$. In particular, as $b_i^{(d(i)+1)}(\tau_0) < 0$, for some small enough $\varepsilon > 0$, we see that for $t \in (\tau_0, \tau_0 + \varepsilon)$, $b_i(t) < b_i(\tau_0) = B(\tau_0)$. This concludes the proof of Lemma 2.2 in the case where $[\![m, \overline{m}]\!] \subset [\![2, k]\!]$.

Case 2: $m = 1$ **or** $\overline{m} = k + 1$.

We only treat the case where $m = 1$, the other case being similar. Moreover, we only sketch the proof, since it uses the same techniques as Case 1 above.
Note first that since $1 \in J(\tau_0)$ and $b_1(\tau_0) = 0$ by (38), it follows that $B(\tau_0) = 0$.
Then, we claim that

$$\overline{m} \leq k - 1. \tag{49}$$

Indeed, if $\overline{m} = k$, then recalling that $b_{k+1}(\tau_0) = 0$ by (38), we see that $J(\tau_0) = [\![1, k+1]\!]$ and $[\![m, \overline{m}]\!] = [\![1, k]\!]$ is not maximal in $J(\tau_0)$, which is a contradiction.
If $\overline{m} = k + 1$, then for all $i = 2, \ldots, k$, $b_i(\tau_0) = 0$ and $\xi_i(\tau_0) = \xi_1(\tau_0) = 0$ by (40), (38) and (35), which is a contradiction by (42). Thus, (49) holds.

If $\overline{m} = 1$, then $[\![m, \overline{m}]\!] = \{1\}$, hence $[\![m, \overline{m}]\!] \cap [\![2, k]\!] = \emptyset$ and we have nothing to prove.

If $\overline{m} \geq 2$ (which means that $k \geq 3$ by (49)), since $\overline{m} + 1 \notin J(\tau_0)$, arguing as for (46), we see that

$$\dot{b}_{\overline{m}}(\tau_0) < 0, \tag{50}$$

and the conclusion of Lemma follows for $i = \overline{m}$. More generally, as in Case 1, the following claim allows us to conclude:

Claim. For all $i = 2, \ldots, \overline{m}$, we have

$$\dot{b}_i(\tau_0) = \cdots = b_i^{(d(i))}(\tau_0) = 0 \text{ and } b_i^{(d(i)+1)}(\tau_0) < 0 \text{ where } d(i) = \overline{m} - i. \tag{51}$$

As in Case 1, the proof of this claim uses the same iterative procedure based on the proof by induction of the following property for all $n = 1, \ldots, \overline{m} + 2$:

$$\overline{P}_n : \quad b^{(n)}_{\overline{m}+1-n}(\tau_0) < 0, \quad \text{and } \ \forall i \in [\![2, \overline{m} - n]\!], \ b_i^{(n)}(\tau_0) = 0.$$

Note that $\overline{P}_1$ follows by (50).
Note also that unlike the property P_n is Case 1 where the negative higher derivative is propagating both from the right and from the left, here, it propagates only from the right. The non propagation from the left is replaced by the information that that $b_1(\tau)$ is identically zero, hence $b_1^{(j)}(\tau_0) = 0$ for all $j \in \mathbb{N}$.
For more details, see Case 1. This concludes the proof of Lemma 2.2 both in Case 1 and in Case 2. □

Since Proposition 2.1 follows from Lemma 2.2 as shown above, this concludes the proof of Proposition 2.1 as well. □

Step 3: A compact stable by the flow of (36) **under condition** (35)

From the definition (38) of $b_i(\tau)$ and the equations (44) and (45), we write for all $i = 2, \ldots, k$ and $\tau \in [0, \tau_\infty)$,

$$\dot{b}_i = (b_i + \sigma_{i-1})(b_{i-1} - 2b_i + b_{i+1}) \quad i = 2, \ldots, k, \tag{52}$$

where we set by convention

$$b_1(\tau) \equiv b_{k+1}(\tau) \equiv 0. \tag{53}$$

Note that thanks to condition (35) and under the condition

$$b_i(\tau) > -\sigma_{i-1}, \tag{54}$$

this system is equivalent to system (36). We claim the following:

Proposition 2.3 (Compacts stable by the flow of system (52))**.** *For all $\eta \in (0, \frac{1}{5}]$ and $A \geq 0$, the compact $\prod_{i=2}^{k}[-\sigma_{i-1}+\eta, A]$ is stable by the flow of system* (52)*. In particular, any solution of system* (52) *whose initial data is in that compact is global.*

Remark. From the equivalence between system (52) and system (36) under conditions (35) and (54), any solution to the Cauchy problem for system (36) under the condition (35) exists globally in time. The same holds for any solution to system (16) too.

Proof. Consider $\eta \in (0, \frac{1}{5}]$ and $A \geq 0$, and consider initial data for system (52) such that

$$\forall i = 2, \dots, k, \;\; -\sigma_{i-1} + \eta \leq b_i(0) \leq A.$$

In particular, we have $B(0) \leq A$ where $B(\tau)$ is defined in (40). Since $B(\tau)$ is nonincreasing by Proposition 2.1, it follows that

$$\forall i = 2, \dots, k, \;\; b_i(\tau) \leq B(\tau) \leq B(0) \leq A.$$

Now assume by contradiction that for some $\tau > 0$, there exists $i = 2, \dots, k$ such that $b_i(\tau) < -\sigma_i + \eta$. Taking the lowest τ, we end-up by continuity with some $\tau^* \geq 0$ such that

$$(i) \qquad \forall \tau \in [0, \tau^*], \;\; \forall i = 1, \dots, k, \;\; b_i(\tau) \geq -\sigma_i + \eta, \tag{55}$$

$$(ii) \qquad b_j(\tau^*) = -\sigma_j + \eta \text{ and } \dot{b}_j(\tau^*) \leq 0 \text{ for some } j = 2, \dots, k, \tag{56}$$

on the one hand.
On the other hand, noting that $(\sigma_i)_i$ is a (strictly) convex family of semi-integers, so that in particular,

$$\forall i = 1, \dots, k, \;\; -\sigma_{i-1} + 2\sigma_i - \sigma_{i+1} \geq 1/2$$

and remarking that

$$b_{j-1}(\tau^*) \geq -\sigma_{j-1} \text{ and } b_{j+1}(\tau^*) \geq -\sigma_{j+1}$$

(use (55) if the indices are between 2 and k, and use (53) and the definition (37) of σ_i if the indices are either 1 or $k+1$), we see from (52) and (56) that

$$\begin{aligned}
\dot{b}_j(\tau^*) &= (b_j(\tau^*) + \sigma_j)(b_{j-1}(\tau^*) - 2b_j(\tau^*) + b_{j+1}(\tau^*)) \\
&\geq \eta(-\sigma_{j-1} + 2\sigma_j - 2\eta - \sigma_{j+1}) \\
&\geq \varepsilon(1/2 - 2\eta) > 0
\end{aligned}$$

since we choose $\eta \in (0, \frac{1}{5}]$. This is a contradiction by (56). This concludes the proof of Proposition 2.3. □

Step 4: Asymptotic behavior of solutions to system (36) **under condition** (35)

Endowing $\mathbb{R}^k$ with the ℓ^2 norm, we show in the following that any solution to system (16) approaches the particular family of solutions given in Step 1:

Proposition 2.4 (Asymptotic behavior for system (36) under condition (35))**.**
(i) Let $(\xi_i(\tau))_{i=1,\dots,k}$ *be a solution to system* (36) *under condition* (35)*, with initial data at* $\tau = 0$ *satisfying*

$$\forall i = 1, \dots, k, \quad |\xi_i(0)| \le C_0 \tag{57}$$

for some $C_0 > 0$*. Then, the solution is defined for all* $\tau \ge 0$ *and there exists* $C_1(C_0) > 0$ *such that*

$$\forall \tau \ge 0, \quad \sup_i |\xi_i(\tau)| \le C_1 e^{-\tau}.$$

(ii) Let $(\zeta_i(s))_{i=1,\dots,k}$ *be a solution to system* (16) *with initial data given at* $s = 1$*. Then, the solution is defined for all* $s \ge 1$ *and*

$$\forall s \ge 1, \quad \sup_i |\zeta_i(s) - (\bar{\zeta}_i(s) + \zeta_0)| \le C s^{-1} \text{ with } \zeta_0 = \frac{1}{k}\sum_{k=1}^{n} \zeta_i(1),$$

where the $(\bar{\zeta}_i(s))$ *is the explicit solution of system* (16) *introduced in Step 1 above.*

Proof. Let us first derive (ii) from (i), then we will prove (i).
(ii) Consider $(\zeta_i(s))_{i=1,\dots,k}$ a solution to system (16) with initial data given at $s = 1$. Since the center of mass is conserved in time, introducing

$$\xi_i(\tau) = \frac{2}{p-1}\left[\zeta_i(s) - \left(\bar{\zeta}_i(s) + \zeta_0\right)\right] \text{ where } \tau = \log s,$$

we see from Step 1 above that

$$\forall \tau \ge 0, \quad \sum_i \xi_i(\tau) = \sum_i \xi_i(0) = 0 \tag{58}$$

and $(\xi_i(\tau))_i$ satisfies (36). Thus, (ii) follows from (i).
(i) From the remark following Proposition 2.3, we know that $(\xi_i(s))_{i=1,\dots,k}$ is globally defined in time.
Introducing $b_i(\tau)$ as in (38), we recall from Step 3 above the equivalence between system (52) and system (36) under conditions (35) and (54). In particular, using Proposition 2.1, we see that $(b_2(\tau), \dots, b_k(\tau)) \equiv (0, \dots, 0)$ is the only critical point of system (52) and that the functional $B - b$ is a Lyapunov functional, strictly decreasing except at the critical point (see Step 3 above). Since Proposition 2.3 provides us with a compact

$$K(C_0) = \prod_{i=2}^{k} [-\sigma_{i-1} + \eta, A] \text{ for some } \eta = \eta(C_0) > 0 \text{ and } A = A(C_0)$$

stable under the flow of system (52), we see that Lyapunov's theorem applies to this system and yields the fact that $b_i(\tau) \to 0$ as $\tau \to 0$ (see Appendix A below for the statement and the proof of the version of Lyapunov's theorem we use). From the relation (38) between ξ_i and b_i together withe zero barycenter condition (35), we see that

$$\begin{array}{ll} (i) & \forall \tau \geq 0, \;\; |\xi_i(\tau)| \leq C_2 = C_2(C_0), \\ (ii) & \xi_i(\tau) \to 0 \text{ as } \tau \to +\infty. \end{array}$$

Since initial data are chosen in a compact (see (57) above), using the continuity with respect to initial data, for solutions of ODEs on a given time interval, we see that this convergence is uniform, in the sense that

$$\forall \epsilon > 0, \; \exists \tau^*(C_0, \epsilon) > 0 \quad \text{such that} \quad \forall \tau \geq \tau^*, \; \|\boldsymbol{\xi}(\tau)\| \leq \epsilon. \tag{59}$$

Linearizing system (36) near the zero solution, we write

$$\forall \tau \geq 0, \quad \left\|\dot{\boldsymbol{\xi}}(\tau) - M\boldsymbol{\xi}(\tau)\right\| \leq C_3 \|\boldsymbol{\xi}(\tau)\|^2 \quad \text{for some} \quad C_3 = C_3(C_0) > 0, \tag{60}$$

where the $k \times k$ matrix $M = (m_{i,j})_{(i,j) \in [\![1,k]\!]}$, with

$$m_{i,i-1} = \sigma_{i-1}, \quad m_{i,i} = -(\sigma_{i-1} + \sigma_i), \quad m_{i,i+1} = \sigma_i, m_{i,j} = 0 \text{ if } |i-j| \geq 2 \tag{61}$$

and σ_i is defined in (37). We claim the following:

Lemma 2.5 (Eigenvalues of M)**.** *The matrix M is diagonalizable, with real eigenvalues*

$$-\, m_i \equiv -\frac{i(i-1)}{2}, \; \textit{for } i = 1, ..., k, \tag{62}$$

and the associated eigenvectors $\boldsymbol{e}_i$ normalized for the ℓ^∞ norm. If $i = 1$, then $e_1 = {}^t(1, \ldots, 1)$.

Remark.

Proof. Since M is symmetric, it is diagonalizable with real eigenvalues. Furthermore, we can compute

$$(M\boldsymbol{\xi}, \boldsymbol{\xi}) = -\sum_{i=1}^{k-1} \sigma_i (\xi_{i+1} - \xi_i)^2.$$

In particular, $M(x, x) = 0$ if and only if $(\boldsymbol{\xi}, {}^t(1, \ldots, 1))$ is linearly dependent, so that M has 0 as an eigenvalue with eigenvector ${}^t(1, \ldots, 1)$, and the other eigenvalues are negative.
The proof of the exact value (62) of the eigenvalues relies on clever transformation of the matrix M, which are somehow long. We leave them to Appendix C. See Appendix C for the end of the proof of Lemma 2.5. □

With this lemma, we carry on the proof of Proposition 2.4. Now, as $\sum_i \xi_i(\tau) = 0$, we have from Lemma 2.5 that

$$(M\boldsymbol{\xi}, \boldsymbol{\xi}) \leq -\|\boldsymbol{\xi}\|^2,$$

so that

$$\frac{d}{d\tau}\|\boldsymbol{\xi}(\tau)\|^2 \leq -2\|\boldsymbol{\xi}(\tau)\|^2 + C_4\|\boldsymbol{\xi}(\tau)\|^3$$

for some $C_4 = C_4(C_0) > 0$. Since $\boldsymbol{\xi}(\tau) \to 0$ as $\tau \to \infty$, uniformly with respect to initial data satisfying (57) (see (59) above), we see that $\|\boldsymbol{\xi}(\tau)\| \leq C_1 e^{-\tau} = C_1 s^{-1}$ for some $C_1 = C_1(C_0)$. This concludes the proof of Proposition 2.4. □

Part 2: Proof for the perturbed ODE

We now turn to the equation (28) satisfied by $(\zeta_i(s))_i$, which is a perturbation of the autonomous system (16) studied in Part 1. We will prove in fact that when $s \to \infty$, $(\zeta_i(s))_i$ approaches one of the particular solutions of the autonomous system (16) introduced in Step 1 of Part 1. More precisely, we will prove a more accurate version of (26), by showing the existence of $\zeta_0 \in \mathbb{R}$ such that

$$\forall i \in [\![1, k]\!],\ \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 + O(s^{-\eta}) \text{ as } s \to \infty, \tag{63}$$

where $(\bar{\zeta}_i(s))_i$ is introduced in (15).

