Asymptotic stability of solitons for the Zahkarov-Kuznetsov equation

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We consider the Zahkarov-Kuznetsov equation in dimension $d \geq 2$

$$\text{(ZK)} \qquad \partial_t u + \partial_{x_1}\big(\Delta u + u^2\big) = 0,$$

where $u = u(x,t)$ is a real-valued function, $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $t \in \mathbb{R}$. (ZK) was introduced by Zakharov and Kuznetsov [5] to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma in dimensions 2 and 3. (ZK) was derived from the Euler-Poisson system with magnetic field in the long wave limit was carried out by Lannes, Linares and Saut [6]; and from the Vlasov-Poisson system in a combined cold ions and long wave limit, by Han-Kwan [4].
The Cauchy problem for (ZK) in $H^s(\mathbb{R}^d)$ has been extensively studied: let us mention the currently optimal results for local well posedness in dimension 2, for $s > 1/2$ by Grünrock and Herr [3], and by Molinet and Pilod [10] (solutions in $H^1(\mathbb{R}^2)$ are global); and in dimension 3 for $s > 1$ by Ribaud and Vento [11].
We are interested in studying the flow of (ZK) around special travelling wave solutions called solitons. They are solutions fo the form

$$Q_c(x_1 - ct, x_2) \quad \text{with} \quad Q_c(x) \underset{|x|\to+\infty}{\longrightarrow} 0\,, \quad c > 0$$

(the travelling speed must lie along the privileged direction x_1 if one expects the travelling wave to be in H^1), where $Q_c(x) = c^{1/(p-1)}Q(c^{1/2}x)$ and $Q > 0$ satisfies

$$-\Delta Q + Q - Q^p = 0.$$

A. de Bouard [1] proved that the L^2-subcritical solitons are orbitally stable (using concentrate compactness *à la* Cazenave-Lions). The main result of [2] is the *asymptotic stability* of solitons of (ZK) in the case $d = 2$.

Theorem 1 (Asymptotic stability)**.** *Assume $d = 2$. Let $c_0 > 0$. For any $\beta > 0$, there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2))$ is a solution of* (ZK) *satisfying*

$$\|u(0) - Q_{c_0}\|_{H^1} \leqslant \varepsilon,$$

then the following holds true.
There exist $c_+ > 0$ with $|c_+ - c_0| \leq K_0\varepsilon$, for some positive constant K_0 independent of ε_0, and $\rho = (\rho_1, \rho_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ such that

$$(1) \qquad u(\cdot, t) - Q_{c_+}(\cdot - \rho(t)) \to 0 \quad \textit{in } H^1(x_1 > \beta t) \quad \textit{as} \quad t \to +\infty,$$

$$(2) \qquad \rho_(t) \to (c_+, 0) \quad \textit{as} \quad t \to +\infty.$$

In fact, the convergence (1) can also be obtained in regions of the form

$$\mathcal{AS}(t,\theta) := \Big\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 - \beta t + (\tan\theta)\, x_2 > 0\Big\}, \text{ where } \theta \in (-\frac{\pi}{3}, \frac{\pi}{3})\,.$$

The maximal angle $\pi/3$, which appears in the crucial monotonicity formula (in physical space), also occurs in Fourier space: for example, when proving the following Strichartz estimate – used in [10] to improve the well-posed results for ZK at low regularity –

$$\big\| |K(D)|^{\frac{1}{8}} e^{-t\partial_{x_1}\Delta}\varphi \big\|_{L^4_{xt}} \leqslant C\|\varphi\|_{L^2} ,$$

where $|K(D)|^{\frac{1}{8}}$ is the Fourier multiplier associated to the symbol $|K(k_1,k_2)|^{\frac{1}{8}} = |3k_1^2 - k_2^2|^{\frac{1}{8}}$. Observe that the multiplier $|K(k_1,k_2)|^{\frac{1}{8}}$ cancels out along the cone $|k_2| = \tan(\frac{\pi}{3})|k_1|$.

The above result follows the framework developed by Martel and Merle [7, 8] for (gKdV), and which we extend to higher space dimension. The proof does neither rely on the structure of the nonlinearity $\partial_{x_1}(u^2)$ of (ZK) – which is *not* integrable – neither on the dimension d. Indeed, it could be extended to different d and p as long as

- local well posedness holds in H^1 (not available in dimension 3).
- a sign condition holds, namely that $\langle \mathcal{L}^{-1}\Lambda Q, \Lambda Q\rangle_{L^2} < 0$. Here $\mathcal{L} = -\Delta + 1 - pQ^{p-1}$ is the linearized operator around Q, and $\Lambda Q = \frac{d}{dc}Q_c\big|_{c=1}$ is the scaling operator.

The above sign spectral condition ensures a certain coercivity in a crucial Virial identity: it has been numerically checked in dimension 2 for $p < 2.1491$ and in dimension 3 for $p < 1.8333$. We however believe that the coercivity could be obtained for a large class of p (observe nonetheless that (ZK) is L^2 critical when $d = 3$, so that solitons are expected to be unstable).

The tools developed for Theorem 1 also apply in the context of a sum of decoupled solitons: they allow to show stability and asymptotic stability of multisolitons in the sense of (1) and (2).

Let us finally mention a few open problems. We would be interested in understanding the behavior in the whole space (i.e. convergence in $H^1(\mathbb{R}^d)$ in (1)): this would require improved dispersion estimates and revisit the theory of global well posedness for small data. A second natural question regards the long time dynamics of large solution in dimension 3. More precisely, we conjecture the existence of finite-time blowup solution for 3D (ZK), as it is expected in the L^2 critical context. Also, (ZK) admits another nonlinear solution: the line soliton. Although it does not lie in $H^1(\mathbb{R}^d)$, this object is worth studying: this was initiated by Mizumachi [9] for the Kadomtsev-Petviashvili II equation, another extension of (KdV) in dimension 2.

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