Recalling that we already have from (14) a less accurate estimate, namely that

$$\forall i \in [\![1, k]\!],\ \zeta_i(s) = \bar{\zeta}_i(s) + O(1) \text{ as } s \to \infty,$$

we write from system (28) that

$$\sum_{i=1}^{k} \dot{\zeta}_i(s) = O\left(\frac{1}{s^{1+\eta}}\right), \text{ hence } \frac{1}{k}\sum_{i=1}^{k} \zeta_i(s) = l + O\left(\frac{1}{s^{\eta}}\right) \text{ as } s \to \infty \tag{64}$$

for some $l \in \mathbb{R}$. Introducing

$$\xi_i(\tau) = \frac{2}{p-1}\left[\zeta_i(s) - (\bar{\zeta}_i(s) + \frac{1}{k}\sum_{i=1}^{k}\zeta_i(s))\right] \text{ with } \tau = \log s, \tag{65}$$

we see from (64) and the definition (15) of $\bar{\zeta}_i(s)$ that $\boldsymbol{\xi} = {}^t(\xi_1, \ldots, \xi_k)$ satisfies

$$\forall \tau \geq \tau_0,\ \left\|\dot{\boldsymbol{\xi}}(\tau) - \tilde{f}(\boldsymbol{\xi}(\tau))\right\| \leq C_0 e^{-\eta\tau},\ \frac{1}{k}\sum_i \xi_i(\tau) = 0,\ \|\boldsymbol{\xi}(\tau)\| \leq C_0, \tag{66}$$

for some positive C_0 and τ_0, where $\tilde{f}$ is the autonomous nonlinearity in the right-hand side of system (36).

In particular, as we will show below, $(\xi_i(\tau))_{i=1,\ldots,k}$ will be close to some solution of system (36) for τ large enough. Since solutions to (36) converge uniformly to 0 by Proposition 2.4, $(\xi_i(\tau))$ will be as close to 0 as we wish, provided that τ is large enough. More precisely, we claim the following:

Claim. For any $\varepsilon > 0$, there exists $\hat{\tau}(\epsilon) > 0$ such that

$$\forall i = 1, \dots, k, \ \ |\xi_i(\hat{\tau})| \le \varepsilon. \tag{67}$$

Let us use first this claim to finish the proof, then we will prove it. From the analysis carried out for the autonomous system (36) in the proof of Proposition 2.4, we linearize system (66) then use the spectral properties of the matrix M (61) to write

$$\forall \tau \ge \hat{\tau}, \ \ \frac{d}{d\tau}\|\boldsymbol{\xi}(\tau)\|^2 \le -2\|\boldsymbol{\xi}(\tau)\|^2 + C\|\boldsymbol{\xi}(\tau)\|^3 + Ce^{-\eta\tau}.$$

Taking ε small enough and starting from the estimate (67) at $\tau = \hat{\tau}$, we get by a classical integration

$$\forall \tau \ge \hat{\tau}, \ \ \|\boldsymbol{\xi}(\tau)\| \le Ce^{-\eta\tau}$$

(we recall here that already in [26], the constant $\eta = \eta(p) > 0$ was chosen small enough). Using the definition (65) of $\xi_i(s)$ together with (64), we see that (63) holds and so does the conclusion of Theorem 1 too. It remains then to prove the Claim in order to conclude the proof of Theorem 1.
For any $\bar{\tau} \ge \tau_0$, let us introduce $(\bar{\xi}(\tau))_{\bar{\tau}, i=1,\dots,k}$ the solution of the unperturbed system (36) with initial data

$$\bar{\xi}_{\bar{\tau},i}(0) = \xi_i(\bar{\tau}). \tag{68}$$

Using the continuity of solutions to ODEs with respect to the coefficients of the equations, we write from (68) and (66) for any $L > 0$,

$$\sup_{i=1,\dots,k;\tau\in[\bar{\tau},\bar{\tau}+L]} |\xi_i(\tau) - \bar{\xi}_{\bar{\tau},i}(\tau - \bar{\tau})| \le C(L)e^{-\eta\bar{\tau}}. \tag{69}$$

Since we have from (66),

$$\forall i = 1, \dots, k, \ \ |\bar{\xi}_{\bar{\tau},i}(0)| \le C_0, \ \ \sum_{j=1}^{k} \bar{\xi}_{\bar{\tau},j}(0) = 0, \ \text{hence} \ \sum_{j=1}^{k} \xi_{\bar{\tau},j}(\tau) = 0 \text{ for all } \tau \ge 0,$$

given $\epsilon > 0$, we see from (i) of Proposition 2.4 that for some $\tau^*(C_0, \varepsilon) > 0$, we have

$$\forall i = 1, \dots, k, \ \ |\bar{\xi}_{\bar{\tau},i}(\tau^*)| \le \frac{\varepsilon}{2}.$$

Using (69) with $L = \tau^*(C_0, \epsilon)$, we see that

$$\forall i = 1, \dots, k, \ \ |\xi_i(\bar{\tau} + \tau^*)| \le |\bar{\xi}_{\bar{\tau},i}(\tau^*)| + C(\tau^*)e^{-\eta\bar{\tau}} \le \frac{\varepsilon}{2} + C(\tau^*)e^{-\eta\bar{\tau}} \le \varepsilon$$

provided that we take $\bar{\tau} = \hat{\tau}(C_0, \epsilon)$ large enough. Taking $\hat{\tau} = \bar{\tau} + \tau^*$, we see that the Claim is proved, and so is (63), (26) and Theorem 1 too, thanks to the reduction we wrote after giving (26). □

2.2 Refined geometrical estimates for the blow-up set

This section is devoted to the proof of Corollary 2, which consists in a refinement of estimate (11) itself coming from [27].

Proof of Corollary 2. From translation invariance of equation (1), we may assume that $x_0 = 0$ and $T(x_0) = 0$. Up to replacing u by $-u$, we know from Theorem 1 that

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^i \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty, \tag{70}$$

where $k = k(0) \geq 2$,

$$d_i(s) = -\tanh \zeta_i(s), \quad \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 \tag{71}$$

for some $\zeta_0 \in \mathbb{R}$, and $(\bar{\zeta}_i(s))_i$ introduced above in (15) is the solution of system (16) with zero center of mass. From symmetry invariance, we may treat the case $x < 0$ first, then, at the end of the proof, we will give indications on how to recover the case $x > 0$.
Case $x < 0$: All that we need to do is to review the proof of estimate (11) in [27] and mechanically improve its estimates thanks to the new refined blow-up behavior we have just proved with Theorem 1.
In [27], we prove the following estimate for w_x, where $x < 0$ with $|x|$ small:

Lemma 2.6. *For all $\epsilon > 0$, there exists $\delta > 0$ and $L > 0$ such that for all $x \in (-\delta, 0)$ and $L_k \geq L$, we have*

$$\left\| \begin{pmatrix} w_x(s_k) \\ \partial_s w_x(s_k) \end{pmatrix} + \begin{pmatrix} \kappa\left(\bar{d}_1^*(s_k)\right) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} + |\bar{\lambda}_1 - 1| \leq \epsilon,$$

where

$$\bar{d}_1^*(s_k) = \frac{\bar{d}_1(s_k)}{1 + \bar{\nu}_1(s_k)}, \quad \bar{d}_1(s_k) = d_1(S_k), \quad \bar{\nu}_i(s_k) = [b - (1 - \bar{d}_i(s_k))] x e^{s_k}, \tag{72}$$

$$s_k = |\log|x|| + L_k, \quad S_k = -\log[|x|(1-b) + e^{-s_k}] \tag{73}$$

and

$$\lambda_1 = \frac{(1 - \bar{d}_1^2)^{\frac{1}{p-1}}}{[(1 + \bar{\nu}_1)^2 - \bar{d}_1^2]^{\frac{1}{p-1}}}. \tag{74}$$

Proof. For the proof, see Section 3 in [27], in particular the proof of Proposition 3.10 in that paper. Nevertheless, let us summarize in the following the 3 main arguments of the proof, and refer the interested reader to [27] for more details:
- applying the similarity variables' transformation (4) twice, we first recover an estimate on $u(x,t)$, then on $w_x(s)$, but only on the interval $y \in (y_1(x,s), 1)$, for some $y_1(x,s) > -1$;
- using a very good understanding of the dynamics of equation (5) near the sum of decoupled solitons (the same dynamical study that we use later in this paper for the

proof of Theorem 3, see Appendix B below), we recover the same estimate on the whole interval $y \in (-1,1)$;
- the estimate we recover on $w_x(s)$ shows in fact that, like w_0, $w_x(s)$ is still a sum of k decoupled solitons, though the solitons are no longer "pure" (i.e. given by $\kappa(d)$ defined in (10)), but generalized, given by the family $\kappa^*(d,\nu)$ defined in (80). As time increases, this family starts to loose its members, starting from the right soliton (with index $i=k$) up to the second soliton (with index $i=2$) which is lost at time $s=s_k$ given above in (73). Thanks to an energy argument, we show that this unique left soliton is a "pure" soliton, in other words given by $-\kappa(\bar{d}_1^*)$ where $\bar{d}_1^*(s_k)$ is defined above in (72). □

This lemma shows that $w_x(s_k)$ is close to $-\kappa(\bar{d}_1^*)$. As a matter of fact, we have the following trapping result from Merle and Zaag [22] which asserts that w_x will eventually converge to a nearby soliton, with a near parameter:

Proposition 2.7 (A trapping criterion for non-characteristic points)**.** *There exist $\epsilon_0>0$ and $C_0>0$ such that if for some $x_0\in\mathbb{R}$, $s_0\ge -\log T(x_0)$, $\theta\in\{\pm1\}$, $d\in(-1,1)$ and $\epsilon\in(0,\epsilon_0]$, we have*

$$\left\|\begin{pmatrix} w_{x_0}(s_0)\\ \partial_s w_{x_0}(s_0)\end{pmatrix}-\theta\begin{pmatrix}\kappa(d)\\ 0\end{pmatrix}\right\|_{\mathscr{H}}\le\epsilon,$$

then $x_0\in\mathscr{R}$, $w_{x_0}(s)\to\theta\kappa(T'(x_0))$ as $s\to\infty$ and $|\arg\tanh T'(x_0)-\arg\tanh d|\le C_0\epsilon$.

Proof. The original statement of this result was proved in Theorem 3 in [22]. The version we are citing comes from (ii) of Proposition 1.1. in [27]. See this latter paper for the precise justification. □

From Lemma 2.6 and this trapping criterion, we already derived in [27] the fact that x is non-characteristic and that

for all $\epsilon>0$, there exists $\delta>0$ and $L>0$ such that for all $x\in(-\delta,0)$ and $L_k\ge L$, we have

$$\left|\arg\tanh(T'(x))-\arg\tanh\left(\bar{d}_1^*(s_k)\right)\right|\le\epsilon. \tag{75}$$

Starting from this estimate and Lemma 2.6, we still follow the proof of Proposition 3.10 in [27], adding however the following new ingredient, which directly follows from estimate (17) proved in Theorem 1, and makes the only novelty with respect to [27]:

$$1-d_1(s)\sim 2e^{2(\bar{\alpha}_1+\zeta_0)}s^{-\gamma_1}\text{ as }s\to\infty.$$

Recall first from (11) and the definition (29) of γ_1 that for $|x|$ small enough, we have

$$\frac{1}{C|\log|x||^{\gamma_1}}\le b\le\frac{C}{|\log|x||^{\gamma_1}}\text{ and }\frac{1}{C|\log|x||^{\gamma_1}}\le|T'(x)-1|\le\frac{C}{|\log|x||^{\gamma_1}}. \tag{76}$$

Therefore, from the definitions given in Lemma 2.6 above, we have as $x\to0$,

$$S_k=-\log|x|-\log(1+e^{-L_k})+O(|\log|x||^{-\gamma_1}),$$

$$1 - \bar{d}_1(s_k) \sim 2e^{2(\bar{\alpha}_1+\zeta_0)}|\log|x||^{-\gamma_1}, \quad (77)$$

$$\bar{\nu}_1(s_k) = O(|\log|x||^{-\gamma_1}). \quad (78)$$

Consider then $\epsilon > 0$. Since we have by definition (74) of $\bar{\lambda}_1$:

$$\bar{\lambda}_1^{-(p-1)} = \left(1 + \frac{\bar{\nu}_1}{1-\bar{d}_1}\right)\left(1 + \frac{\bar{\nu}_1}{1+\bar{d}_1}\right),$$

using (75), (77) and (78), we see that for $|x|$ small and L_k large, we have

$$\frac{|\bar{\nu}_1|}{1-\bar{d}_1} \le C\epsilon.$$

Since we have $1 - \bar{d}_1^* = 1 - \bar{d}_1 + O(\bar{\nu}_1)$ for small $\bar{\nu}_1$ from (72), using (77), the last line gives for $|x|$ small and L_k large:

$$\left|1 - \bar{d}_1^* - 2e^{2(\bar{\alpha}_1+\zeta_0)}|\log|x||^{-\gamma_1}\right| \le C\epsilon|\log|x||^{-\gamma_1}. \quad (79)$$

Therefore, from (75) together with (79) and (76), we write for $x < 0$, $|x|$ small and L_k large:

$$\begin{aligned} \left|T'(x) - \bar{d}_1^*\right| &\le \max\left(1 - (T'(x))^2, 1 - (\bar{d}_1^*)^2\right)\left|\arg\tanh(T'(x)) - \arg\tanh\left(\bar{d}_1^*(s_k)\right)\right| \\ &\le C\epsilon|\log|x||^{-\gamma_1}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, using (79), we see that (20) follows when $x < 0$. By integration, we get (21), also when $x < 0$. It remains then to treat the case $x > 0$.
Case $x > 0$: Introducing $u^\sharp(x^\sharp, t) = (-1)^k u(-x^\sharp, t)$, we see that $u^\sharp$ is also a solution of (1) with 0 as a characteristic point and that $T^\sharp(x^\sharp) = T(-x^\sharp)$. Thus, we reduce to the study of $u^\sharp$ for $x^\sharp < 0$.
Since we have from the definition of the similarity variables' transformation (4) that $w_0^\sharp(y^\sharp, s) = (-1)^k w_0(-y^\sharp, s)$, we derive from (70) the following estimate (after reversing the order of the solitons):

$$\left\|\begin{pmatrix} w_0^\sharp(s) \\ \partial_s w_0^\sharp(s)\end{pmatrix} - \begin{pmatrix} \sum_{i=1}^k (-1)^i \kappa(D_i(s)) \\ 0 \end{pmatrix}\right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty,$$

where $D_i(s) = -d_{k-i}(s)$ satisfies the following from (71) and the symmetry relation (33) on $\bar{\zeta}_i(s)$:

$$D_i(s) = -\tanh \Xi_i(s) \text{ and } \Xi_i(s) = -\zeta_{k-i}(s) = -\bar{\zeta}_{k-i}(s) - \zeta_0 = \bar{\zeta}_i(s) - \zeta_0.$$

Thus, up to replacing ζ_0 by $-\zeta_0$, we see that we are in the case "$x < 0$" already treated above, and the result follows for $u^\sharp$ with $x^\sharp < 0$, hence for u with $x > 0$. This concludes the proof of Corollary 2. □

3 Construction of a multi-soliton solution in similarity variables

In this section, we construct a multi-soliton solution in similarity variables for equation (5). Technically, we use the dynamical system formulation introduced in [27]. For that reason, we introduce for all $d \in (-1,1)$ and $\nu > -1 + |d|$, $\kappa^*(d,\nu,y) = (\kappa_1^*, \kappa_2^*)(d,\nu,y)$ where

$$\kappa_1^*(d,\nu,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{2}{p-1}}}, \quad \kappa_2^*(d,\nu,y) = \nu\partial_\nu \kappa_1^*(d,\nu,y) = -\frac{2\kappa_0\nu}{p-1}\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{p+1}{p-1}}}. \tag{80}$$

In our paper, we refer to these functions as "generalized solitons" or solitons for short. Note also that for any $\mu \in \mathbb{R}$, $\kappa^*(d,\mu e^s,y)$ is a solution to equation (5). Note that $\kappa^*(d,\mu e^s,y) \to (\kappa(d),0)$ in $\mathscr{H}$ as $s \to -\infty$. Moreover ,
- when $\mu = 0$, we recover the stationary solutions $(\kappa(d),0)$ defined in (10);
- when $\mu > 0$, the solution exists for all $(y,s) \in (-1,1) \times \mathbb{R}$ and converges to 0 in $\mathscr{H}$ as $s \to \infty$ (it is a heteroclinic connection between $(\kappa(d),0)$ and 0);
- when $\mu < 0$, the solution exists for all $(y,s) \in (-1,1) \times \left(-\infty, \log\left(\frac{|d|-1}{\mu}\right)\right)$ and blows up at time $s = \log\left(\frac{|d|-1}{\mu}\right)$.

We also introduce for $l = 0$ or 1, for any $d \in (-1,1)$ and $r \in \mathscr{H}$,

$$\Pi_l^d(r) = \phi\left(W_l(d), r\right) \tag{81}$$

where

- $\phi(q,r) = \int_{-1}^{1} \left(q_1 r_1 + q_1' r_1'(1-y^2) + q_2 r_2\right)\rho dy = \int_{-1}^{1} \left(q_1\left(-\mathscr{L} r_1 + r_1\right) + q_2 r_2\right)\rho dy,$
- $W_l(d,y) = (W_{l,1}(d,y), W_{l,2}(d,y))$

with

$$W_{1,2}(d,y)(y) = {}_1(d)\frac{(1-d^2)^{\frac{1}{p-1}}(1-y^2)}{(1+dy)^{\frac{2}{p-1}+1}}, \quad W_{0,2}(d,y) = {}_0\frac{(1-d^2)^{\frac{1}{p-1}}(y+d)}{(1+dy)^{\frac{2}{p-1}+1}}, \tag{82}$$

for some positive ${}_1(d)$ and ${}_0$, and $W_{l,1}(d,y) \in \mathscr{H}_0$ is uniquely determined as the solution of

$$-\mathscr{L} r + r = \left(l - \frac{p+3}{p-1}\right) W_{l,2}(d) - 2y\partial_y W_{l,2}(d) + \frac{8}{p-1}\frac{W_{l,2}(d)}{1-y^2} \tag{83}$$

normalized by the fact that $\Pi_l^d(F_l(d)) = \phi\left(W_l(d), F_l(d)\right)$, where

$$F_1(d,y) = (1-d^2)^{\frac{p}{p-1}}\begin{pmatrix}(1+dy)^{-\frac{2}{p-1}-1}\\(1+dy)^{-\frac{2}{p-1}-1}\end{pmatrix}, \quad F_0(d,y) = (1-d^2)^{\frac{1}{p-1}}\begin{pmatrix}\frac{y+d}{(1+dy)^{\frac{2}{p-1}+1}}\\0\end{pmatrix}$$

(see estimate (3.57) in [26] for more details).

Given $k \geq 2$ and $s_0 > 0$, we will construct the multi-solution as a solution to the Cauchy problem of equation (5) with initial data

$$w(y, s_0) = \sum_{i=1}^{k} (-1)^i \kappa^* \left(\bar{d}_i(s_0), \nu_{i,0}\right) \text{ with } |\nu_{i,0}| \leq s_0^{-\frac{1}{2}-|\gamma_i|}, \tag{84}$$

where $\bar{d}_i(s_0)$ is fixed by

$$\bar{d}_i(s_0) = -\tanh \bar{\zeta}_i(s_0),$$

$\bar{\zeta}_i(s_0)$ is defined in (15) and γ_i is defined in (29). Such a solution will be denoted by $w(s_0, (\nu_{i,0})_i, y, s)$, or, when there is no ambiguity, by $w(y, s)$ or $w(s)$ for short. We will show that when s_0 is fixed large enough, we can fine-tune the parameters $\nu_{i,0}$ in the intervals $[-s_0^{-\frac{1}{2}-|\gamma_i|}, s_0^{-\frac{1}{2}-|\gamma_i|}]$ so that the solution $w(s_0, (\nu_{i,0})_i, y, s)$ (or $w(y, s)$ for short) will decompose as a sum of k decoupled solitons. This is the aim of the section:

Proposition 3.1 (A multi-soliton solution in the $w(y, s)$ setting)**.** *For any integer $k \geq 2$, there exist $s_0 > 0$, $\nu_{i,0} \in \mathbb{R}$ for $i = 1, \dots, k$ and $\zeta_0 \in \mathbb{R}$ such that equation* (5) *with initial data (at $s = s_0$) given by* (84) *is defined for all $(y, s) \in (-1, 1) \times [s_0, \infty)$, satisfies $(w(s), \partial_s w(s)) \in \mathscr{H}$ for all $s \geq s_0$, and*

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty, \tag{85}$$

for some continuous $d_i(s) = -\tanh \zeta_i(s)$ satisfying

$$\zeta_i(s) - \bar{\zeta}_i(s) \to \zeta_0 \text{ as } s \to \infty \text{ for } i = 1, ..., k \tag{86}$$

where the $\bar{\zeta}_i(s)$ are introduced in (15).

Remark. Note from (84) that initial data are in $H^1 \times L^2(-1, 1)$. Going back to the $u(x, t)$ formulation, we see that initial data is also in $H^1 \times L^2(-1, 1)$ of the initial section of the backward light-cone. Therefore, from the solution to the Cauchy-problem in light-cones, we see that the solution stays in $H^1 \times L^2$ of any section.

As one can see from (84), at the initial time $s = s_0$, $w(y, s_0)$ is a pure sum of solitons. From the continuity of the flow associated with equation (5) in $\mathscr{H}$ (this continuity comes from the continuity of the flow associated with equation (1) in $H^1 \times L^2$ of sections of backward light-cones), $w(y, s)$ will stay close to a sum of solitons, at least for a short time after s_0. In fact, we can do better, and impose some orthogonality conditions, killing the zero and expanding directions of the linearized operator of equation (5) around the sum of solitons. The following modulation technique from Merle and Zaag in [27] is crucial for that:

Proposition 3.2 (A modulation technique; Proposition 2.1 of [27]). *For all $A \geq 1$, there exist $E_0(A) > 0$ and $\epsilon_0(A) > 0$ such that for all $E \geq E_0$ and $\epsilon \leq \epsilon_0$, if $v \in \mathscr{H}$ and for all $i = 1, ..., k$, $(\hat{d}_i, \hat{\nu}_i) \in (-1,1) \times \mathbb{R}$ are such that*

$$-1 + \frac{1}{A} \leq \frac{\hat{\nu}_i}{1 - |\hat{d}_i|} \leq A, \;\; \hat{\zeta}^*_{i+1} - \hat{\zeta}^*_i \geq E \text{ and } \|\hat{q}\|_{\mathscr{H}} \leq \epsilon$$

where $\hat{q} = v - \sum_{j=1}^{k} (-1)^j \kappa^*(\hat{d}_j, \hat{\nu}_j)$ and $\hat{d}^*_i = \frac{\hat{d}_i}{1+\hat{\nu}_i} = -\tanh \hat{\zeta}^*_i$, then, there exist (d_i, ν_i) such that for all $i = 1, \ldots, k$ and $l = 0, 1$,

- $\Pi_l^{d_i^*}(q) = 0$ *where* $q = v - \sum_{j=1}^{k} (-1)^j \kappa^*(d_j, \nu_j)$,
- $\left| \frac{\nu_i}{1 - |d_i|} - \frac{\hat{\nu}_i}{1 - |\hat{d}_i|} \right| + |\zeta^*_i - \hat{\zeta}^*_i| \leq C(A) \|\hat{q}\|_{\mathscr{H}} \leq C(A)\epsilon$,
- $-1 + \frac{1}{2A} \leq \frac{\nu_i}{1 - |d_i|} \leq A + 1, \;\; \zeta^*_{i+1} - \zeta^*_i \geq \frac{E}{2}$ *and* $\|q\|_{\mathscr{H}} \leq C(A)\epsilon$

where $d^*_i = \frac{d_i}{1+\nu_i} = -\tanh \zeta^*_i$.

Let us apply this proposition with $r = w(y, s_0)$ (84), $\hat{d}_i = \bar{d}_i(s_0)$ and $\hat{\nu}_i = \nu_{i,0}$. Clearly, we have $\hat{q} = 0$. Then, from (84), (15) and straightforward calculations, we see that

$$\frac{|\hat{\nu}_i|}{1 - |\hat{d}_i|} \leq \frac{C}{\sqrt{s_0}} \text{ and } \hat{\zeta}^*_{i+1} - \hat{\zeta}^*_i \geq \frac{(p-1)}{4} \log s_0$$

for s_0 large enough. Therefore, Proposition 3.2 applies with $A = 2$ and from the continuity of the flow associated with equation (5) in $\mathscr{H}$, we have a maximal $\bar{s} = \bar{s}(s_0, (\nu_{i,0})_i) > s_0$ such that w exists for all time $s \in [s_0, \bar{s})$ and w can be modulated in the sense that

$$w(y, s) = \sum_{i=1}^{k} (-1)^i \kappa^*(d_i(s), \nu_i(s)) + q(y, s) \tag{87}$$

where the parameters $d_i(s)$ and $\nu_i(s)$ are such that for all $s \in [s_0, \bar{s}]$,

$$\Pi_l^{d_i^*(s)}(q(s)) = 0, \;\; \forall l = 0, 1, \;\; i = 1, \ldots, k$$

and

$$\frac{|\nu_i(s)|}{1 - |d_i(s)|} \leq s_0^{-1/4}, \;\; \zeta^*_{i+1}(s) - \zeta^*_i(s) \geq \frac{(p-1)}{8} \log s_0 \text{ and } \|q(s)\|_{\mathscr{H}} \leq \frac{1}{\sqrt{s_0}}. \tag{88}$$

Two cases then arise:

- either $\bar{s}(s_0, (\nu_{i,0})_i) = +\infty$;

- or $\bar{s}(s_0, (\nu_{i,0})_i) < +\infty$ and one of the $\leq$ symbol in (88) has to be replaced by a $=$ symbol.

At this stage, we see that controlling the solution $w(s) \in \mathscr{H}$ is equivalent to controlling $q \in \mathscr{H}$, $(d_i(s))_i \in (-1,1)^k$ and $(\nu_i(s))_i \in \mathbb{R}^k$. Introducing

$$J = \sum_{i=2}^{k} e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})}, \quad \bar{J} = \sum_{i=1}^{k} \frac{|\nu_i|}{1-d_i^2}, \quad \hat{J} = \sum_{i=1}^{k} e^{-\frac{\bar{p}}{p-1}(\zeta_i - \zeta_{i-1})} \tag{89}$$

where

$$\bar{p} = \begin{cases} p & \text{if } p < 2, \\ 2 - 1/100 & \text{if } p = 2, \\ 2 & \text{if } p > 2, \end{cases} \tag{90}$$

we recall from [27] and [26] differential and integral equations satisfied by those components:

Proposition 3.3 (Dynamics of the parameters). *There exists $\delta > 0$ such that for s_0 large enough and for all $s \in [s_0, \bar{s})$, we have*

$$\frac{|\dot{\nu}_i - \nu_i|}{1-d_i^2} \leq C\left(\|q\|_{\mathscr{H}}^2 + J + \|q\|_{\mathscr{H}} \bar{J}\right) \tag{91}$$

$$\left|\frac{\dot{\zeta}_i}{c_1(p)} - (e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)})\right| \leq C(\|q\|_{\mathscr{H}}^2 + (J + \|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}), \tag{92}$$

$$\|q(s)\|_{\mathscr{H}}^2 \leq Ce^{-\delta(s-s_0)}\|q(s_0)\|_{\mathscr{H}}^2 + C\hat{J}(s)^2, \tag{93}$$

where $\zeta_i(s) = -\arg\tanh d_i(s)$, $c_1(p) > 0$ was already introduced in Proposition 3.2 of [26], J, $\bar{J}$ and $\hat{J}$ are introduced in (89).

Proof. This statement is a small refinement of Claims 3.8 and 3.9 in [26] and Proposition 3.2 in [27] where the authors handle the same equation. For that reason, we leave the proof to Appendix B. $\square$

From (92) and (88), we see that $(\zeta_i(s))_i$ satisfies a perturbed version of the system (16) studied in Section 2. Moreover, as one can see from our purpose stated in Proposition 3.1, our aim is to show the existence of a solution with $\zeta_i(s) \sim \bar{\zeta}_i(s)$ as $s \to \infty$ (at least when $i \neq \frac{k+1}{2}$). Hence, it is natural to do as in (34) in Section 2 and linearize system (92) around $(\bar{\zeta}_i(s))_i$ by introducing

$$\xi_i(s) = \frac{2}{p-1}(\zeta_i(s) - \bar{\zeta}_i(s)) \tag{94}$$

If $\boldsymbol{\xi}(s) = (\xi_1(s), \ldots, \xi_k(s))$, then we obtain the following perturbed version of system (60): For all $s \in [s_0, \bar{s})$:

$$\left|\dot{\boldsymbol{\xi}}(s) - \frac{1}{s}M\boldsymbol{\xi}(s)\right| \leq \frac{C}{s}|\boldsymbol{\xi}(s)|^2 + C(\|q(s)\|_{\mathscr{H}}^2 + (J(s) + \|q(s)\|_{\mathscr{H}})\bar{J}(s) + J(s)^{1+\delta}), \tag{95}$$

where the self-adjoint $k \times k$ matrix M is introduced in (61) and is diagonalizable as stated in Lemma 2.5 (note that here we keep the time variable s and don't work with $\tau = \log s$). It is then natural to work in the basis defined by its eigenvectors $(\boldsymbol{e}_i)_i$ by introducing $\boldsymbol{\phi}(s) = (\phi_1(s), \dots, \phi_k(s))$ defined by

$$\boldsymbol{\xi}(s) = \sum_{i=1}^{k} \phi_i(s) \boldsymbol{e}_i. \tag{96}$$

Note that thanks to all these changes of variables, controlling w is equivalent to the control of $(q, \boldsymbol{\phi}, (\nu_i)_i)$. As a matter of fact, in order to control w near multi-solitons, we introduce the following set:

Definition 3.4 (Definition of a shrinking set for the parameters)**.** *We say that $w \in \mathscr{V}(s_0, s)$ if and only if*

$$\begin{array}{l} s^{1/2+\eta}\|q\|_{\mathscr{H}} \le 1, \quad \forall i = 1, \dots, k, \quad s^{1/2+|\gamma_i|}|\nu_i| \le 1, \\ \forall i = 2, \dots, k, \quad s^{\eta}|\phi_i| \le 1, \quad \textit{and} \quad s_0^{\eta}|\phi_1| \le 1, \end{array}, \tag{97}$$

where

$$\eta = \frac{1}{4} \min\left\{1, \delta, \frac{\bar{p}}{2} - \frac{1}{2}\right\}, \tag{98}$$

$\delta > 0$ is defined in Proposition 3.3 and $\bar{p}$ is defined in (90)*.*

From the existence of $\bar{s}$, we know that there is a maximal $s^*(s_0, (\nu_{i,0})_i) \in [s_0, \bar{s})$ such that for all $s \in [s_0, s^*)$, $w(s) \in \mathscr{V}(s_0, s)$ and:
- either $s^* = +\infty$,
- or $s^* < +\infty$ and from continuity, $w(s^*) \in \partial\mathscr{V}(s_0, s^*)$, in the sense that one $\le$ symbol in (97) has to be replaced by the $=$ symbol.

Our aim is to show that for s_0 large enough, one can find a parameter $(\nu_{i,0})_i$ in $\prod_{i=1}^{k}[-s_0^{-\frac{1}{2}-|\gamma_i|}, s_0^{-\frac{1}{2}-|\gamma_i|}]$ such that

$$s^*(s_0, (\nu_{i,0})_i) = +\infty. \tag{99}$$

Introducing

$$\tilde{J} = \sum_{j=2}^{k} \phi_j^2 \tag{100}$$

(note that the sum's index runs from 2 to k, and not from 1 to k), we derive from (92) the following differential inequality satisfied by $\boldsymbol{\phi}(s)$:

Corollary 3.5 (Dynamics for ϕ_i)**.** *For all $s \in [s_0, s^*)$,*

$$\left|\dot{\phi}_i + \frac{m_i}{s}\phi_i\right| \le C\frac{\tilde{J}}{s} + C\left(\|q\|_{\mathscr{H}}^2 + (J + \|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}\right). \tag{101}$$

Remark. This corollary is more subtle than one may think. Indeed, if we project the differential inequality (95) on the eigenvalues of M, then we trivially obtain almost the same identity as (101), except that with this trivial way, we have an additional term in the right-hand side : $C\frac{\phi_1^2}{s}$ (remember that in the definition of $\tilde{J}$, the index runs from 2 to k, and not from 1 to k). With more work, we get the more subtle version, as one can see from the proof below.

Proof. This is a direct consequence of the estimate (92). First recall from the definitions (15), (31) and (37) of $\bar{\zeta}_i(s)$, $\bar{\alpha}_i$ and σ_i that

$$e^{-\frac{2}{p-1}(\bar{\zeta}_i - \bar{\zeta}_{i-1})} = \frac{p-1}{2c_1}\frac{\sigma_{i-1}}{s}.$$

Then, as $w \in \mathscr{V}(s_0, s)$, ξ_i defined in (94) are bounded, so that we have the expansion (uniform in s):

$$\begin{aligned} e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} &= e^{-\frac{2}{p-1}(\bar{\zeta}_i - \bar{\zeta}_{i-1}) - (\xi_i - \xi_{i-1})} \\ &= e^{-\frac{2}{p-1}(\bar{\zeta}_i - \bar{\zeta}_{i-1})}\left(1 - (\xi_i - \xi_{i-1}) + O((\xi_i - \xi_{i-1})^2\right) \\ &= \frac{p-1}{2c_1}\frac{\sigma_{i-1}}{s}\left(1 - (\xi_i - \xi_{i-1}) + O((\xi_i - \xi_{i-1})^2\right). \end{aligned}$$

Hence, from the differential identities (16) and (92) satisfied by $\bar{\zeta}_i$ and ζ_i, the equation on ξ_i writes (recall from (37) that $\sigma_0 = \sigma_k = 0$)

$$\begin{aligned} \dot{\xi}_i &= \frac{2}{p-1}(\dot{\zeta}_i - \dot{\bar{\zeta}}_i) \\ &= -\frac{\sigma_{i-1}}{s}\left(\xi_i - \xi_{i-1} + O(\xi_i - \xi_{i-1})^2\right) + \frac{\sigma_i}{s}(\xi_i - \xi_{i-1} + O(\xi_i - \xi_{i-1})^2) \\ &\quad + O(\|q\|_{\mathscr{H}}^2 + (J + \|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}) \\ &= \frac{1}{s}\left(\sigma_{i-1}\xi_{i-1} - (\sigma_{i-1} + \sigma_i)\xi_i + \sigma_i\xi_{i+1} + O\left(\sum_{i=2}^k |\xi_i - \xi_{i-1}|^2\right)\right) \\ &\quad + O(\|q\|_{\mathscr{H}}^2 + (J + \|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}), \end{aligned}$$

so that we have from the definition (61) of the matrix M

$$\dot{\boldsymbol{\xi}} = \frac{1}{s}M\boldsymbol{\xi} + O\left(\frac{\sum_{i=2}^k |\xi_i - \xi_{i-1}|^2}{s}\right) + O(\|q\|_{\mathscr{H}}^2 + (J + \|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}).$$

Note that this differential inequality is already more accurate than (95) which was obtained by a rough Taylor expansion of (92).
Now if we denote $\boldsymbol{e}_i = {}^t(e_{i,1}, \ldots, e_{i,k})$ the eigenvector of M defined in Lemma 2.5 (recall that $e_1 = {}^t(1, \ldots, 1)$), we see from the definition (96) of $\phi(s)$ that

$$\xi_i - \xi_{i-1} = \sum_{j=1}^k \phi_j(e_{j,i} - e_{j,i-1}) = \sum_{j=2}^k \phi_j(e_{j,i} - e_{j,i-1}),$$

and from this, we deduce:

$$\sum_{i=2}^{k} |\xi_i - \xi_{i-1}|^2 = O(\tilde{J})$$

where $\tilde{J}$ is defined in (100). Hence, when projecting the last relation of $\dot{\boldsymbol{\xi}}$ on the $\boldsymbol{e}_i$, we obtain the desired relation. This concludes the proof of Corollary 3.5. □

With the differential and integral inequalities in Proposition 3.3 and Corollary 3.5, we are in a position to prove the following proposition, which directly implies Proposition 3.1:

Proposition 3.6 (A solution $w(y,s) \in \mathscr{V}(s_0, s)$)**.** *For s_0 large enough, there exists $(\nu_{i,0})_i \in \prod_{i=1}^{k}[-s_0^{-\frac{1}{2}-|\gamma_i|}, s_0^{-\frac{1}{2}-|\gamma_i|}]$ such that equation* (5) *with initial data (at $s = s_0$) given by* (84) *is defined for all $(y,s) \in (-1,1) \times [s_0, \infty)$ and satisfies $w(s) \in \mathscr{V}(s_0, s)$ for all $s \geq s_0$.*

Proof of Proposition 3.6. In fact, we started the proof of this proposition right after the statement of Proposition 3.1. For the sake of clearness, we summarize here all the previous arguments, and conclude the proof thanks to a topological argument.
Let s_0 be large enough. For simplicity in the exposition, we work with rescaled functions. Define $\mathbb{B}$ (resp. $\mathbb{S}$) the unit ball (resp. sphere) in $(\mathbb{R}^k, \ell^\infty)$, and the rescaling function

$$\Gamma_s : \boldsymbol{\nu} = {}^t(\nu_1, \ldots, \nu_k) \mapsto {}^t(s^{-1/2-|\gamma_1|}\nu_1, \ldots, s^{-1/2-|\gamma_k|}\nu_k), \tag{102}$$

For all $\boldsymbol{\nu} \in \mathbb{B}$, we consider the solution $w(s_0, \boldsymbol{\nu}, y, s)$ (or $w(y,s)$ for short) to the equation (5), with initial condition at time s_0 given by (84) with

$$(\nu_{i,0})_i = \Gamma_{s_0}(\boldsymbol{\nu})_i.$$

As we showed after the statement of Proposition 3.2, $w(y,s)$ can be modulated (up to some time $\bar{s} = \bar{s}(s_0, \boldsymbol{\nu}) > s_0$) into a triplet $(q(s), (d_i(s))_i, (\nu_i(s))_i)$. From the uniqueness of such a decomposition (which is a consequence of the application of the implicit function theorem, see the proof of Proposition 2.1 in [27]), we obviously have

$$q(s_0) = 0, \quad d_i(s_0) = \bar{d}_i(s_0) \text{ and } \nu_i(s_0) = \Gamma_{s_0}(\boldsymbol{\nu})_i. \tag{103}$$

Performing the change of variables (94) and (96), we reduce the control of $w(s)$ to the control of $(q(s), (\nu_i(s))_i, (\phi_i(s))_i)$ and we see from (103) that

$$\forall i = 1, \ldots, k, \;\; \phi_i(s_0) = 0. \tag{104}$$

Introducing

$$N(\boldsymbol{\nu}, s) := \max\left\{ s^{1/2+\eta}\|q(s)\|_{\mathscr{H}},\; \sup_i s^{1/2+|\gamma_i|}|\nu_i(s)|,\; \sup_{i\geq 2} s^{\eta}|\phi_i(s)|,\; s_0^{\eta}|\phi_1(s)| \right\}, \tag{105}$$

we see that the set $\mathscr{V}(s_0, s)$ introduced in Definition 3.4 is simply the unit ball of the norm $N(\boldsymbol{\nu}, s)$.

As we asserted in (99), our goal is to find $\boldsymbol{\nu}$ such that the associated $w \in \mathscr{C}([s_0,\infty),\mathscr{H})$ is globally defined for forward times and for all $s \geq s_0$, $N(\boldsymbol{\nu}, s) \leq 1$, i.e. $w(s) \in \mathscr{V}(s_0, s)$.

We argue by contradiction. Assume that the conclusion of Proposition 3.6 does not hold. In particular, for all $\boldsymbol{\nu}$, the exit time $s^*(s_0,\boldsymbol{\nu})$ is finite, where

$$s^*(s_0,\boldsymbol{\nu}) = \sup\{s \geq s_0 \mid \forall \tau \in [s_0, s],\ N(\boldsymbol{\nu},\tau) \leq 1\}. \tag{106}$$

Then by continuity, notice that

$$N(\boldsymbol{\nu}, s^*(s_0,\boldsymbol{\nu})) = 1, \tag{107}$$

and that the supremum defining $s^*(s_0,\boldsymbol{\nu})$ is in fact a maximum.
We now consider the (rescaled) flow for the ν_i, that is

$$\Phi : (s,\boldsymbol{\nu}) \mapsto \Gamma_s^{-1}({}^t(\nu_1(s),\dots,\nu_k(s))). \tag{108}$$

By the properties of the flow, Φ is a continuous function of $(s,\boldsymbol{\nu}) \in [s_0, s^*(s_0,\boldsymbol{\nu})] \times \mathbb{B}$. By definition of the exit time $s^*(s_0,\boldsymbol{\nu})$, we have that for all $s \in [s_0, s^*(s_0,\boldsymbol{\nu})]$, $\Phi(s,\boldsymbol{\nu}) \in \mathbb{B}$. The following claim allows us to conclude:

Claim. For s_0 large enough, we have:
(i) For all $\boldsymbol{\nu} \in \mathbb{B}$, $\Phi(s^*(s_0,\boldsymbol{\nu}),\boldsymbol{\nu}) \in \mathbb{S}$.
(ii) The flow $s \mapsto \Phi(s,\boldsymbol{\nu})$ is transverse (outgoing) when it hits $\mathbb{B}$ for $s \in [s_0, s^*(s_0,\boldsymbol{\nu})]$.
(iii) If $\boldsymbol{\nu} \in \mathbb{S}$, then $s^*(s_0,\boldsymbol{\nu}) = s_0$ and $\Phi(s^*(s_0,\boldsymbol{\nu}),\boldsymbol{\nu}) = \boldsymbol{\nu}$.

Indeed, from (ii) of this claim, $\boldsymbol{\nu} \to s^*(s_0,\boldsymbol{\nu})$ is continuous, hence from (i) and (iii),

$$\boldsymbol{\nu} \mapsto \Phi(s^*(s_0,\boldsymbol{\nu}),\boldsymbol{\nu})$$

is a continuous map from $\mathbb{B}$ to $\mathbb{S}$ whose restriction to $\mathbb{S}$ is the identity. By index theory, this is contradiction. Thus, there exists $\boldsymbol{\nu} \in \mathbb{B}$ such that for all $s \geq s_0$, $N(s_0,\boldsymbol{\nu}) \leq 1$, hence $w(s_0,\boldsymbol{\nu},\cdot,s) \in \mathscr{V}(s_0,s)$ which is the desired conclusion of Proposition 3.6. It remains to prove the Claim in order to conclude.

Remark. Note that we use (ii) of the Claim either with $s = s^*$, in order to prove the continuity of the exit time, or with $\boldsymbol{\nu} \in \mathbb{S}$ and $s = s_0$ to show (iii) of the same claim.

Proof of the Claim. In the following, the constant C stands for $C(s_0)$.
(i) Since for all $s \in [s_0, s^*(s_0,\boldsymbol{\nu})]$, $N(s_0,s) \leq 1$, it follows that $|\boldsymbol{\phi}(s)| \leq C$, hence from the change of variables (94) and (96) together with the definition (15) of $\bar{\zeta}_i(s)$, we see that

$$|\xi_i(s)| = \frac{2}{p-1}|\zeta_i(s) - \bar{\zeta}_i(s)| \leq C \text{ so that } |\zeta_i(s) - \zeta_{i-1}(s) - \frac{p-1}{2}\log s| \leq C.$$

This in turns implies that $1/(Cs^{|\gamma_i|}) \leq 1 - d_i^2 \leq C/s^{|\gamma_i|}$, except for $i = (k+1)/2$ if k is odd, where $1 - d_i(s)^2 \geq \frac{1}{C}$. This leads also to the bounds

$$J \leq \frac{C}{s}, \quad \bar{J} \leq \frac{C}{s^{1/2}}, \quad \hat{J} \leq \frac{C}{s^{\bar{p}/2}}, \quad \tilde{J} \leq \frac{C}{s^{2\eta}},$$

where the different quantities are defined in (89) and (100).
Hence, the estimates (93), (103), (91) and (101) read as follows: for all $s \in [s_0, s^*(s_0, \boldsymbol{\nu})]$

$$\|q(s)\|_{\mathscr{H}} \leq \frac{C}{s^{\bar{p}/2}} \leq \frac{1}{2s^{1/2+\eta}}, \text{ and from this} \tag{109}$$

$$|\dot{\nu}_i - \nu_i| \leq C\left(\frac{1}{s^{|\gamma_i|+\bar{p}}} + \frac{1}{s^{|\gamma_i|+1}} + \frac{1}{s^{1/2+|\gamma_i|+\bar{p}/2}}\right) \leq \frac{C}{s^{|\gamma_i|+1}} \tag{110}$$

$$\left|\dot{\phi}_i + \frac{m_i}{s}\phi_i\right| \leq C\left(\frac{1}{s^{1+2\eta}} + \frac{1}{s^{\bar{p}}} + \frac{1}{s^{3/2}} + \frac{1}{s^{(\bar{p}+1)/2}} + \frac{1}{s^{1+\delta}}\right) \leq \frac{C}{s^{1+2\eta}}, \tag{111}$$

provided that s_0 is large enough, where we used the definition (98) of η in the first and last line above.
Now, if $i = 2, \dots, k$, recall from Lemma 2.5 and the definition (98) of η that $0 < 2\eta < m_i$. Considering $g_i(s) = s^{m_i}\phi_i(s)$, we see that $|\dot{g}_i(s)| \leq Cs^{m_i-(1+2\eta)}$. Since $\phi_i(s_0) = 0$ by (104), we write

$$|\phi_i(s)| \leq \left(\frac{s_0}{s}\right)^{m_i}|\phi_i(s_0)| + \frac{C}{s^{2\eta}} = \frac{C}{s^{2\eta}} \leq \frac{1}{2s^{\eta}} \tag{112}$$

for s_0 large enough.
For ϕ_1, directly integrating the relation (111) and using the fact that $\phi_1(s_0) = 0$ (see (104)) gives

$$|\phi_1(s)| \leq |\phi_1(s_0)| + \frac{C}{s_0^{2\eta}} = \frac{C}{s_0^{2\eta}} \leq \frac{1}{2s_0^{\eta}} \tag{113}$$

for s_0 large enough.
Since $N(\boldsymbol{\nu}, s^*(s_0, \boldsymbol{\nu})) = 1$ by (107), we see from the definition (105) of the norm N together with (109), (112) and (113) that necessarily there exists $i = 1, \dots, k$ such that

$$s^*(s_0, \boldsymbol{\nu})^{1/2+|\gamma_i|}|\nu_i(s^*(s_0, \boldsymbol{\nu}))| = 1.$$

Using the definitions (108) and (102) of the flow Φ and the rescaling function Γ_s, we get to the conclusion of (i) of the Claim.

(ii) Assume that $\Phi(s, \boldsymbol{\nu}) \in \mathbb{S}$ for some $s \in [s_0, s^*(s_0, \boldsymbol{\nu})]$. Therefore, there exists $i = 1, \dots, k$ such that

$$s^{1/2+|\gamma_i|}|\nu_i(s)| = 1. \tag{114}$$

Using (110), we write

$$\begin{aligned}\frac{d}{ds}s^{1/2+|\gamma_i|}\nu_i(s) &= s^{1/2+|\gamma_i|}\left(\left(\frac{1}{2} + |\gamma_i|\right)\frac{\nu_i(s)}{s} + \dot{\nu}_i(s)\right)\\ &= s^{1/2+|\gamma_i|}\left(\nu_i(s)\left(1 + \frac{1}{2s} + \frac{|\gamma_i|}{s}\right) + O\left(\frac{1}{s^{1+|\gamma_i|}}\right)\right)\\ &= s^{1/2+|\gamma_i|}\left(\nu_i(s) + O\left(\frac{1}{s^{1+|\gamma_i|}}\right)\right)\end{aligned}$$

Using (114), we deduce that for s_0 large enough,

$$\frac{d}{ds}s^{1/2+|\gamma_i|}\nu_i(s)\cdot\frac{1}{s^{1/2+|\gamma_i|}\nu_i(s)}\geq\frac{1}{2}.$$

The same computation holds for any j such that $\nu_j(s^*)$ reaches one extremity of the interval. Thus, the flow is transverse on $\mathbb{B}$ and (ii) of the Claim holds.

(iii) Take $\boldsymbol{\nu}\in\mathbb{S}$. From (103) and the definition (108) of the flow Φ, we see that

$$\Phi(s_0,\boldsymbol{\nu})=\boldsymbol{\nu}. \tag{115}$$

Since $\boldsymbol{\nu}\in\mathbb{S}$, we can use (ii) of the Claim and see that the flow Φ is transverse to $\mathbb{B}$ at $s=s_0$. By definition of the exit time, we see that

$$s^*(s_0,\boldsymbol{\nu})=s_0.$$

Using (115), we get to the conclusion of (iii) of the Claim. This concludes the proof of the Claim. □

Since a contradiction follows from the Claim and index theory, this concludes the proof of Proposition 3.6 too. □

It remains to give the proof of Proposition 3.1 in order to conclude this section. Let us first recall from Lemma A.2 in [27] the following continuity result for the family of solitons $\kappa^*(d,\nu)$:

Lemma 3.7 (Continuity of κ^*)**.** *For all $A\geq 2$, there exists $C(A)>0$ such that if (d_1,ν_1) and (d_2,ν_2) satisfy*

$$\frac{\nu_1}{1-|d_1|},\frac{\nu_2}{1-|d_2|}\in[-1+\frac{1}{A},A], \tag{116}$$

then

$$\|\kappa^*(d_1,\nu_1)-\kappa^*(d_2,\nu_2)\|_{\mathscr{H}}\leq C(A)\left(\left|\frac{\nu_1}{1-|d_1|}-\frac{\nu_2}{1-|d_2|}\right|+|\arg\tanh d_1-\arg\tanh d_2|\right). \tag{117}$$

Remark. Since $\kappa(d,y)=\kappa^*(d,0,y)$ by definitions (10) and (80), this statement is a generalization of the continuity identity for the family $\kappa(d,y)$ given in (27).

With this lemma, we can give the proof of Proposition 3.1.

Proof of Proposition 3.1. Let us consider the solution constructed in Proposition 3.6. Since $w(s)\in\mathscr{V}(s_0,s)$ for all $s\geq s_0$, from Corollary 3.5 and the definition 3.4 of $\mathscr{V}(s_0,s)$, we see that (111) holds. In particular, for $i=1$, we see that

$$\forall s\geq s_0,\ \ |\phi_1'(s)|\leq\frac{C}{s^{1+2\eta}}.$$

Therefore, $\phi_1(s)$ converges to some $l_0 \in \mathbb{R}$ as $s \to \infty$. Since $\phi_i(s) \to 0$ for $i = 2, \dots, k$, using (96) and the fact that $e_1 =^t (1, \dots, 1)$ (see Lemma 2.5), we see that $\xi_i(s) \to l_0$. From (94), we see that $\zeta_i(s) - \bar{\zeta}_i(s) \to \zeta_0 \equiv \frac{(p-1)}{2} l_0$ for all $i = 1, ..., k$ and (86) follows. In particular,

$$1 - |d_i(s)| \sim C_i s^{-|\gamma_i|} \text{ as } s \to \infty,$$

hence, from the definition 3.4 of $\mathscr{V}(s_0, s)$, we have

$$\forall s \geq s_0, \quad \frac{|\nu_i|}{1 - |d_i(s)|} \leq C(s_0) s^{-\frac{1}{2}}.$$

Therefore, Lemma 3.7 applies and since $\kappa^*(d_i(s), 0, y) = \kappa(d_i(s), y)$ by definitions (10) and (80), we write

$$\|\kappa^*(d_i(s), \nu_i(s)) - (\kappa(d_i(s), 0))\|_{\mathscr{H}} \leq C(s_0) \frac{|\nu_i|}{1 - |d_i(s)|} \leq C(s_0) s^{-\frac{1}{2}}.$$

Since $\|q(s)\|_{\mathscr{H}} \leq \frac{C}{s^{\frac{1}{2}+\eta}}$ by definition 3.4 of $\mathscr{V}(s_0, s)$, we write from the definition (87) of $q(y, s)$,

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \begin{pmatrix} \displaystyle\sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leq \|q(s)\|_{\mathscr{H}} + C(s_0) s^{-\frac{1}{2}} \leq C(s_0) s^{-\frac{1}{2}}$$

and (85) follows. This concludes the proof of Proposition 3.1. □

4 Multi-solitons solution in the $u(x,t)$ setting

In this section, we use the multi-soliton solution in similarity variables given in Proposition 3.1 together with the Lorentz transform to prove Theorem 3 and Corollary 4. We divide this section in two subsections, each devoted to the proof of one statement.

4.1 Prescribing only one characteristic point

We prove Theorem 3 here. We proceed in 2 parts:
- in Part 1, we translate the construction of the previous section into the $u(x,t)$ setting, and recover a solution to our purpose, without the possibility of prescribing the center of mass. This part contains straightforward and obvious arguments which may be skipped by specialists. We give them for the reader's convenience;
- in Part 2, we apply the Lorentz transform to the solution constructed in Part 1, making the center of mass of the solitons equal to any prescribed value.

Proof of Theorem 3.
Part 1: A multi-soliton solution in the $u(x,t)$ without prescribing the center of mass

This part has straightforward arguments. It may be skipped by specialists.
Consider an integer $k \geq 2$ and consider $w(y,s)$ the solution of (5) constructed in Proposition 3.1.
Then, let us define $u(x,t)$ as the solution of equation (1) with initial data in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ whose trace in $(-1,1)$ is given by

$$u(x,0) = w(x,s_0) \text{ and } \partial_t u(x,0) = \partial_s w(x,s_0) + \frac{2}{p-1} w(x,s_0) + x\partial_y w(x,s_0). \tag{118}$$

We will see that $u(x,t)$ satisfies all the requirements in Theorem 3, except for subscribing the center of mass. More precisely, using the definition of similarity variables' transformation (4) in the other way, we translate the properties of $w(y,s)$ in the following properties of $u(x,t)$:

(i) *For all* $t \in [0,1)$ *and* $|x| < 1-t$,

$$u(x,t) = (1-t)^{-\frac{2}{p-1}} w\left(\frac{x}{1-t}, s_0 - \log(1-t)\right). \tag{119}$$

Indeed, by definition (4) of similarity variables, the function on the right-hand side of (119) is a solution to equation (1) with the same initial data (118) as $u(x,t)$. Since that initial data is in $H^1 \times L^2(-1,1)$ and equation (1) is well-posed in $H^1 \times L^2$ of sections of backward light cones, both solutions are equal from the uniqueness to the Cauchy problem and the finite speed of propagation, hence (119) holds. In particular, from (4), we have

$$\forall s \geq 0, \ \ \forall y \in (-1,1), \ \ w_{0,1}(y,s) = w(y, s+s_0). \tag{120}$$

(ii) *u is a blow-up solution.* Indeed, if not, then u is global and $u \in L^\infty_{loc}([0,\infty), \mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}))$. In particular, we write from the Sobolev injection, for all $s \geq 0$ and $\epsilon > 0$,

$$\|w_{0,1}(s)\|_{L^2_\rho} \leq C\|u\|_{L^\infty(|x|<1+\epsilon-t)} e^{-\frac{2s}{p-1}} \to 0 \text{ as } s \to \infty. \tag{121}$$

This is in contradiction with (120) and (85).

(iii) $T(0) = 1$. Indeed, from (120) we see that $u(x,t)$ is defined in the cone $|x| < 1-t$, $t \geq 0$, hence $T(0) \geq 1$. From (121), we see that if $T(0) > 1+\epsilon$ for some $\epsilon > 0$, then the same contradiction follows. Thus $T(0) = 1$.

(iv) From above, we can use the simplified notation for (4) and write w_0 instead of $w_{0,1}$, and rewrite (120) as follows:

$$\forall s \geq 0, \ \ \forall y \in (-1,1), \ \ w_0(y,s) = w(y, s+s_0).$$

Using (85) and (86), we see that (23) follows for w_0 with

$$\zeta_i(s) - \bar{\zeta}_i(s) \to \zeta_0 \text{ as } s \to \infty \text{ for } i = 1, ..., k$$

where $\zeta_0 \in \mathbb{R}$ and $(\bar{\zeta}_i(s))_i$ (15) is the explicit solution of system (16). Using the continuity result (27) for $\kappa(d,y)$, we see that (23) still holds if we slightly modify the $\zeta_i(s)$ by

putting $\zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0$ as required by (24). Finally, from the classification of the blow-up behavior for general solutions given in page 3, we clearly see that the origin is a characteristic point.
Thus, we have a solution obeying all the requirements of Theorem 3, except that we cannot prescribe the center of mass ζ_0 in (24).

Part 2: Prescribing the center of mass of the solitons

Now, we take the solution constructed in Part 1 and perform a Lorentz transform to be able to prescribe the center of mass of the solitons.

More precisely, given an integer $k \geq 2$, Part 1 gives a blow-up solution $u^\sharp(x^\sharp, t^\sharp)$ of equation (1) with 0 as a characteristic point such that $T^\sharp(0) = 1$ and

$$\left\| \begin{pmatrix} w_0^\sharp(s) \\ \partial_s w_0^\sharp(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i^\sharp(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty, \tag{122}$$

where $w_0^\sharp$ is its similarity variables' version around $(0, T^\sharp(0))$ introduced in (4),

$$d_i^\sharp(s) = -\tanh \zeta_i^\sharp(s), \quad \zeta_i^\sharp(s) = \bar{\zeta}_i(s) + \zeta_0^\sharp, \tag{123}$$

$(\bar{\zeta}_i(s))_i$ is the explicit solution of system (16) introduced in (15) and $\zeta_0^\sharp \in \mathbb{R}$. In particular, $u^\sharp$ is defined (at least) in the truncated cone $\mathscr{C}_{0,1,1} \cap \{t^\sharp \geq 0\}$ defined in (3).

Given an arbitrary $\zeta_0 \in \mathbb{R}$, our goal now is to construct u a blow-up solution of equation (1) with 0 as a characteristic point such that its similarity variables' version w_0 (4) has a profile decomposing into k solitons as in the following:

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty, \tag{124}$$

with

$$d_i(s) = -\tanh \zeta_i(s) \text{ and } \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0. \tag{125}$$

Since this would imply by definitions (15) and (31) of $\bar{\zeta}_i(s)$ and $\bar{\alpha}_i$ that

$$\frac{\zeta_i(s) + \cdots + \zeta_k(s)}{k} = \zeta_0,$$

we call this part of the proof "prescription of the center of mass".

For this, consider Lorentz transforms of $u^\sharp$: given $d \in (-1, 1)$ we consider

$$u(d; x, t) = u^\sharp(x^\sharp, t^\sharp) \text{ with } x^\sharp = \frac{x - d(t-1)}{\sqrt{1-d^2}} \text{ and } t^\sharp = 1 + \frac{t - 1 - dx}{\sqrt{1-d^2}}.$$

Note first that $u(d; x, t)$ is still a solution of equation (1). Note also that the cone $\mathscr{C}_{0,1,1}$ is preserved by the Lorentz transform, and that the image of the truncated cone $\mathscr{C}_{0,1,1} \cap$

$\{t^\sharp \geq 0\}$ is included in the truncated cone $\mathscr{C}_{0,1,1} \cap \{t \geq 1 - \sqrt{\frac{1-|d|}{1+|d|}}\}$.
Now, in self-similar variables, the Lorentz transform reads as follows:

$$w_0(d;y,s) := \mathscr{T}_d w_0^\sharp(y,s) = \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} w^\sharp(y^\sharp, s^\sharp), \quad y^\sharp = \frac{y+d}{1+dy}, \quad s^\sharp = s + \log \frac{\sqrt{1-d^2}}{1+dy}. \tag{126}$$

The following claim allows us to conclude:

Claim. We have the following:
(i) It holds that

$$\sup_{|y|<1} \left| (1-y^2)^{\frac{1}{p-1}} \left(w_0(y,s) - \sum_{i=1}^{k} (-1)^{i+1} \kappa(d * d_i^\sharp(s), y) \right) \right| \to 0 \quad \text{as} \quad s \to \infty,$$

where $d_1 * d_2 = \frac{d_1+d_2}{1+d_1 d_2}$.
(ii) There exists $t_d > 1 - \sqrt{\frac{1-|d|}{1+|d|}}$ such that $(u(t_d), \partial_t u(t_d)) \in H^1 \times L^2(|x| < 1 - t_d)$.

Indeed, let us define from u a solution $\hat{u}$ satisfying the requirements of Theorem 3. From translation invariance of equation (1), we may take the time origin at $t = t_d$. Take

$$d = \tanh(\zeta_0^\sharp - \zeta_0). \tag{127}$$

From (ii), we can consider $\hat{u}(x,t)$ the solution of equation (1) with initial data (at $t = t_d$) in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ whose trace in $(-1,1)$ is given by

$$\hat{u}(x,t_d) = u(d;x,t_d) \text{ and } \partial_t \hat{u}(x,t_d) = \partial_t u(d;x,t_d).$$

Following Part 1, we see that $\hat{u}$ is a blow-up solution, $\hat{T}(0) = 1$ and

$$\forall t \in [t_d, 1) \text{ and } |x| < 1-t, \quad \hat{u}(x,t) = u(d;x,t).$$

In particular,

$$\forall s \geq 0 \text{ and } |y| < 1, \quad \hat{w}_0(y,s) = w_0(y,s)$$

and (i) of the Claim provides us with an asymptotic expansion for $\hat{w}_0$ in L^∞ with the weight $(1-y^2)^{\frac{1}{p-1}}$ on the one hand. On the other hand, using the classification of all blow-up solutions of equation (1) given in page 3, we see that

$$\left\| \begin{pmatrix} \hat{w}_0(s) \\ \partial_s \hat{w}_0(s) \end{pmatrix} - \hat{\theta}_1 \begin{pmatrix} \sum_{i=1}^{\hat{k}} (-1)^{i+1} \kappa(\hat{d}_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ as } s \to \infty, \tag{128}$$

with $\hat{\theta}_1 = \pm 1$ and:
- either 0 is a non-characteristic point, hence $\hat{k} = 1$, $\hat{d}_1(s) \equiv \hat{T}'(0)$,
- or 0 is a characteristic point, with $\hat{k} \geq 2$, and $\hat{d}_i(s) = -\tanh \hat{\zeta}_i(s)$ with

$$\left| \hat{\zeta}_i(s) - \bar{\zeta}_i(s) \right| \leq C$$

for s large enough (use the definition (15) of $\bar\zeta_i(s)$ to derive this from (14)). Recalling the following Hardy-Sobolev inequality from Lemma 2.2 page 51 in [22]:

$$\forall h \in \mathscr{H}_0,\ \ \|h(1-y^2)^{\frac{1}{p-1}}\|_{L^\infty(-1,1)} \le C\|h\|_{\mathscr{H}_0}, \tag{129}$$

we write from (128) that

$$\sup_{|y|<1}\left|(1-y^2)^{\frac{1}{p-1}}\left(\hat w_0(y,s) - \hat\theta_1\sum_{i=1}^{\hat k}(-1)^{i+1}\kappa(\hat d_i(s),y)\right)\right| \to 0 \quad \text{as} \quad s\to\infty, \tag{130}$$

Introducing

$$\hat W_0(\xi,s) = (1-y^2)^{\frac{1}{p-1}}\hat w_0(y,s) \text{ with } y=\tanh\xi$$

and $\hat\kappa_0(\xi) = \kappa_0\cosh^{-\frac{2}{p-1}}\xi$, we write from (i) of the Claim and (130) two expansions of $\hat w_0$ as $s\to\infty$ as follows:

$$\sup_{\xi\in\mathbb{R}}|\hat W_0(\xi,s) - \sum_{i=1}^{k}(-1)^{i+1}\hat\kappa_0(\xi-\zeta_i^\sharp(s)+\arg\tanh d)| \to 0,$$

$$\sup_{\xi\in\mathbb{R}}|\hat W_0(\xi,s) - \hat\theta_1\sum_{i=1}^{\hat k}(-1)^{i+1}\hat\kappa_0(\xi-\hat\zeta_i(s))| \to 0 \text{ as } s\to\infty.$$

Comparing these two expansions for the same function, we immediately see that

$$\hat k = k,\ \ \hat\theta_1 = 1 \text{ and } \hat\zeta_i(s) = \zeta_i^\sharp(s) - \arg\tanh d + o(1) \text{ as } s\to\infty.$$

In particular, $\hat k \ge 2$, hence 0 is a characteristic point for $\bar u$. Moreover, using the continuity estimate (27) on $\kappa(d,y)$, we see that (128) holds also with

$$\hat\zeta_i(s) = \zeta_i^\sharp(s) - \arg\tanh d = \bar\zeta_i(s) + \zeta_0^\sharp - \arg\tanh d = \bar\zeta_i(s)+\zeta_0.$$

where we used the definitions (123) and (127) of $\zeta_i^\sharp(s)$ and d in this last line. This is the desired estimate in Theorem 3. It remains to prove the claim in order to conclude.

Proof of the Claim.
(i) Using the Hardy-Sobolev inequality of (129), we see that (122) yields the fact that

$$\sup_{s^\sharp\ge\tau^\sharp,|y^\sharp|<1}\left|(1-{y^\sharp}^2)^{\frac{1}{p-1}}\left(w_0^\sharp(y^\sharp,s^\sharp) - \sum_{i=1}^{k}(-1)^{i+1}\kappa(d_i^\sharp(s^\sharp),y^\sharp)\right)\right| \to 0 \quad \text{as} \quad \tau^\sharp\to\infty. \tag{131}$$

In the following, we will apply the Lorentz transform in the w version (126) to this estimate to get the desired conclusion.
Note first that straightforward calculations give the fact that $\mathscr{T}_d$ has a group structure, in the sense that

$$\mathscr{T}_{d_1}\circ\mathscr{T}_{d_2} = \mathscr{T}_{d_1 * d_2} \text{ where } d_1 * d_2 = \frac{d_1+d_2}{1+d_1d_2}.$$

Therefore, since we have $\kappa(d) = \mathscr{T}_d(\kappa_0)$ from the definition (10) of $\kappa(d,y)$, we see that

$$\kappa(d * d_i^\sharp(s), y) = \mathscr{T}_{d*d_i^\sharp(s)}(\kappa_0) = \mathscr{T}_d \circ \mathscr{T}_{d_i^\sharp(s)}(\kappa_0) = \mathscr{T}_d \kappa(d_i^\sharp(s)). \tag{132}$$

Since we have from (126) the fact that

$$\frac{(1-y^2)(1-d^2)}{(1+dy)^2} = 1 - {y^\sharp}^2,$$

we write from (126) and (132) for $s \geq -\log(1-t_d)$ and $|y| < 1$,

$$\begin{aligned}
&\left|(1-y^2)^{\frac{1}{p-1}} \left(w_0(y,s) - \sum_{i=1}^k (-1)^{i+1} \kappa(d * d_i^\sharp(s), y)\right)\right| \\
&= \left|(1-y^2)^{\frac{1}{p-1}} \left(\mathscr{T}_d w_0^\sharp(y,s) - \sum_{i=1}^k (-1)^{i+1} \mathscr{T}_d \kappa(d_i^\sharp(s), y)\right)\right| \\
&= \left|\left(\frac{(1-y^2)(1-d^2)}{(1+dy)^2}\right)^{\frac{1}{p-1}} \left(w_0^\sharp(y^\sharp, s^\sharp) - \sum_{i=1}^k (-1)^{i+1} \kappa(d_i^\sharp(s), y^\sharp)\right)\right| \\
&\leq \left|(1-{y^\sharp}^2)^{\frac{1}{p-1}} \left(w_0^\sharp(y^\sharp, s^\sharp) - \sum_{i=1}^k (-1)^{i+1} \kappa(d_i^\sharp(s^\sharp), y^\sharp)\right)\right| \\
&\qquad + \sum_{i=1}^k \left|(1-{y^\sharp}^2)^{\frac{1}{p-1}} \left(\kappa(d_i^\sharp(s^\sharp), y^\sharp) - \kappa(d_i^\sharp(s), y^\sharp)\right)\right|,
\end{aligned} \tag{133}$$

where $y^\sharp$ and $s^\sharp$ defined in (126) satisfy

$$|s^\sharp - s| \leq \frac{1}{2} \log \frac{1+|d|}{1-|d|}. \tag{134}$$

Using (129) and the continuity relation (27) of $\kappa(d,y)$, we write

$$\begin{aligned}
\sup_{|y^\sharp|<1} \left|(1-{y^\sharp}^2)^{\frac{1}{p-1}} \left(\kappa(d_i^\sharp(s^\sharp), y^\sharp) - \kappa(d_i^\sharp(s), y^\sharp)\right)\right| &\leq \left\|\kappa(d_i^\sharp(s^\sharp)) - \kappa(d_i^\sharp(s))\right\|_{\mathscr{H}_0} \\
&\leq C|\arg\tanh d_i^\sharp(s^\sharp) - \arg\tanh d_i^\sharp(s)|.
\end{aligned}$$

From the definition (123) of $d_i^\sharp(s)$ and (134), we see that

$$\sup_{|y^\sharp|<1} \left|(1-{y^\sharp}^2)^{\frac{1}{p-1}} \left(\kappa(d_i^\sharp(s^\sharp), y^\sharp) - \kappa(d_i^\sharp(s), y^\sharp)\right)\right| \leq \frac{C|s^\sharp - s|}{s} \leq \frac{C(d)}{s}. \tag{135}$$

Therefore, using (133) and (135), we write

$$\sup_{|y|<1} \left|(1-y^2)^{\frac{1}{p-1}} \left(w_0(y,s) - \sum_{i=1}^k (-1)^{i+1} \kappa(d * d_i^\sharp(s), y)\right)\right|$$

$$\leq \sup_{|y^\sharp|<1,\ s^\sharp \geq s-\frac{1}{2}\log\frac{1+|d|}{1-|d|}} \left|(1-{y^\sharp}^2)^{\frac{1}{p-1}}\left(w_0^\sharp(y^*,s^*)-\sum_{i=1}^k(-1)^{i+1}\kappa(d_i^\sharp(s^\sharp),y^\sharp)\right)\right| + \frac{C(d)}{s}.$$

Using (131), we conclude the proof of (i) of the Claim.

(ii) It is enough to prove that for some $s_2 > s_1 > 0$, we have

$$I_{d,s_1,s_2} := \int_{s_1}^{s_2}\int_{-1}^{1}(w_0(y,s))^2 + (\partial_y w_0(y,s))^2 + (\partial_s w_0(y,s))^2 dyds \leq C(s_2,s_1,d). \quad (136)$$

Indeed, if this is true, then, by the mean value theorem, there exists $s_d \in (s_1,s_2)$ such that

$$\int_{-1}^{1}(w_0(y,s_d))^2 + (\partial_y w_0(y,s_d))^2 + (\partial_s w_0(y,s_d))^2 dy = \frac{1}{s_2-s_1}I_{d,s_1,s_2} \leq \frac{C(s_2,s_1,d)}{s_2-s_1}.$$

Using the similarity transformation (4) in the other way, we get the desired estimate with $t_d = 1-e^{-s_d}$. Let us prove (136) then.
From the transformation (126), (134) and the similarity variables definition (4), we see that

$$\begin{aligned} I_{d,s_1,s_2} &\leq C(d)\int_{s_1-\frac{1}{2}\log\frac{1+|d|}{1-|d|}}^{s_2+\frac{1}{2}\log\frac{1+|d|}{1-|d|}}\int_{-1}^{1}(w_0^\sharp(y^\sharp,s^\sharp))^2 + (\partial_y w_0^\sharp(y^\sharp,s^\sharp))^2 + (\partial_s w_0^\sharp(y^\sharp,s^\sharp))^2 dy^\sharp ds^\sharp \\ &\leq C(d,s_2,s_1)\int_{t_1(d)}^{t_2(d)}\int_{|x^\sharp|<1-t^\sharp}(u^\sharp(x^\sharp,t^\sharp))^2 + (\partial_x u^\sharp(x^\sharp,t^\sharp))^2 + (\partial_t u^\sharp(x^\sharp,t^\sharp))^2. \end{aligned}$$

Since initial data for $u^\sharp$ is in $H^1 \times L^2(-1,1)$ and equation (1) is well-posed in $H^1 \times L^2$ of sections of the backward light cone with vertex $(0,1)$ (see the paragraph right after (119)), this latter integral is bounded in terms of d, s_1 and s_2. This concludes the proof of the Claim. □

Since the Claim implies Theorem 3, this concludes the proof of Theorem 3 too. □

4.2 Prescribing more characteristic points

We use the finite speed of propagation to derive the multiple characteristic points case (Corollary 4) from the one characteristic point case (Theorem 3).

Proof of Corollary 4. Let us first remark that thanks to the invariance of equation 1 under space and time translations together with the following dilation

$$\lambda \mapsto u_\lambda(\xi,\tau) = \lambda^{\frac{2}{p-1}}u(\lambda\xi,\lambda\tau),$$

we can prescribe the characteristic point and the blow-up time, in addition to the number of solitons and the center of mass. More precisely, given $\bar{x} \in \mathbb{R}$, $\bar{T} > 0$, $\bar{k} \geq 2$ and $\bar{\zeta}_0 \in \mathbb{R}$, there exists a blow-up solution $u_{\bar{x},\bar{T},\bar{k},\bar{\zeta}}$ of equation (1) in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ such that $\bar{x}$

is a characteristic point, $T(\bar{x}) = \bar{T}$ and $w_{\bar{x}}$ behaves as in (23) and (24) with $k = \bar{k}$ and $\zeta_0 = \bar{\zeta}_0$.

Let us now consider $I = \{1, ..., n_0\}$ or $I = \mathbb{N}$ and for all $n \in I$, $x_n \in \mathbb{R}$, $T_n > 0$, $k_n \geq 2$ and $\zeta_{0,n} \in \mathbb{R}$ such that

$$x_n + T_n < x_{n+1} - T_{n+1}.$$

From this condition, we can define a solution $u(x,t)$ of equation (1) in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R})$ by taking its initial data such that

$$\forall n \in I, \ \ \forall x \in (x_n - T_n, x_n + T_n), \ \ u(x,0) = u_{x_n,T_n,k_n,\zeta_{0,n}}(x,0)$$

and the same for time derivatives. From the finite speed of propagation, this identity propagates in the cone $\mathscr{C}_{x_n,T_n,1}$ for positive times, in the sense that

$$\forall n \in I, \ \ \forall t \in [0, T_n), \ \ \forall x \in (x_n - (T_n - t), x_n + (T_n - t)), \ \ u(x,t) = u_{x_n,T_n,k_n,\zeta_{0,n}}(x,t)$$

and the same for time derivatives. As we did in Part 1 of the proof of Theorem 3 above, we see by obvious arguments that $u(x,t)$ satisfies all the requirements of Corollary 4. □

A Lyapunov's Theorem

Lyapunov's Theorem is a classical result which we crucially use in the proof of Theorem 1 given in Section 2 above.
In this section, we give the statement we use and for the reader's convenience, its proof.

Theorem (Lyapunov's Theorem)**.** *Let K be a compact set of $\mathbb{R}^k$, $X : K \to \mathbb{R}^k$ be a vector field, and $x_0 \in \mathring{K}$, the interior of K. Denote by $(t,x) \mapsto \varphi(t,x)$ the flow of X (at time t, starting at point x at time 0).*
Assume that K is stable by the flow (in particular, for all $x \in K$, the flow is globally defined), and that there exists $L : K \to \mathbb{R}$, a continuous function (Lyapunov) such that

$$\forall x \in K \setminus \{x_0\}, \quad t \mapsto L(\varphi(t,x)) \text{ is (strictly) decreasing.}$$

Then, x_0 is a critical point for X (the only one in K), and for all $x \in K \setminus \{x_0\}$, $L(x) > L(x_0)$ (so that L reaches its infimum on K at x_0 only).
Furthermore, for all $x \in K$, $\varphi(t,x) \to x_0$ as $t \to +\infty$ (x_0 is a global attractor).

Remark. The stability of K can follow from various assumptions on L, for example if L is defined on a neighbourhood of K where the decreasing assumption holds on, and $K = L^{-1}((-\infty, \ell])$ for some $\ell \in \mathbb{R}$.

Proof. Let $c = \inf\{L(x) \mid x \in K\}$ and $I = L^{-1}(\{c\})$ be the set of points where L reaches its infimum. $I \neq \varnothing$ because L is continuous and K is compact. Now if $x \in K \setminus \{x_0\}$, then $L(\varphi(1,x)) < L(\varphi(0,x)) = L(x)$, so that $x \notin I$. Hence $I = \{x_0\}$ and for all $x \in K \setminus \{x_0\}$, $L(x) > L(x_0)$.

We now prove that x_0 is a critical point. We claim that it is enough to prove that

$$\text{there exists } t_0 > 0 \text{ such that } \forall t \in [0, t_0],\ \varphi(t, x_0) = x_0. \tag{137}$$

Indeed, if (137) is true, then by the uniqueness in the Theorem of Cauchy-Lipschitz, we see that $\varphi(t, x_0) = x_0$ for all $t \geq 0$ and x_0 is a critical point. Let us then prove (137).
Assume by contradiction that (137) does not hold. Then, there exists a decreasing sequence of times $t_n \to 0$ such that $\varphi(t_n, x_0) \neq x_0$. As $\varphi(t_{n-1}, x_0) = \varphi(t_{n-1} - t_n, \varphi(t_n, x_0))$, the sequence $L(\varphi(t_n, x_0))$ is strictly increasing, hence, it has a limit $a > \varphi(t_1, x_0) > c$. But $t_n \to 0$ so that $\varphi(t_n, x_0) \to x_0$ by continuity of the flow, and by continuity of L, $L(\varphi(t_n, x_0)) \to L(x_0) = c$, and we reached a contradiction. Thus, (137) holds and x_0 is a critical point.

Let us finally prove that $\varphi(t, x) \to x_0$. Note that it is enough to prove that

$$L(\varphi(t, x)) \to c \text{ as } t \to \infty. \tag{138}$$

Indeed, if a sequence $(y_n) \subset K$ is such that $L(y_n) \to c$, then $y_n \to x_0$, otherwise, there exists a subsequence z_n of y_n and $\varepsilon > 0$ such that for all n, $z_n \in K \setminus B(x_0, \varepsilon)$. This latter set is compact, so that up to a subsequence which we also denote z_n, z_n converges to some $z \in K \setminus B(x_0, \varepsilon)$. By continuity, $L(z) = c = L(x_0)$, so that $z = x_0$ from the fact that $I = \{x_0\}$, and we reached a contradiction. Let us then prove (138).
Assume by contradiction that (138) does not hold. Then, there exists $\delta > 0$ such that the nonincreasing function

$$L(\varphi(t, x)) \to c + \delta \text{ as } t \to +\infty. \tag{139}$$

As $(\varphi(t, x))_t$ remains in $K_\delta := K \cap L^{-1}([c + \delta, \infty))$ which is a compact, there exists an increasing sequence of times $t_n \to +\infty$ and $\bar{x} \in K_\delta$ such that $\varphi(t_n, x) \to \bar{x}$. Note in particular that

$$\bar{x} \neq x_0 \tag{140}$$

Now let $t \in \mathbb{R}$ and consider the flow starting from $\bar{x}$. By continuity of the flow, $\varphi(t, \bar{x}) = \lim_n \varphi(t, \varphi(t_n, x)) = \lim_n \varphi(t + t_n, x)$. As L is continuous, $L(\varphi(t + t_n, x)) \to L(\varphi(t, \bar{x}))$ on the one hand. On the other hand, from (139), we have $L(\varphi(t + t_n, x)) \to c + \delta$. Hence, for any $t \in \mathbb{R}$, $L(\varphi(t, \bar{x})) = c + \delta$. This is a contradiction because we are not on the stationary trajectory (see (140)) and L is strictly decreasing outside that trajectory. Thus, $\varphi(t, x) \to x_0$ as $t \to \infty$. This concludes he proof of Lyapunov's Theorem. □

B Dynamics of equation (5) near multi-solitons

This section is devoted to the proof of Proposition 3.3. Since the proof needs only minor refinements with respect to the proofs of Claims 3.8 and 3.9 in [27] and Proposition 3.2 in [26], we only give indications on the refinements. Hence, this section is not self-contained, since making it self-contained would add many pages with no new techniques with respect to [26] and [27].

Proof of Proposition 3.3. We first recall from Appendix C in [27] the equation satisfied by q defined in (87) for all $s \in [s_0, \bar{s})$:

$$\frac{\partial}{\partial s}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \hat{L}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} - \sum_{j=1}^{k}(-1)^j\left[(\nu_j'(s) - \nu_j(s))\partial_\nu\kappa^* + d_j'(s)\partial_d\kappa^*\right](d_j(s), \nu_j(s), y)$$
$$+ \begin{pmatrix} 0 \\ R \end{pmatrix} + \begin{pmatrix} 0 \\ f(q_1) \end{pmatrix} \tag{141}$$

where

$$\hat{L}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 + \psi q_1 - \frac{p+3}{p-1}q_2 - 2y\partial_y q_2 \end{pmatrix},$$
$$\psi(y,s) = p|K_1^*(y,s)|^{p-1} - \frac{2(p+1)}{(p-1)^2}, \qquad K_1^*(y,s) = \sum_{j=1}^{k}(-1)^i\kappa_1^*(d_j(s), \nu_j(s), y),$$
$$f(q_1) = |K_1^* + q_1|^{p-1}(K_1^* + q_1) - |K_1^*|^{p-1}K_1^* - p|K_1^*|^{p-1}q_1,$$
$$R = |K_1^*|^{p-1}K_1^* - \sum_{j=1}^{k}(-1)^j\kappa_1^*(d_j(s), \nu_j(s), y)^p.$$

We now proceed to the justification of the 3 estimates of Proposition 3.3, based on Lemma C2., Claims 3.8 and 3.9 of [27], together with Proposition 3.2 in [26].
Since estimate (88) holds for all $s \in [s_0, \bar{s})$, it is easy to see that all those results hold provided that s_0 is large enough. As a matter of fact, we take below $s \in [s_0, \bar{s})$ and s_0 large enough.
- Estimate (91) follows from (i) of Lemma C.2 in [27] (use in particular the last identity of the proof of (i) of Lemma C.2 there).
- Estimate (93) follows directly from (ii) and (iii) of Claim 3.8 in [27] (for details, see in particular the proof of (ii) of Claim 3.9 there).
- With respect to the analysis in [27], (92) needs some refinements, which can be found in [26]. Note first that we have a rough estimate from the statement of (i) of Lemma C.2 in [27] which we recall in the following:

$$\frac{|d_i'(s)|}{1 - d_i(s)^2} = |\zeta_i'(s)| \le C\left(\|q\|_{\mathscr{H}}^2 + J + \|q\|_{\mathscr{H}}\frac{|\nu_i|}{1 - d_i^2}\right). \tag{142}$$

Looking in the proof of this statement in that paper and using Appendix C of [26], we derive the fact that

$$\left|(-1)^{i+1}d_i'\Pi_0^{d_i^*}(\partial_d\kappa^*(d_i,\nu_i)) + \Pi_0^{d_i^*}((0,R))\right| \le C\left(\|q\|_{\mathscr{H}}^2 + J^{1+\delta_1} + \|q\|_{\mathscr{H}}\frac{|\nu_i|}{1 - d_i^2}\right) \tag{143}$$

for some $\delta_1 > 0$. It remains to estimate the two terms on the left-hand side of (143) in order to conclude.
The term $\Pi_0^{d_i^*}(\partial_d\kappa^*(d_i,\nu_i))$ has been evaluated in Claim 2.2 in [27], but we need to further

refine that estimate, given the fact that $\frac{|\nu_i|}{1-|d_i|}$ is small (see (88)). Using estimates (2.36) and (2.27) which are given in the proof of Claim 2.2 in [27], we see that

$$\frac{1}{c_0 L_i}\Pi_0^{d_i^*}(\partial_d \kappa^*(d_i,\nu_i)) \tag{144}$$
$$= -\frac{4}{p-1}\int_{-1}^{1} Y^2(1-Y^2)^{\frac{2}{p-1}-1}dY + (1-x_i)\left(x_i d_i^2 + \frac{p+1}{p-1}\right)\int_{-1}^{1}\frac{Y^2(1-Y^2)^{\frac{2}{p-1}}}{1-x_i^2 d_i^2 Y^2}dY$$

where $c_0 > 0$,

$$L_i = \frac{2\kappa_0(1-d_i^2)^{\frac{1}{p-1}-1}(1+\nu_i)^{-\frac{p+1}{p-1}}}{(p-1)(1-{d_i^*}^2)^{\frac{1}{p-1}}} \quad \text{and} \quad x_i = \frac{1}{\nu_i+1}.$$

Since $x_i d_i = \frac{d_i}{1+\nu_i} = d_i^*$, we write

$$\int_{-1}^{1}\frac{Y^2(1-Y^2)^{\frac{2}{p-1}}}{1-x_i^2 d_i^2 Y^2}dY \le \frac{1}{1-{d_i^*}^2}\int_{-1}^{1} Y^2(1-Y^2)^{\frac{2}{p-1}}dY. \tag{145}$$

Using (88) and the definition (89) of $\bar{J}$, we see that for s_0 large enough, we have

$$\frac{1}{1-{d_i^*}^2} \le \frac{C}{1-d_i^2} \quad \text{and} \quad |L_i - \frac{2\kappa_0}{(p-1)(1-d_i^2)}| \le C\frac{|\nu_i|}{1-d_i^2} \le C\bar{J}. \tag{146}$$

Using (144), (145) and (146), we see that

$$\left|\Pi_0^{d_i^*}(\partial_d \kappa^*(d_i,\nu_i)) + \frac{8\kappa_0 c_0}{(p-1)^2(1-d_i^2)}\int_{-1}^{1} Y^2(1-Y^2)^{\frac{2}{p-1}-1}dY\right| \le C\bar{J}. \tag{147}$$

Now, we estimate the second term of (143).

Proceeding as for the proof of Proposition 3.2 of [26] given in Section 3.3 of that paper, we derive the fact that

$$\left|\Pi_0^{d_i^*}((0,R)) - c_2(p)(-1)^i\lambda_i^{p-1}\left(\lambda_{i-1}e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - \lambda_{i+1}e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right)\right| \le CJ^{1+\delta_2} \tag{148}$$

for some $c_2(p) > 0$, $\delta_2 > 0$ and $\lambda_j(s) = \frac{(1-d_j(s)^2)^{\frac{1}{p-1}}}{[(1+\nu_j(s))^2-d_j(s)^2]^{\frac{1}{p-1}}}$, where by convention $\zeta_0(s) \equiv -\infty$ and $\zeta_{k+1}(s) \equiv +\infty$.

Since we have from (88) and (89), $|\lambda_j(s)-1| \le C\frac{|\nu_j(s)|}{1-|d_j(s)|} \le C\bar{J}(s)$ for s_0 large enough, we see from (148) that

$$\left|\Pi_0^{d_i^*}((0,R)) - c_2(p)(-1)^i\left(e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right)\right| \le CJ^{1+\delta_2} + CJ\bar{J}. \tag{149}$$

Since $d_i = -\tanh\zeta_i$ hence $\zeta_i' = -\frac{d_i'}{1-d_i^2}$, using (143), (147), (142) and (149), we see that (92) is proved. Finally, we would like to stress the fact that since our computations are based on those appearing in the proof of Proposition 3.2 of [26], the constant $c_1(p) > 0$ we get in (92) is the same as in that statement. This concludes the proof of Proposition 3.3. □

C Computation of the eigenvalues of the matrix M (61)

ggg We finish the proof of Lemma 2.5 here. It remains to prove the formula (62) for the m_i, $i = 1, \dots, k$ in order to conclude. Recall from (37) that

$$\sigma_i = \frac{i(k-i)}{2}, \quad i = 1, \dots, k-1.$$

For $j \in [\![1, k]\!]$, we define the tridiagonal squared matrix of size $k - j + 1$

$$M_k^j = \begin{pmatrix} -\sigma_j & \sigma_1 & & & & \\ \sigma_j & -\sigma_1 - \sigma_{j+1} & \sigma_2 & & & \\ 0 & \sigma_{j+1} & -\sigma_2 - \sigma_{j+1} & \sigma_3 & & \\ & \ddots & & \ddots & \ddots & \\ & & & & \sigma_{k-j+1} & \\ & & & \sigma_{k-2} & -\sigma_{k-j-1} - \sigma_{k-1} & \sigma_{k-j} \\ & & & & \sigma_{k-1} & -\sigma_{k-j} \end{pmatrix} \tag{150}$$

that is, on the upper diagonal lie σ_1, σ_2, $\dots$, σ_{k-1}, on the lower diagonal σ_j, σ_{j+1}, $\dots$, σ_{k-1}, and the sum of the coefficient of each *column* is zero. Note that M_k^k is the zero 1×1 matrix and that $M = M_k^1$. We now prove the following Lemma, which give the result for $j = 1$.

Lemma C.1. *M_k^j is similar to the diagonal matrix*

$$\mathrm{diag}(0, -j, -(j + (j+1)), -(j + (j+1) + (j+2)), \dots, -(j + (j+1) + \dots + (k-1)).$$

Indeed, when $j = 1$, we see that the eigenvalues of $M_k^1 = M$ are given by 0, -1, $-\frac{2\times 3}{2}$,....,$-\frac{(k-1)\times k}{2}$, which is the desired conclusion announced in (62). Let us prove the lemma now.

Proof. We will prove the result by decreasing induction on j. The result is obvious for $j = k$. If it holds for $j+1$, let us prove it for M_k^j. Let A be the $(k-j+1) \times (k-j+1)$ matrix

$$A = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}, \quad \text{so that} \quad A^{-1} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ -1 & & \ddots & & \vdots \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

Let us conjugate M_j^k by A and compute:

$$
\begin{aligned}
AM_k^jA^{-1} &= \begin{pmatrix} -\sigma_j & \sigma_1 & 0 & \cdots & 0 \\ 0 & -\sigma_{j+1} & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\sigma_{k-1} & \sigma_{k-j} \\ 0 & & \cdots & 0 & 0 \end{pmatrix} A^{-1} \\
&= \begin{pmatrix} -\sigma_1-\sigma_j & \sigma_1 & 0 & \cdots & 0 & 0 \\ \sigma_{j+1} & -\sigma_2-\sigma_{j+1} & \sigma_2 & \ddots & \vdots & \vdots \\ 0 & \sigma_{j+2} & -\sigma_3-\sigma_{j+2} & \sigma_3 & & \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ & & & & \sigma_{k-j-1} & 0 \\ 0 & \cdots & 0 & \sigma_{k-1} & -\sigma_{k-j}-\sigma_{k-1} & \sigma_{k-j} \\ 0 & & \cdots & 0 & 0 & 0 \end{pmatrix} \\
&= -j\,\mathrm{Id}_{k-j+1} + \left(\begin{array}{ccc|c} & & & 0 \\ & & & \vdots \\ & M_k^{j+1} & & 0 \\ & & & \sigma_{k-1} \\ \hline 0 & \cdots & 0 & j \end{array}\right),
\end{aligned}
$$

where the last line comes from the fact that

$$
\begin{aligned}
&-\sigma_1-\sigma_j+\sigma_{j+1} = \sigma_1-\sigma_2-\sigma_{j+1}+\sigma_{j+2} = \cdots \\
&\qquad = \sigma_{k-j-2}-\sigma_{k-j-1}-\sigma_{k-2}+\sigma_{k-1} = \sigma_{k-j-1}-\sigma_{k-j}-\sigma_{k-1} = -j
\end{aligned}
$$

(because σ_i is quadratic in i with highest order term $-i^2/2$).

The induction hypothesis gives that the right-hand side block matrix is diagonalizable, with eingenvalues j, 0, $-(j+1)$, $-((j+1)+(j+2))$, $\dots$, $-((j+1)+\cdots+(k-1))$. Hence M_k^j is diagonalizable with eigenvalues 0, $-j$, $-(j+(j+1))$, $-(j+(j+1)+(j+2))$, $\dots$, $-(j+(j+1)+\cdots+(k-1))$. This concludes the induction and concludes the proof of Lemma 2.5 too. □

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